

On the strict monotonicity of spectral radii for classes of bounded positive linear operators

Chris Guiver¹ 

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Abstract Strict monotonicity of the spectral radii of bounded, positive, ordered linear operators is investigated. It is well-known that under reasonable assumptions, the spectral radii of two ordered positive operators enjoy a non-strict inequality. It is also well-known that a “strict” inequality between operators does not imply strict monotonicity of the spectral radii in general—some additional structure is required. We present a number of sufficient conditions on both the cone and the operators for such a strict ordering to hold which generalise known results in the literature, and have utility in comparison arguments, ubiquitous in positive systems theory.

Keywords Comparison argument · Ordered Banach space · Positive linear operator · Spectral radius

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1 Introduction

We investigate the following problem pertaining to the spectral radii of bounded positive linear operators which admit a certain ordering. Here, and throughout the manuscript, \mathcal{X} denotes a real Banach space, with positive cone $\mathcal{K} \subset \mathcal{X}$ which induces the partial order \leq or \geq and $A_1, A_2 \in \mathcal{B}(\mathcal{X})$ denote bounded, positive linear operators with $r(A_1) > 0$. Given these hypotheses, we seek to investigate when

✉ Chris Guiver
c.guiver@bath.ac.uk

¹ Department of Mathematical Sciences, University of Bath, Bath, UK

$$A_2 \leq A_1 \text{ and } A_2 \neq A_1 \Rightarrow r(A_2) < r(A_1), \tag{1.1}$$

holds. It is known in the finite-dimensional case that irreducibility of A_2 is sufficient for (1.1), see [22, Theorem 9]. For general cones, it is known that if \mathcal{K} is normal and reproducing, then

$$A_2 \leq A_1 \Rightarrow r(A_2) \leq r(A_1), \tag{1.2}$$

(see, for example, [19, Theorem 4.2] or [5, Theorem 1.1]). The non-strict inequality (1.2) has been considered for more general cones and positive operators in [5], which builds on earlier work such as [23], and shown not to hold in general when the assumptions of normality or reproducing are dropped, and not replaced with suitable alternatives. The strict inequality (1.1) has been considered in [19], and sufficient conditions given. By way of further background, we mention that there is also a body of work on monotonicity of the spectral radius for commuting ordered Banach algebras, see [21] and the references therein.

The trivial example wherein $\mathcal{X} = \mathbb{R}^2, \mathcal{K} = \mathbb{R}_+^2$ (which is normal and reproducing), with the usual partial ordering of componentwise inequality, and

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } A_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad x > 0, \tag{1.3}$$

shows that (1.1) need not hold in general. As another example, consider the bounded linear operators

$$A_2(x_1, x_2, \dots) = (0, x_1, x_2, \dots) \text{ and } A_1(x_1, x_2, \dots) = (x_1, x_1, x_2, \dots),$$

defined on the space of convergent sequences with zero limit, equipped with the supremum norm, and positive cone consisting of component wise nonnegative sequences. Clearly $A_2 \leq A_1, A_2 \neq A_1$, but both A_2 and A_1 are isometries and so $r(A_1) = r(A_2) = 1$.

We acknowledge that there are elementary sufficient conditions for (1.1), such as if \mathcal{K} is normal and reproducing and

$$\exists \rho > 0 : A_2 + \rho I \leq A_1 \Rightarrow r(A_2) < r(A_1), \tag{1.4}$$

or

$$\exists \gamma \in (0, 1) : A_2 \leq \gamma A_1 \Rightarrow r(A_2) < r(A_1). \tag{1.5}$$

Both (1.4) and (1.5) follow from straightforward adjustments to (1.2), using the known equalities $r(A_2 + \rho I) = r(A_2) + \rho$ and $r(\gamma A_1) = \gamma r(A_1)$. The assumptions in (1.4) and (1.5) are too conservative for many applications, however. We are interested in the strict inequality (1.1) owing to its utility for discrete-time positive dynamical systems, where comparison arguments are readily applicable, such as [10], particularly in the infinite-dimensional case. For example, much attention has been devoted in theoretical ecology to discrete-time dynamical systems specified by certain classes of integral operators, so-called Integral Projection Models [7,8]. Here the spectral radius gives a theoretical long-term exponential growth (or decline) rate of a population.

There is considerable overlap between the present work and aspects of [19], where (1.1) is also considered. Briefly, we derive sufficient conditions for (1.1) which are distinct to those in [19] and we highlight the differences in the manuscript. Finally, we draw heavily on the textbook [12] but, to the best of our knowledge, the results presented here do not appear in [12] or indeed elsewhere in the literature.

2 Notation and preliminaries

There are a number of conventions pertaining to terminology in the positive operator and positive systems literature, which are not all equivalent, and we use those in [12]. We briefly recall some key terms. Let $(\mathcal{X}, \|\cdot\|)$ denote a real Banach space. A (positive) cone $\mathcal{K} \subseteq \mathcal{X}$ is a closed subset of \mathcal{X} such that $\mathcal{K} + \mathcal{K} \subseteq \mathcal{K}$, $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$ and $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$. The cone is called reproducing (also sometimes known as generating) if $\mathcal{X} = \mathcal{K} - \mathcal{K}$ and normal if $0 \leq x \leq y$ implies that $\|x\| \leq a\|y\|$ for some constant $a > 0$ which is independent of x and y . A cone is called solid if it has non-empty interior. Solid cones are reproducing.

For $u \in \mathcal{K} \setminus \{0\}$ we shall require the set

$$\mathcal{X}_u := \{x \in \mathcal{X} : -\gamma u \leq x \leq \gamma u, \text{ for some } \gamma \geq 0\},$$

(see [12, p. 42]). It is clear that $u \in \mathcal{X}_u$ and hence $\mathcal{X}_u \neq \emptyset$. Furthermore, as \mathcal{X}_u is closed under addition and scalar multiplication, it follows that $\mathcal{X}_u \subseteq \mathcal{X}$ is a subspace. Thus \mathcal{X}_u is a normed space when equipped with

$$\|x\|_u := \inf\{\gamma \geq 0 : -\gamma u \leq x \leq \gamma u\}, \quad x \in \mathcal{X}_u,$$

which has the elementary properties:

- i. $\|u\|_u = 1$;
- ii. for $x, y \in \mathcal{X}_u \cap \mathcal{K}$, $x \leq y$ implies that $\|x\|_u \leq \|y\|_u$;
- iii. for $x \in \mathcal{X}_u$, $-\|x\|_u u \leq x \leq \|x\|_u u$.

For Banach spaces \mathcal{X} and \mathcal{Y} , we let $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{B}(\mathcal{X})$ denote the set of bounded linear operators $\mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{X} \rightarrow \mathcal{X}$, respectively. We let $\mathcal{C}(\mathcal{X}) \subseteq \mathcal{B}(\mathcal{X})$ denote the subset of compact operators. The (continuous) dual of \mathcal{X} , the set of bounded, real-valued linear functionals on \mathcal{X} , is denoted $\mathcal{X}' = \mathbb{B}(\mathcal{X}, \mathbb{R})$ and equipped with usual norm

$$\|f\|_{\mathcal{X}'} = \sup_{\substack{x \in \mathcal{X} \\ \|x\|=1}} |f(x)|.$$

Given $A \in \mathcal{B}(\mathcal{X})$, we recall that the adjoint operator $A' \in \mathcal{B}(\mathcal{X}')$ is defined by $(A'f)(x) = f(Ax)$ for all $f \in \mathcal{X}'$ and all $x \in \mathcal{X}$ (see, for example, [13, Definition 4.5-1, p. 232]). The adjoint operator A' is bounded with respect to the induced operator norm

$$\|A'\|_{\mathcal{X}'} = \sup_{\substack{f \in \mathcal{X}' \\ f \neq 0}} \frac{\|A'f\|_{\mathcal{X}'}}{\|f\|_{\mathcal{X}'}}.$$

If \mathcal{Y} has cone \mathcal{L} , then the operator $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is called positive if $A\mathcal{K} \subseteq \mathcal{L}$. A positive operator $A \in \mathcal{B}(\mathcal{X})$ is called u -bounded if there exist functions $\alpha, \beta : \mathcal{X} \rightarrow \mathbb{R}_+$ such that

$$\alpha(x)u \leq Ax \leq \beta(x)u \quad \forall x \in \mathcal{K}, \tag{2.1}$$

and $\alpha(x), \beta(x) > 0$ if $x \neq 0$. The term u -bounded from above means that only the second inequality in (2.1) holds. If \mathcal{K} is reproducing and $A \in \mathcal{B}(\mathcal{X})$ is a positive operator which is u -bounded from above, then $A : \mathcal{X} \rightarrow \mathcal{X}_u$ is well-defined. Moreover, $A|_{\mathcal{X}_u} \in \mathcal{B}(\mathcal{X}_u)$ and the induced operator norm satisfies

$$\|A\|_{\mathcal{X}_u} = \sup_{\substack{x \in \mathcal{X}_u \\ x \neq 0}} \frac{\|Ax\|_u}{\|x\|_u} = \|Au\|_{\mathcal{X}_u}. \tag{2.2}$$

The above claims are all easily established from their definitions.

We shall make use of the following properties without further reference. First, if the three conditions all hold: (i) $\mathcal{W} \subseteq \mathcal{X}$ is continuously embedded in \mathcal{X} ; (ii) $A \in \mathcal{B}(\mathcal{X})$ has positive spectral radius, and; (iii) additionally $A \in \mathcal{B}(\mathcal{X}, \mathcal{W})$, then the spectral radii of $A \in \mathcal{B}(\mathcal{X})$ and $A|_{\mathcal{W}} \in \mathcal{B}(\mathcal{W})$ are equal.

Second, if $\mathcal{X} = \overline{\mathcal{K} - \mathcal{K}}$, $A_1, A_2 \in \mathcal{B}(\mathcal{X})$, $A_1 \leq A_2$ and $A_1 \neq A_2$, then there exists $x^* \in \mathcal{K}$ such that

$$(A_1 - A_2)x^* \neq 0, \tag{2.3}$$

which is readily established by contraposition. Indeed, if (2.3) fails, that is,

$$(A_1 - A_2)v = 0 \quad \forall v \in \mathcal{K}, \tag{2.4}$$

then, for arbitrary $x \in \mathcal{X}$, there exists $(u_n)_{n \in \mathbb{N}} \subset \mathcal{K}$, $(v_n)_{n \in \mathbb{N}} \subset \mathcal{K}$ such that

$$x = \lim_{n \rightarrow \infty} (u_n - v_n).$$

Thus, by (2.4) and continuity of A_1 and A_2

$$\begin{aligned} A_1x &= A_1 \lim_{n \rightarrow \infty} (u_n - v_n) = \lim_{n \rightarrow \infty} (A_1u_n - A_1v_n) = \lim_{n \rightarrow \infty} (A_2u_n - A_2v_n) \\ &= A_2 \lim_{n \rightarrow \infty} (u_n - v_n) = A_2x, \end{aligned}$$

implying that $A_1 = A_2$.

Following the convention of [12], we say that positive $A \in \mathcal{B}(\mathcal{X})$ is irreducible if $Ax \leq \kappa x$ for some $\kappa \geq 0$ and $x \in \mathcal{K} \setminus \{0\}$ implies that x is a quasi-interior point of \mathcal{K} . Recall that $x \in \mathcal{K}$ is a quasi-interior point if $f(x) > 0$ for all non-zero, positive functionals $f \in \mathcal{X}'$. A positive u -bounded operator $A \in \mathcal{B}(\mathcal{X})$ is irreducible if u is a quasi-interior point. We comment that for non-solid cones, the definition of quasi-interior point used here is *not* equivalent to that used in [23], see [12, p. 36]. If \mathcal{K}

is solid, then the sets of quasi-interior points and interior points coincide. We let \mathcal{K}' denote the set of positive functionals in $\mathcal{B}(\mathcal{X}, \mathbb{R})$, which is a positive cone if, and only if, $\mathcal{X}' = \overline{\mathcal{K} - \mathcal{K}}$.

Finally, we note that the arguments which follow make assertions about the spectrum of an operator $A \in \mathcal{B}(\mathcal{X})$, and so strictly speaking we extend A to the complexification of \mathcal{X} , denoted \mathcal{X}_c in the usual way; see, for example [6, p.79].

3 Strict monotonicity of spectral radii

Our main results are contained here. The first subsection considers estimates in the spirit of (1.4) and (1.5). The second appeals to spectral properties of positive operators. For notational convenience throughout, let $r_1 := r(A_1)$ and $r_2 := r(A_2)$.

Strict monotonicity of spectral radii by direct estimates

Lemma 3.1 *Let \mathcal{X} denote a real Banach space, with a cone $\mathcal{K} \subset \mathcal{X}$ and positive operators $A_1, A_2 \in \mathcal{B}(\mathcal{X})$ which satisfy $A_2 \leq A_1$ and $A_1 \neq A_2$. If either of the following:*

- (1) *there exist $w_2 \in \mathcal{K} \setminus \{0\}$ such that $A_2 w_2 = r_2 w_2$, $\varepsilon > 0$ and $n \in \mathbb{N}$ such that*

$$r_2^n w_2 \leq (A_1 - \varepsilon I)^n w_2, \tag{3.1}$$

or $\gamma \in (0, 1)$ such that

$$r_2^n w_2 \leq \gamma^n A_1^n w_2, \tag{3.2}$$

- (2) *there exist $f_2 \in \mathcal{K}' \setminus \{0\}$ such that $A'_2 f_2 = r_2 f_2$, $\varepsilon > 0$ and $n \in \mathbb{N}$ such that*

$$r_2^n f_2 \leq (A'_1 - \varepsilon I)^n f_2,$$

or $\gamma \in (0, 1)$ such that

$$r_2^n f_2 \leq \gamma^n (A'_1)^n f_2,$$

hold, then $r(A_2) < r(A_1)$.

Proof We consider the hypotheses in (1). If (3.1) holds, then $r_2^n w_2 = A_2^n w_2 \leq (A_1 - \varepsilon I)^n w_2$. It follows from [12, Lemma 9.1, p. 89] that

$$r_2^n \leq r((A_1 - \varepsilon I)^n) = (r(A_1 - \varepsilon I))^n,$$

whence

$$r(A_2) \leq r(A_1 - \varepsilon I) = r(A_1) - \varepsilon < r(A_1).$$

If (3.2) holds instead, then we again invoke [12, Lemma 9.1, p. 89] to see that

$$r_2^n \leq r((\gamma A_1)^n) = \gamma^n (r(A_1))^n,$$

yielding

$$r(A_2) \leq \gamma r(A_1) < r(A_1),$$

as required. The proof for (2) is identical, save using $r(A') = r(A)$ for $A \in \mathcal{B}(\mathcal{X})$. \square

Although Lemma 3.1 is appealing in that no conditions are placed on the cone \mathcal{K} , the difficulty with using the lemma in practice is verifying that the inequalities in (1) or (2) hold, which requires knowledge of w_2 or f_2 , respectively. It is well-known that for general operators, the assumptions in (1) and (2) are not symmetric—existence of eigenfunctionals need not imply existence of eigenvectors, for instance, see Example 4.1.

Lemma 3.2 *Let \mathcal{X} denote a real Banach space, with a reproducing and normal cone $\mathcal{K} \subset \mathcal{X}$ and positive linear operators $A_1, A_2 \in \mathcal{B}(\mathcal{X})$ which satisfy $A_2 \leq A_1$ and $A_1 \neq A_2$. Assume that $w_1 \in \mathcal{K} \setminus \{0\}$ satisfies $A_1 w_1 = r_1 w_1$ and that one of the following:*

- A_1 is w_1 -bounded from above;
- A_1 is u -bounded, for some $u \in \mathcal{K} \setminus \{0\}$;

hold. It follows that $r(A_2) < r(A_1)$ if, and only if, there exist $\gamma \in (0, 1)$ and $N \in \mathbb{N}$ such that

$$A_2^n w_1 \leq (\gamma r_1)^n w_1 \quad \forall n \in \mathbb{N}, n \geq N. \tag{3.3}$$

Clearly, $A_2 w_1 \leq \gamma r_1 w_1$ is sufficient for (3.3) to hold with $N = 1$. However, consider the simple example

$$\mathcal{X} = \mathbb{R}^3, \quad \mathcal{K} = \mathbb{R}_+^3, \quad A_2 = \begin{pmatrix} 0 & 0 & \kappa \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where $\kappa \in (0, 1)$. We have that $r_2 < r_1 = 1$ and may choose $w_1 = (1 \ 1 \ 1)^T$. By considering $A_2^n w_1$ for $n \in \mathbb{N}$ we see that (3.3) holds for $N \geq 3$, but not for $N \in \{1, 2\}$.

Proof of Lemma 3.2 The assumption that \mathcal{K} is normal implies that \mathcal{X}_{w_1} is continuously embedded in \mathcal{X} , as for $x \in \mathcal{X}_{w_1}$

$$-\|x\|_{w_1} w_1 \leq x \leq \|x\|_{w_1} w_1 \quad \Rightarrow \quad 0 \leq x + \|x\|_{w_1} w_1 \leq 2\|x\|_{w_1} w_1.$$

By the reverse triangle inequality and normality, it follows that

$$\|x\| - \|x\|_{w_1} \|w_1\| \leq \|x + \|x\|_{w_1} w_1\| \leq 2a\|x\|_{w_1} \|w_1\|,$$

for some $a > 0$. We conclude that

$$\|x\| \leq (2a + 1)\|w_1\|\|x\|_{w_1},$$

as required. Our assumptions imply that A_1 (and so also A_2) are w_1 -bounded from above. Indeed, if A_1 is u -bounded, then combining (2.1) with $A_1 w_1 = r_1 w_1$, yields

$$\alpha(w_1)u \leq A_1 w_1 = r_1 w_1 \implies u \leq \frac{r_1}{\alpha(w_1)} w_1.$$

Here we have used that $\alpha(w_1) > 0$ as $w_1 \neq 0$. We conclude that

$$A_1 x \leq \beta(x)u \leq \frac{r_1 \beta(x)}{\alpha(w_1)} w_1 = \gamma(x)w_1 \quad \forall x \in \mathcal{K}, \tag{3.4}$$

where $\gamma(x) := r_1 \beta(x) / \alpha(w_1)$.

Therefore, by an abuse of notation we consider $A_1, A_2 : \mathcal{X}_{w_1} \rightarrow \mathcal{X}_{w_1}$, which are bounded operators with the same respective spectral radii as $A_1, A_2 \in \mathcal{B}(\mathcal{X})$.

Assume first that $r(A_2) < r(A_1)$. It follows that $r(A_2) < \gamma r(A_1) = r(\gamma A_1)$, for some $\gamma \in (0, 1)$. Therefore, by the Gelfand formula for the spectral radius, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} \|A_2^n w_1\|_{w_1} &= \|A_2^n\|_{w_1} \leq \|(\gamma A_1)^n\|_{w_1} = \|(\gamma A_1)^n w_1\|_{w_1} \\ &= \gamma^n r_1^n \|w_1\|_{w_1} = \gamma^n r_1^n, \quad \forall n \geq N. \end{aligned}$$

Thus, by definition of the norm on \mathcal{X}_{w_1}

$$A_2^n w_1 \leq \|A_2^n w_1\|_{w_1} w_1 \leq \gamma^n r_1^n w_1 \quad \forall n \geq N,$$

which is (3.3). The converse argument reverses these steps, using the property (ii). \square

In its simplest form Lemma 3.2 requires verifying that inequality $A_2 w_1 \leq \gamma r_1 w_1$, for some $\gamma \in (0, 1)$ and where w_1 satisfies $A_1 w_1 = r_1 w_1$. More assumptions are placed on the cone than in Lemma 3.1. Moreover, an obvious corollary follows by applying Lemma 3.2 to the adjoint operator, which we state next, and crucially use that normal and reproducing are dual notions between \mathcal{K} and \mathcal{K}' , see [12, Theorems 4.5, 4.6, p. 40].

Corollary 3.3 *Imposing the notation and assumptions of Lemma 3.2, assume that $f_1 \in \mathcal{K}' \setminus \{0\}$ satisfies $A_1' f_1 = r_1 f_1$ and that one of the following:*

- A_1' is f_1 -bounded from above;
- A_1' is g -bounded, for some $g \in \mathcal{K}' \setminus \{0\}$;

hold. It follows that $r(A_2) < r(A_1)$ if, and only if, there exist $\gamma \in (0, 1)$ and $N \in \mathbb{N}$ such that

$$(A_2')^n f_1 \leq (\gamma r_1)^n f_1 \quad \forall n \in \mathbb{N}, n \geq N.$$

A drawback of Lemmas 3.1, 3.2 and 3.3 is the requirement that the spectral radius of A_2 or A_1 is an eigenvalue with positive eigenvector or eigenfunctional, respectively. Our next lemma relaxes that requirement for u -upper bounded operators. We recall that every bounded positive operator with respect to a solid cone is u -bounded from above for any interior point u of the cone, because $\mathcal{X} = \mathcal{X}_u$ for such u . The proof is the same as one direction of Lemma 3.2, and so is omitted.

Lemma 3.4 *Let \mathcal{X} denote a real Banach space, with a reproducing and normal cone $\mathcal{K} \subset \mathcal{X}$ and positive linear operators $A_1, A_2 \in \mathcal{B}(\mathcal{X})$ which satisfy $A_2 \leq A_1$ and $A_1 \neq A_2$. Assume that A_1 is u -bounded from above, for some $u \in \mathcal{K} \setminus \{0\}$. If there exist $\gamma \in (0, 1)$ and $N \in \mathbb{N}$ such that*

$$A_2^n u \leq \gamma^n A_1^n u \quad \forall n \in \mathbb{N}, n \geq N,$$

then $r(A_2) < r(A_1)$.

Finally, Lemma 3.4 may be formulated for the adjoint operator as well, mutatis mutandis, and so we do not give a formal statement.

Strict monotonicity of spectral radii by spectral theory

Here we derive sufficient conditions for (1.1) in terms of the operators A_1, A_2 and the cone $\mathcal{K} \subset \mathcal{X}$ which avoid checking estimates of the form (3.1), (3.2) or (3.3). To that end, we formulate the sequential assumptions:

- (A.1) There exists $w_1 \in \mathcal{K}, w_1 \neq 0$, such that $A_1 w_1 \leq r(A_1) w_1$.
- (A.2) There exists $f_2 \in \mathcal{K}', f_2 \neq 0$ such that $A_2' f_2 \geq r(A_2) f_2$.
- (A.3) $f_2(y) > 0$ where $y := (A_1 - A_2)w_1 \in \mathcal{K}$.

Note that $y \in \mathcal{K}$ in (A.3) follows as $w_1 \in \mathcal{K}$ and by our standing assumption that $A_2 \leq A_1$.

Clearly, a necessary condition for (A.3) is that $y \neq 0$, that is, the operators A_1, A_2 and w_1 satisfy

$$A_2 w_1 \neq r_1 w_1. \tag{3.5}$$

The assumptions (A.1)–(A.3) are sufficient for strict monotonicity of the spectral radii, recorded in our main result.

Theorem 3.5 *Let \mathcal{X} denote a real Banach space, with a cone $\mathcal{K} \subset \mathcal{X}$ and positive linear operators $A_1, A_2 \in \mathcal{B}(\mathcal{X})$ which satisfy $A_2 \leq A_1$ and $A_1 \neq A_2$. If (A.1)–(A.3) are satisfied, then $r(A_2) < r(A_1)$.*

Proof Combining (A.1)–(A.3), we see that

$$0 < f_2(y) = f_2((A_1 - A_2)w_1) \leq f_2((r_1 I - A_2)w_1) \leq (r_1 - r_2)f_2(w_1),$$

from which we conclude that $r_1 > r_2$, as required. □

We proceed to gather sufficient conditions for (A.1)–(A.3) and (3.5) to hold, formulated as the following three lemmas. We do not claim that the following lists are exhaustive. Obviously, (A.1) holds (with equality) if the spectral radius of A_1 is an eigenvalue of A_1 , with associated positive eigenvector $w_1 \in \mathcal{K}$. This is the approach we take. Although it is known that under mild assumptions the spectral radius belongs to the spectrum of a positive operator, it need not be an eigenvalue in general. Akin to (A.1), assumption (A.2) is satisfied with equality if A_2 admits a positive eigenfunctional corresponding to the spectral radius—a positive left eigenvector in the finite-dimensional case. We highlight that assumptions (A.1) and (A.2) are themselves not sufficient for (1.1), as the counter-example (1.3) demonstrates. Thus, the third assumption (A.3) is crucial and is a coupling condition between A_1 and A_2 where strict positivity plays a role.

Lemma 3.6 *Imposing the notation of Theorem 3.5, if any one of the following:*

- (a) $\mathcal{X} = \mathbb{R}^N$, for some $N \in \mathbb{N}$;
- (b) \mathcal{K} is reproducing, A_1^k is compact for some $k \in \mathbb{N}$, with $r(A_1) > 0$;
- (c) \mathcal{K} is reproducing, normal and minihedral, A_1 is monotonically compact and u -bounded;
- (d) \mathcal{K} is reproducing and normal, A_1 is focussing, non-degenerate and u -bounded;
- (e) \mathcal{K} is reproducing and normal, A_1 is Riesz with respect to $\mathcal{C}(\mathcal{X})$, with $r(A_1) > 0$;

are satisfied, then (A.1) holds.

We recall that a positive cone \mathcal{K} is minihedral if every finite subset of \mathcal{X} which is bounded with respect to the partial order induced by \mathcal{K} has a supremum. The reader is referred as well to [12, Theorems 9.8, 9.9] for the special cases that A_1 is an integral operator. Further, recall that an element a of a Banach algebra \mathcal{A} is called Riesz with respect to a closed ideal \mathcal{I} if the spectrum of the element $a + \mathcal{I}$ in the quotient algebra \mathcal{A}/\mathcal{I} is zero, see [20] or [2].

Proof of Lemma 3.6 (a) See [12, Theorem 9.1, p.87].

(b) See [12, Theorem 9.3, p.87].

(c) See [12, Theorem 9.7, p.92].

(d) See [12, Theorem 10.2, p. 105] and the second, unnamed, result in [12, Section 11.4, p. 115].

(e) It follows from [3, Theorem 1.7.3] that \mathcal{K} is reproducing and normal if, and only if, the cone of positive operators is normal in the Banach algebra $\mathcal{B}(\mathcal{X})$. This cone is a closed algebra cone, see [20], which is semi-simple. The set of compact operators $\mathcal{C}(\mathcal{X})$ is readily shown to be a closed, inessential ideal of $\mathcal{B}(\mathcal{X})$. Thus, the claim follows by [20, Theorem 3.7] which gives that there exists nonzero, positive $U \in \mathcal{B}(\mathcal{X})$ such that $A_1 U = r(A_1)U$. Taking $v \in \mathcal{K}$ such that $w_1 := Uv \neq 0$ yields (A.1). \square

The next lemma contains sufficient conditions for (A.2) and (A.3).

Lemma 3.7 *Imposing the notation of Theorem 3.5, any one of the following:*

- (f) there exists $m \in \mathbb{N}$ such that A_2^m is compact;
- (g) \mathcal{K} is normal and reproducing and A_2 is Riesz with respect to $\mathcal{C}(\mathcal{X})$, with $r(A_2) > 0$;

- (h) \mathcal{K} is normal and solid;
- (i) \mathcal{K} is reproducing and normal, and A_2 is u -bounded;

is sufficient for (A.2). Define $y := (A_1 - A_2)w_1$. If, in addition to one of (f)–(i) above, (A.1), (3.5) and any one of the following:

- (j) y is a quasi-interior point of \mathcal{K} ;
- (k) A_2 is irreducible;
- (l) A_2 is u -bounded;

hold, then (A.3) is satisfied.

Proof (f) The claim follows from [12, Theorem 9.2, p. 87] applied to $\mathcal{A}' \in \mathcal{B}(\mathcal{X}')$, where we have used that A' is compact if A is, Schauder’s Theorem (see, for example, [14, Theorem 7, p.243]).

(g) If A_2 is Riesz with respect to the compact linear operators $\mathcal{C}(\mathcal{X})$, then $A'_2 \in \mathcal{B}(\mathcal{X}')$ is Riesz with respect to the compact linear operators $\mathcal{C}(\mathcal{X}')$, and the claim follows from [20, Theorem 3.7] applied to A'_2 . Here we have used that \mathcal{K} is a normal and reproducing cone implies that \mathcal{K}' is as well (see [12, Theorems 4.5, 4.6, p. 40]), and so from [3, Theorem 1.7.3] the cone of positive operators $\mathcal{X}' \rightarrow \mathcal{X}'$ is normal in the semi-simple Banach algebra $\mathcal{B}(\mathcal{X}')$.

(h) and (i) The claim follows from [12, Theorem 9.12, pp. 99–100].

That any of (j)–(l) and (A.1) and (3.5) are sufficient for (A.3) follows from [12, Theorem 16.3, p. 171], once we notice from (A.1) that

$$A_2w_1 \leq A_1w_1 \leq r_1w_1 .$$

□

We next provide sufficient conditions for the inequality (3.5) to hold.

Lemma 3.8 *Imposing the notation of Theorem 3.5, let $u \in \mathcal{K} \setminus \{0\}$ and assume that (A.1) holds. The conditions:*

- (m) $\mathcal{X} = \overline{\mathcal{K} - \mathcal{K}}$ (for example, \mathcal{K} is reproducing) and w_1 is a quasi-interior point of \mathcal{K} ;
- (n) \mathcal{K} is reproducing, A_1 is u -bounded and $(A_1 - A_2)^2 \neq 0$;
- (p) \mathcal{K} is reproducing, A_1 is u -bounded and there exists $x \in \mathcal{X}$ such that $-nw_1 \leq x \leq nw_1$ for some $n \in \mathbb{N}$ with the property that $(A_1 - A_2)x \neq 0$;

are each sufficient for (3.5).

We note that irreducibility of A_2 is sufficient for irreducibility of A_1 which in turn is sufficient for irreducibility of $A_1 + A_2$. If this latter condition holds, then, in light of (A.1) and the estimates

$$(A_1 + A_2)w_1 \leq 2A_1w_1 \leq 2r_1w_1,$$

it follows that w_1 is a quasi-interior point of \mathcal{K} .

Proof of Lemma 3.8 (m) Choose $x^* \in \mathcal{K}$ such that $w^* := (A_1 - A_2)x^* \in \mathcal{K} \setminus \{0\}$. Thus, by [12, Theorem 2.2, pp. 20–21], there exists a positive functional $g \in \mathcal{K}'$ such that $g(w^*) > 0$. Therefore, $h \in \mathcal{K}'$ defined by

$$h := g \circ (A_1 - A_2) : \mathcal{K} \rightarrow \mathbb{R}_+,$$

satisfies

$$h(x^*) = g((A_1 - A_2)x^*) = g(w^*) > 0,$$

and so h is non-zero. As w_1 is a quasi-interior point, and by (A.1),

$$0 < h(w_1) = g((A_1 - A_2)w_1) = g(y),$$

whence $y \neq 0$.

(n) In Lemma 3.2 we proved the inequality (3.4), that u -boundedness of A_1 implies that A_1 is w_1 -bounded from above. Clearly, $A_1 - A_2$ is w_1 -bounded from above as well.

Next, the reproducing property of \mathcal{K} implies that there exists $x^* \in \mathcal{K}$ such that

$$y^* = (A_1 - A_2)^2 x^* \in \mathcal{K} \setminus \{0\},$$

so that clearly both $x^* \neq 0$ and $\gamma^* := \gamma((A_1 - A_2)x^*) > 0$. Thus, invoking the w_1 -upper boundedness of $A_1 - A_2$ with $x = x^*$,

$$0 \leq (A_1 - A_2)x^* \leq \gamma((A_1 - A_2)x^*)w_1 = \gamma^*w_1,$$

and applying $A_1 - A_2$ to both sides yields that

$$y^* = (A_1 - A_2)^2 x^* \leq \gamma^*(A_1 - A_2)w_1 \leq \gamma^*y.$$

Since $y^*/\gamma^* \neq 0$, we conclude that $y \neq 0$.

(p) Similarly to (n), $A_1 - A_2$ is w_1 -bounded from above. Thus, $(A_1 - A_2)|_{\mathcal{X}_{w_1}} \in \mathcal{B}(\mathcal{X}_{w_1})$ and, by assumption, is not equal to the zero operator. Therefore, invoking (2.2) and (ii), we see that

$$0 < \|(A_1 - A_2)|_{\mathcal{X}_{w_1}}\|_{w_1} = \|(A_1 - A_2)w_1\|_{w_1} = \|y\|_{w_1},$$

demonstrating that $y \neq 0$. □

To summarise briefly, Lemmas 3.7 and 3.8 place assumptions on A_2 which, via Theorem 3.5, ensure that (1.1) holds. Our final result is inspired by [19, Theorem 4.3], and instead places more assumptions on A_1 .

Proposition 3.9 *Let \mathcal{X} denote a real Banach space, with reproducing and normal cone $\mathcal{K} \subset \mathcal{X}$ which induces a Riesz space, and positive linear operators $A_1, A_2 \in \mathcal{B}(\mathcal{X})$ which satisfy $A_2 \leq A_1$ and $A_1 \neq A_2$. If there exist*

- (I) $w_1 \in \mathcal{K} \setminus \{0\}$ such that $A_1 w_1 = r_1 w_1$;
- (II) $f_1, f_2 \in \mathcal{K}' \setminus \{0\}$ such that $A'_i f_i = r_i f_i$;

and one of:

- (III) (1) A_1 is u -bounded, for some $u \in \mathcal{K} \setminus \{0\}$, a quasi-interior point of \mathcal{K} ;
- (2) A_1 is irreducible;

hold, then $r(A_2) < r(A_1)$.

Recall that an ordered Banach space \mathcal{X} is a Riesz space if for each $u, v \in \mathcal{X}$, the supremum and infimum of u and v (with respect to the ordering induced by the cone \mathcal{K} in this instance) also are elements of \mathcal{X} . Riesz spaces are well-studied objects; see, for example [1, p. 2] or [18, p. 48].

With reference to assumption (III) (1), in light of the inequalities

$$\alpha(w_1)u \leq A_1 w_1 = r_1 w_1 \leq \beta(w_1)u, \tag{3.6}$$

it follows that u is a quasi-interior point of \mathcal{K} if, and only if, w_1 is.

Proof of Proposition 3.9 The assumption that \mathcal{K} is normal and reproducing implies that $r_2 \leq r_1$ by, for example, [5, Theorem 1.1]. Seeking a contradiction, assume that $r_2 = r_1 =: r$. If A_1 is irreducible, then w_1 is quasi-interior point, and as for all $x \in \mathcal{K}$, $x \neq 0$

$$0 < f_1(A_1^n x) = r^n f_1(x),$$

for some $n \in \mathbb{N}$ by [12, Theorem 11.2, p.113], we conclude that $r > 0$ and f_1 is strictly positive. Alternatively, if A_1 is u -bounded by a quasi-interior point, then $f_1(u) > 0$. It now follows from (2.1) that

$$0 < \alpha(x) f_1(u) \leq f_1(A_1 x) = r f_1(x) \quad \forall x \in \mathcal{K} \setminus \{0\},$$

meaning $r > 0$ and f_1 is strictly positive. Using (II), we now estimate that

$$r f_2 = A'_2 f_2 \leq A'_1 f_2.$$

Since $\phi := A'_1 f_2 - r f_2 \in \mathcal{K}'$, and w_1 is a quasi-interior point, the equality

$$\phi(w_1) = f_2(A_1 w_1) - r f_2(w_1) = (r - r) f_2(w_1) = 0,$$

implies that $\phi = 0$, that is, $A'_1 f_2 = r f_2$.

We claim that

$$f_2 = c f_1, \tag{3.7}$$

for some $c > 0$. The arguments which follow are based on those of [12, pp.112–113]. To that end, consider $g := t f_1 - f_2 \in \mathcal{X}'$, for $t > 0$. If $g = 0$ for some $t > 0$, then there is nothing to prove. We consider two exhaustive possibilities. Either we may choose $t > 0$ sufficiently large such that $g \in \mathcal{K}' \setminus \{0\}$ and $g(x) = 0$ for some $x \in \mathcal{K} \setminus \{0\}$

or, for these $t > 0$, $g \notin \mathcal{K}' \cup (-\mathcal{K}')$. In the first case we reach the contradiction that the obvious equality $A'_1 g = rg$ implies that g must be strictly positive. In the second, the element $g_+ := \sup\{0, g\} \in \mathcal{K}'$ is well-defined by the minhedrality of \mathcal{K}' , see [12, Theorem 6.4, p. 61] and is not strictly positive. Moreover, as $A'_1 g_+ \geq 0$ and $A'_1 g_+ \geq rg_+$, it follows that $A'_1 g_+ \geq rg_+$, yet $A'_1 g_+ \neq rg_+$ (else g_+ would be strictly positive). Therefore, on the one hand, the functional

$$h := \sum_{k=0}^{\infty} (2\|A'\|)^{-k} (A'_1)^k (A'_1 g_+ - rg_+) \in \mathcal{K}' \setminus \{0\},$$

satisfies

$$h(w_1) = \sum_{k=0}^{\infty} (2\|A'\|)^{-k} (A'_1)^k \underbrace{(g_+(A_1 w_1) - rg_+(w_1))}_{=0} = 0.$$

However, on the other hand, the easily established estimate

$$A'_1 h \leq 2\|A'_1\|h,$$

implies that h is strictly positive, and so $h(w_1) > 0$, a contradiction. We have established (3.7), and so f_2 is also strictly positive, as f_1 is.

Let $v \in \mathcal{K} \setminus \{0\}$ be such that $(A_1 - A_2)v \neq 0$, so that $f_i(v) > 0$, as $v \neq 0$. Thus, we arrive at the contradiction

$$\begin{aligned} r = r_1 &= \frac{f_1(A_1 v)}{f_1(v)} = \frac{f_2(A_1 v)}{f_2(v)} = \frac{f_2(A_2 v)}{f_2(v)} + \frac{f_2((A_1 - A_2)v)}{f_2(v)} \\ &= r_2 + \frac{f_2((A_1 - A_2)v)}{f_2(v)} > r. \end{aligned} \tag{3.8}$$

□

We conclude this section with some commentary, first on the assumptions of Proposition 3.9, and then make some comparisons with [19].

Remark 3.10 Inspection of the proof of Proposition 3.9 shows where the assumptions made are applied, and how these may be substituted. The assumption that \mathcal{K} is normal and reproducing is used to ensure that $r_2 \leq r_1$, from which a contradiction argument is used. If $\mathcal{X} = \overline{\mathcal{K}} - \overline{\mathcal{K}}$ and either A_2 is compact or $r(A_2)$ is a pole of the resolvent of (the complexification of) A_2 , then $r(A_2) \leq r(A_1)$, see [5, Theorem 1.2, Corollary 1.3]. The Riesz property and normality are together used in the proof of (3.7) to ensure that \mathcal{K}' is minhedral, and so the element $g_+ = \sup\{g, 0\}$ is well-defined. If the cone is \mathcal{K}' is solid, then a different argument may be used to establish (3.7), see [12, Theorem 11.1]. By [12, Theorems 5.6 and 5.10], solidity of \mathcal{K}' is equivalent to the existence of a uniformly positive functional $f \in \mathcal{K}'$, that is, there exists $\theta > 0$ such that $f(x) \geq \theta\|x\|$ for all $x \in \mathcal{K}$. Irreducibility or u -boundedness is used to establish

that f_1 is strictly positive and that (3.7) holds, so that f_2 is strictly positive as well. Strict positivity is required so that the arguments in (3.8) make sense. \square

Remark 3.11 There is overlap between our results and those of [19], namely [19, Theorems 4.3, 4.4]. In both of these results the cone \mathcal{K} is assumed closed, and so the cones in [19, Theorems 4.3, 4.4] are assumed reproducing (although that assumption is not made throughout [19]). Our Proposition 3.9, including its proof, is based on [19, Theorem 4.3], and strengthens it slightly by permitting that A_1 is u -bounded. Irreducibility as used here goes by the term semi-nonsupporting operator in [19], and is assumed of A_1 in [19, Theorem 4.3], but the concepts are equivalent via [12, Theorem 11.2, p. 113].

Theorem 3.5 is comparable with, but generalises, [19, Theorem 4.4], where it is assumed that A_1 is strongly positive (non-zero positive elements are mapped to quasi-positive ones), which ensures that (l) holds. Finally, the assumption that $(A_1 - A_2)x$ is a quasi-interior point whenever $x \in \mathcal{K} \setminus \{0\}$ is stronger than our assumption (i). Finally, we do note that [19, Theorem 4.4] proves other assertions than solely (1.1). \square

4 Examples

Example 4.1 Let $\mathcal{X} = C([0, 1])$ denote the Banach space of continuous real-valued functions $[0, 1] \rightarrow \mathbb{R}$ equipped with the supremum norm, and let \mathcal{K} denote the cone of nonnegative-valued functions. This cone is solid (and so reproducing) and normal. The operator

$$A_2 : \mathcal{X} \rightarrow \mathcal{X}, \quad (A_2x)(t) = tx(t) \quad \forall x \in \mathcal{X}, \quad \forall t \in [0, 1],$$

is linear, continuous and positive. Evidently, for $x \in \mathcal{X}$ with $\|x\|_\infty = 1$

$$\|A_2x\|_\infty = \sup_{t \in [0,1]} |tx(t)| \leq \sup_{t \in [0,1]} |x(t)| = \|x\|_\infty = 1,$$

and the bound is achieved when $x \equiv 1$, so $\|A_2\|_\infty = 1$. Moreover, $\|A_2^n\| = 1$ for all $n \in \mathbb{N}$, and so $r(A_2) = 1$. However, it is clear that $A_2x = \lambda x$ has no non-zero solutions, and so A_2 has no eigenvalues and eigenvectors. For $\rho > 1$ and $\omega > 0$ define

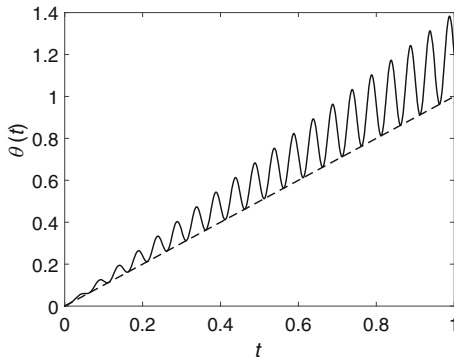
$$A_1 : \mathcal{X} \rightarrow \mathcal{X}, \quad (A_1x)(t) = t(\rho + (\rho - 1) \sin(\omega t))x(t) \quad \forall x \in \mathcal{X}, \quad \forall t \in [0, 1],$$

which is also linear, bounded and positive. The coefficients of x in A_1 and A_2 are plotted in Fig. 1, which visualises the readily established properties that $A_2 \leq A_1$ and $A_2 \neq A_1$. As with A_2 , A_1 does not have any eigenvalues, but noting that for all $s \in [0, 1]$, $g_s \in \mathcal{K}'$ defined by $g_s(x) = x(s)$ clearly satisfies

$$(A_2'g_s)(x) = sx(s) = sg_s(x)$$

and $(A_1'g_s)(x) = s\theta(s)x(s) = s\theta(s)g_s(x),$

Fig. 1 Coefficient of A_2 dashed line and coefficient of A_1 , solid line. Here $\rho = 1.1$ and $\omega = 40\pi$



where $\theta : [0, 1] \rightarrow \mathbb{R}_+$ is given by

$$\theta(t) := \rho + (\rho - 1) \sin(\omega t) \quad \forall t \in [0, 1],$$

we see that both A'_1 and A'_2 have positive eigenfunctionals.

For ease of exposition, assume that $\theta(1) > 1$ and choose $\rho \in (1/\theta(1), 1)$. In this case, we have that

$$A'_2 g_1 = g_1 \leq \rho \theta(1) g_1 = \rho A'_1 g_1.$$

An application of Lemma 3.1 yields that $r(A_2) = 1 < r(A_1)$. □

Example 4.2 Let $\mathcal{X} = \mathbb{R}^N$ for some $N \in \mathbb{N}$, $\mathcal{K} = \mathbb{R}_+^N$ which induces the partial order of componentwise inequality. Let $A_1, A_2 \in \mathbb{R}_+^{N \times N}$ denote nonnegative matrices. If $A_2 \leq A_1$, $A_2 \neq A_1$ and A_2 is irreducible, then $r(A_2) < r(A_1)$. The claim is known from [22, Theorem 9], but in the present context follows from Theorem 3.5, after noting that assumptions (a), (h), (j) and (m) are satisfied. If A_1 is irreducible, then $r(A_2) < r(A_1)$ is known from [4, Corollary 1.5, p.27], but in the present setting follows from Proposition 3.9.

In [4, Corollary 1.5, p.27], it is proven that irreducibility of $A_1 + A_2$ is sufficient for $r(A_2) < r(A_1)$, which is seemingly weaker than requiring A_1 is irreducible. We comment that there is no generality gained by assuming that $\alpha A_1 + (1 - \alpha)A_2$ is irreducible or u -bounded, for some $\alpha \in (0, 1]$, at least under our standing assumption that $A_2 \leq A_1$, even in more general operator settings. Indeed, the trivially established inequalities

$$\alpha A_1 \leq \alpha A_1 + (1 - \alpha)A_2 \leq A_1,$$

shows that $\alpha A_1 + (1 - \alpha)A_2$ is irreducible or u -bounded if, and only if, A_1 is—where Proposition 3.9 applies. □

Example 4.3 Let $\mathcal{X} = L^p(\Omega; \mathbb{R})$ for $1 \leq p < \infty$ and where Ω is a compact metric space. Let \mathcal{K}_+ denote the cone of functions $\Omega \rightarrow \mathbb{R}$ which are nonnegative almost

everywhere on Ω , which is reproducing and normal, but not solid. Consider $A_1, A_2 \in \mathcal{B}(\mathcal{X})$ defined by

$$(A_i x)(t) = \int_{\Omega} k_i(t, s)x(s) ds, \quad i \in \{1, 2\},$$

for some kernels $k_i : \Omega \times \Omega \rightarrow \mathbb{R}$. The operators A_i are positive if

$$k_i(t, s) \geq 0 \quad \text{for almost all } (t, s) \in \Omega \times \Omega, \tag{4.1}$$

and moreover, by [12, Theorem 2.1, p.19], nonnegativity of the kernel (4.1) is sufficient for A_i to be bounded. By linearity, the inequality $A_2 \leq A_1$ is equivalent to

$$k_2(t, s) \leq k_1(t, s) \quad \text{for almost all } (t, s) \in \Omega \times \Omega, \tag{4.2}$$

and $A_2 \neq A_1$ means that there exist sets of positive measure $\Omega_i^* \subset \Omega$ such that

$$k_2(t, s) < k_1(t, s) \quad \text{for almost all } (t, s) \in \Omega_1^* \times \Omega_2^*.$$

It is well-known that integral operators are compact under rather general assumptions on the kernel. If $k_1 \in L^1(\Omega \times \Omega)$, then A_1 is compact and so is A_2 by (4.2). It follows from (b) and (e) that (A.1) and (A.2) hold. The operators A_i are u_i -bounded if there exists $u_i \in \mathcal{K}$, $u_i \neq 0$ and nonnegative functions $\alpha_i, \beta_i \in L^q(\Omega)$, where $q \in (1, \infty)$ is complimentary to p , such that

$$\alpha_i(s)u_i(t) \leq k_i(t, s) \leq \beta_i(s)u_i(t) \quad \text{for almost all } (t, s) \in \Omega \times \Omega.$$

The operators A_i are irreducible if, for any measurable proper subset $\Omega_1 \subset \Omega$ with positive measure, there exist closed sets $\Gamma_1 \subset \Omega$ and $\Gamma_2 \subset \Omega \setminus \Omega_1$ such that

$$k_i(t, s) > 0 \quad \text{for almost all } (t, s) \in \Gamma_1 \times \Gamma_2.$$

Irreducibility of A_2 (k) is sufficient for (A.3), as is u -boundedness of A_2 (l) and w_1 a quasi-interior point (m). Alternatively, if one of (j)–(l) hold and $\Omega^* := \Omega_1^* \cap \Omega_2^*$ has positive measure, then there exists $x \in \mathcal{K}$ such that $(A_1 - A_2)^2 x \neq 0$, and so statement (m) holds, which together are sufficient for (A.3). To see that (m) holds, we compute that for $x \in \mathcal{K}$ and almost all $t \in \Omega$

$$\begin{aligned} (A_1 - A_2)^2 x(t) &= \int_{s \in \Omega} (k_1 - k_2)(t, s)(A_1 - A_2)x(s) ds \\ &= \int_{s \in \Omega} (k_1 - k_2)(t, s) \int_{\tau \in \Omega} (k_1 - k_2)(s, \tau)x(\tau) d\tau ds \\ &= \iint_{s, \tau \in \Omega} \underbrace{(k_1 - k_2)(t, s)(k_1 - k_2)(s, \tau)}_{>0 \text{ on } \Omega^* \times \Omega^*} x(\tau) d\tau ds. \end{aligned}$$

□

Example 4.4 Consider the discrete-time switched dynamical system

$$x(t+1) = \mathcal{B}(t)x(t) \quad x(0) = x^0 \quad t \in \{0\} \cup \mathbb{N}, \quad (4.3)$$

where $\mathcal{B}(t) \in \{B_1, \dots, B_N\}$ for each $t \in \{0\} \cup \mathbb{N}$ and given positive $B_k \in \mathcal{B}(\mathcal{X})$. Switched systems are a popular and important class of control systems which, for example, arise in dynamical systems with several distinct modes of operation. Their interest is in part motivated by the property that even if $r(B_k) < 1$ for every k , the zero equilibrium of (4.3) may be unstable, depending on the switching between the B_k , captured via \mathcal{B} . We refer the reader to the survey articles [15] or [16] and the references therein, for further background on switched systems. Much attention has been devoted in the systems and control literature to the stability of switched *positive* dynamical systems, particularly by Valcher, Shorten and their collaborators; see, for instance [9, 11].

Here we simply record an elementary application of our results. Namely, if $B_1, \dots, B_N \leq A_2 \leq A_1$, with $A_1 \neq A_2$, $r(A_1) = 1$ and (1.1) holds, then $r(A_2) < 1$ and the zero equilibrium is globally exponentially stable if the cone is normal. Indeed, the solution x of (4.3) satisfies

$$x(t) \leq A_2^t x(0) \quad \Rightarrow \quad 0 \leq \|x(t)\| \leq \|A_2^t\| \cdot \|x(0)\| \leq M r_2^t \|x(0)\| \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

for some $M > 0$ by, for example, [17, Lemma 1]. □

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