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Regularity and Compactness Properties of Integral Hankel Operators and Their Singular Vectors

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Abstract

Integral Hankel operators on vector-valued $L^2(\mathbb{R}_+, U)$-function spaces are considered. Regularity (integrability) and compactness properties of the kernel are shown to give rise to quantifiable regularity and compactness properties of the Hankel operator, and consequently of the associated singular vectors (also called Schmidt pairs), which finds relevance in model order reduction schemes. As demonstrated, strong-Lebesgue and Sobolev spaces naturally arise in the case that U is infinite dimensional. The theory is illustrated with examples.

Keywords Compact operator · Integral Hankel operator · Model order reduction · Systems and control theory · Singular vectors · Schmidt pairs

Mathematics Subject Classification 41A65 · 47B35 · 47A56 · 47G10 · 47N70 · 93B28 · 93C05

1 Introduction

Hankel operators are fundamental objects, and "one of the most important classes of operators on spaces of analytic functions." [23, Preface]. There are numerous definitions of Hankel operators, which generalise in certain senses the property that the Hankel operator matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & \iddots & \\ a_2 & \iddots & & \\ \vdots & & & \end{pmatrix}, \tag{1.1}$$

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has constant anti-diagonals. For instance, Hankel operators are often considered as linear operators defined in terms of multiplication with a symbol and restriction/projection on the Hardy space of analytic functions on the unit disc. Nehari's Theorem (see, for example [23, Theorems 2.1, 2.2]) characterises such operators in terms of the Hankel operator matrix in (1.1) inducing a bounded linear operator on $\ell^2(\mathbb{Z}_+)$. Alternatively, for a Hilbert space K and a shift operator $S \in \mathcal{B}(K)$, as in [24, p.1], the bounded linear operator $\Gamma : K \to K$ is called S-Hankel if it satisfies the entwining property $S^*\Gamma = \Gamma S$ which is the approach is adopted in, for example, [21]. The Hankel operator matrix (1.1) is evidently S-Hankel on $\ell^2(\mathbb{Z}_+)$ with respect to the usual right shift S on $\ell^2(\mathbb{Z}_+)$ given by

$$(z_0, z_1, z_2, \dots) \mapsto Sz := (0, z_0, z_1, z_2, \dots) \quad \forall\, z = (z_0, z_1, z_2, \dots) \in \ell^2(\mathbb{Z}_+).$$

Hankel operators naturally arise in mathematical systems and control theory in the context of so-called realisations of linear control systems and model order reduction [8, 9, 11, 26]. In the setting of linear control systems in continuous time, integral operators of the form

$$(\Gamma u)(t) = \int_0^\infty h(t+s)u(s)\,\mathrm{d}s \quad u \in L^2(\mathbb{R}_+, U) \quad t \geq 0\,, \tag{1.2a}$$

$$\text{or} \quad (\Gamma v)(t) = \int_{-\infty}^0 h(t-s)v(s)\,\mathrm{d}s \quad v \in L^2(\mathbb{R}_-, U) \quad t \geq 0\,, \tag{1.2b}$$

are called Hankel operators (see, for example, [5, Section 8.2]). We postpone for now assumptions on h appearing above, often called the *impulse response*, other than noting that h is supported on $\mathbb{R}_+$, and the independent variable t denotes time. Here, U plays the role of the space of values of the input- and output-variables, and presently we allow for the situation wherein U is infinite dimensional. The above two operators are equivalent up to a reflection $t \mapsto -t$, and the latter has the interpretation of past inputs (controls) v being mapped to future outputs (measurements) Γv. At least formally, Γ in (1.2a) is $\mathcal{S}$-Hankel when $\mathcal{S}$ denotes the Laguerre shift on $L^2(\mathbb{R}_+, U)$. We refer the reader to, for example, [22, Chapters 4–7] or [23, Chapter 11], for more details on the connections between Hankel operators and control theory. We highlight, [6] and [13], and the references therein, as a selection of evidence of the interest that Hankel operators and their role in control theory has generated.

For our purposes, recall that the spectrum of $T \in \mathcal{B}(X)$, for a Hilbert space X, with compact T^*T comprises countably many nonnegative eigenvalues (point spectrum) only. The square roots of these are called the *singular* or *characteristic values* of T, as in [16, p. 330]. The Schmidt pairs (v_i, w_i) are normalised eigenvectors of T^*T and TT^*, respectively, called singular vectors, and which consequently satisfy

$$Tv_i = \sigma_i w_i \quad \text{and} \quad T^*w_i = \sigma_i v_i \quad \forall\, i \in \mathbb{N}.$$

On the other hand, the above definition of singular values coincides with the definition in terms of approximation by finite-rank operators, see [10, Theorem VI. 1.5], and is an essential ingredient in approximation theory. Indeed, the regularity of Schmidt pairs of Hankel operators plays a role in the so-called "AAK theory" [1] — a suite of results on Hankel operators by Adamjam, Arov and Kreĭn — as well as model order reduction of infinite-dimensional linear control systems; see [9, 11], and certain Proper Orthogonal Decomposition (POD) schemes [27]. In this short paper we show how regularity/compactness properties of h in (1.2) ensure both boundedness/compactness properties of Γ and regularity properties of its Schmidt pairs, presented as Proposition 2.1 and Theorem 2.3 below. The argumentation is mostly direct, with one key technical ingredient being the results of Lax [15, Theorems I–III] on operators on Banach spaces embedded in Hilbert spaces. In addition to the connection to model order reduction, the work is partly inspired by [12] where additional regularity properties of certain convolution operators (input–output maps of linear control systems in control theory) are ensured by that of the kernel. Examples are presented in Sect. 3.

Notation: Notation is kept to a minimum and we mention only a few items. We use $|\cdot|_U$ to denote the norm on U, and $\mathcal{B}(U)$ and $\mathcal{K}(U)$ denote the Banach spaces of bounded and compact linear operators $U \to U$, respectively, both equipped with the uniform topology. For $1 \le p \le \infty$, we let $L^p(\mathbb{R}_+, U)$ denote the usual Bochner-Lebesgue spaces of (equivalence classes of Bochner-measurable) functions $\mathbb{R}_+ \to U$ which are p-integrable ($p < \infty$), or essentially bounded ($p = \infty$). We let $W^{m,p}(\mathbb{R}_+, U)$ for $m \in \mathbb{N}$ denote the usual Sobolev spaces with norms

$$\|u\|_{W^{m,p}(\mathbb{R}_+)} := \left(\sum_{k=0}^{m} \|u^{(k)}\|^p_{L^p(\mathbb{R}_+)} \right)^{\frac{1}{p}} \quad \forall\, u \in W^{m,p}(\mathbb{R}_+, U),$$

and set $H^m(\mathbb{R}_+, U) := W^{m,2}(\mathbb{R}_+, U)$. Here the symbol $u^{(j)}$ denotes the j-th (weak) derivative of u for $j \in \mathbb{Z}_+$, with $u^{(0)} = u$. If u has a j-th classical derivative, then this is also denoted by $u^{(j)}$. Observe that we omit the space U from the norm notation for $L^p(\mathbb{R}_+, U)$ and $W^{m,p}(\mathbb{R}_+, U)$ for brevity.

The so-called strong L^2-space, denoted $L^2_{\mathrm{str}}(\mathbb{R}_+, \mathcal{B}(U))$, comprises all $f : \mathbb{R}_+ \to \mathcal{B}(U)$ such that

$$fv \in L^2(\mathbb{R}_+, U) \quad \forall\, v \in U \quad \text{and} \quad \|f\|_{L^2_{\mathrm{str}}(\mathbb{R}_+)} := \sup_{\|v\| \le 1} \|t \mapsto f(t)v\|_{L^2(\mathbb{R}_+)} < \infty.$$

Strong L^p-spaces are studied in some generality in [17, Appendix F] and, as follows from [17, Lemma F.1.5, p.1003], $L^2_{\mathrm{str}}(\mathbb{R}_+, \mathcal{B}(U)) \doteq L^2(\mathbb{R}_+, \mathcal{B}(U))$ (equal with equivalent norms) when U is finite dimensional. The corresponding strong Sobolev spaces are defined by

$$H^m_{\mathrm{str}}(\mathbb{R}_+, \mathcal{B}(U)) := \big\{ u \in L^2_{\mathrm{str}}(\mathbb{R}_+, \mathcal{B}(U)) \; : \; u^{(j)} \in L^2_{\mathrm{str}}(\mathbb{R}_+, \mathcal{B}(U)), \; \forall\, j \in \{1, 2, \ldots, m\} \big\}.$$

We let $L^p(\mathbb{R}_+, \mathbb{C}) \otimes \mathcal{K}(X)$ denote the tensor product of $L^p(\mathbb{R}_+)$ and $\mathcal{K}(X)$, defined as usual as the vector space spanned by all finite linear combinations $\sum_{j=1}^n f_j \otimes M_j$, where $f_j \in L^p(\mathbb{R}_+, \mathbb{C})$, $M_j \in \mathcal{K}(X)$ and $(f_j \otimes M_j)(\cdot) := f_j(\cdot)M_j$. We identify $L^p(\mathbb{R}_+, \mathbb{C}) \otimes \mathcal{K}(X)$ as a subspace of $L^p(\mathbb{R}_+, \mathcal{K}(X))$.

Finally, we write $\lesssim$ to mean less than or equal to up to a general multiplicative constant, independent of the other terms appearing. Its use is intended to reduce the number of constants appearing.

2 Regularity and Compactness Properties of Integral Hankel Operators

Our first result characterises additional boundedness properties of an integral Hankel operator in terms of its kernel.

Proposition 2.1 *Let $h \in L^1(\mathbb{R}_+, \mathcal{B}(U))$, and let $m \in \mathbb{N}$. The Hankel operator $\Gamma : L^2(\mathbb{R}_+, U) \to L^2(\mathbb{R}_+, U)$ in (1.2a)* restricts to a bounded linear operator $H^m(\mathbb{R}_+, U) \to H^m(\mathbb{R}_+, U)$ if, and only if, $h \in H^{m-1}_{\mathrm{str}}(\mathbb{R}_+, \mathcal{B}(U))$.

Proof The proof is inspired by that of [11, Lemma 4.9] and [12, Proposition 3.10]. Observe that a change of variable in (1.2) gives

$$(\Gamma u)(t) = \int_t^\infty h(s)u(s-t)\,\mathrm{d}s \quad \forall\, u \in W^{1,1}(\mathbb{R}_+, U) \cup H^m(\mathbb{R}_+, U), \ \forall\, t \geq 0,$$

which is differentiable, with

$$(\Gamma u)' = hu(0) - \Gamma(u') \quad \forall\, u \in W^{1,1}(\mathbb{R}_+, U) \cup H^m(\mathbb{R}_+, U). \tag{2.1}$$

Repeated differentiation of (2.1) formally gives

$$(\Gamma u)^{(k)} = h^{(k-1)}u(0) - \big(\Gamma(u')\big)^{(k-1)} \quad \forall\, u \in H^k(\mathbb{R}_+, U), \quad k \in \mathbb{N}. \tag{2.2}$$

If $h \in H^{m-1}_{\mathrm{str}}(\mathbb{R}_+, \mathcal{B}(U))$, then $h^{(k-1)} \in L^2_{\mathrm{str}}(\mathbb{R}_+, \mathcal{B}(U))$ and

$$\|h^{(k-1)}u(0)\|_{L^2(\mathbb{R}_+)} \lesssim |u(0)|_U \lesssim \|u\|_{H^m(\mathbb{R}_+)} \quad \text{for } k = 1, 2, \ldots, m-1.$$

An induction argument invoking (2.2) now shows that Γ restricted to $H^m(\mathbb{R}_+, U)$ is continuous $H^m(\mathbb{R}_+, U) \to H^m(\mathbb{R}_+, U)$. Conversely, consider $u(t) := \mathrm{e}^{-t}v$ for $v \in U$, which belongs to $H^r(\mathbb{R}_+, U)$ and satisfies $\|u\|_{H^r(\mathbb{R}_+)} \lesssim |v|_U$ for every $r \in \mathbb{N}$. This choice of u in equation (2.2) yields

$$h^{(k-1)}v = (\Gamma u)^{(k)} + \big(\Gamma(u')\big)^{(k-1)} \quad \forall\, u \in H^k(\mathbb{R}_+, U), \quad k \in \mathbb{N}. \tag{2.3}$$

The conjunction of (2.3) and the assumed regularity of Γ entails that $h^{k-1} \in L^2_{\mathrm{str}}(\mathbb{R}_+, \mathcal{B}(U))$ for $k = 1, 2, \ldots, m$, as required. □

As a corollary of the above proposition and [12, Corollary 3.11] we obtain the following equivalence.

Corollary 2.2 *Let $h \in L^1(\mathbb{R}_+, \mathcal{B}(U))$, and let $m \in \mathbb{N}$. The Hankel operator Γ in (1.2a)* restricts to a bounded linear operator $H^m(\mathbb{R}_+, U) \to H^m(\mathbb{R}_+, U)$ if, and only if, the convolution operator $L^2(\mathbb{R}_+, U) \to L^2(\mathbb{R}_+, U)$ given by $u \mapsto h * u$ does.

Convolution operators of the form $u \mapsto h * u$ are linear and right-shift invariant, so arise as the input–output maps of time-invariant linear control systems. In words, the above corollary states that the associated Hankel operator and input–output map of a time-invariant linear control system admit the same regularity properties in terms of restricting to bounded maps. Our next result considers compactness properties of integral Hankel operators.

Theorem 2.3 *Let $h \in L^1(\mathbb{R}_+, \mathcal{B}(U))$ and assume that $h(t) \in \mathcal{K}(U)$ for almost all $t \geq 0$. Let $m \in \mathbb{N}$, and consider the integral Hankel operator Γ in (1.2a).* The following statements hold:

(1) Γ viewed as an operator $L^q(\mathbb{R}_+, U) \to L^q(\mathbb{R}_+, U)$ is compact for all $1 \leq q < \infty$;
(2) Γ viewed as an operator $W^{1,1}(\mathbb{R}_+, U) \to W^{1,1}(\mathbb{R}_+, U)$ is compact;
(3) The singular vectors of $\Gamma : L^2(\mathbb{R}_+, U) \to L^2(\mathbb{R}_+, U)$ belong to $L^2(\mathbb{R}_+, U) \cap W^{1,1}(\mathbb{R}_+, U)$.

If, additionally, $h \in H^{m-1}(\mathbb{R}_+, \mathcal{K}(U))$, then the following statements hold:

(4) $\Gamma|_{H^m(\mathbb{R}_+,U)} : H^m(\mathbb{R}_+, U) \to H^m(\mathbb{R}_+, U)$ is compact;
(5) The singular vectors of $\Gamma : L^2(\mathbb{R}_+, U) \to L^2(\mathbb{R}_+, U)$ belong to $H^m(\mathbb{R}_+, U)$.

To connect to earlier work, Theorem 2.3 is a substantial generalisation of [11, Lemma 4.9], ensuring additional compactness and regularity in terms of the parameter m, and to the setting wherein U is infinite dimensional. Proposition 2.1 indicates the role of strong L^2- and H^m-spaces in characterising additional boundedness properties of integral Hankel operators. Here we note the requirement that the impulse response takes values in $\mathcal{K}(U)$ (almost everywhere) for compactness properties. This condition is not so surprising in light of the vector-valued version of Hartman's Theorem; see, for instance [23, Theorem 4.1] which, roughly speaking, states that a symbol which is continuous and takes values in $\mathcal{K}(U)$ is a necessary and sufficient condition for a Hankel operator to be compact. Apart from the above considerations, the setting that U is infinite dimensional is treated with little additional difficulty.

By way of further commentary on our hypotheses, recall the right-shift semigroup $(S_\tau)_{\tau\geq 0}$ on $L^2(\mathbb{R}_+, U)$ defined by

$$(S_\tau f)(t) = \begin{cases} 0 & t < \tau \\ f(t-\tau) & t \geq \tau \end{cases} \quad \text{for all } f \in L^2(\mathbb{R}_+, U) \text{ and almost all } t \geq 0.$$

A linear operator $\Gamma : L^2(\mathbb{R}_+, U) \to L^2(\mathbb{R}_+, U)$ is called if S_τ-Hankel if

$$S_\tau^* \Gamma = \Gamma S_\tau \quad \forall\, \tau \geq 0.$$

We note that, at least in the case that U is finite dimensional, a necessary condition for a S_τ-Hankel operator to be nuclear is that it admits a representation as an integral operator of the form (1.2a) with kernel $h \in L^1(\mathbb{R}_+, \mathcal{B}(U))$, see [11, Proposition 3.4, Remark 3.7]. The converse is known to be false from [9, Example 2.3]. A linear operator is nuclear (or trace class) if its singular values form a summable sequence, and is a natural requirement for non-trivial error bounds in terms of sums of singular values, such as those in [11]. Nuclearity of integral Hankel operators has been studied in [4].

The proof of Theorem 2.3 relies on the following lemma.

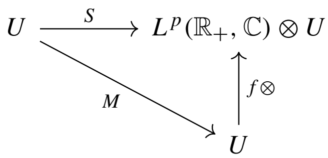
Lemma 2.4 *Let U be a separable Hilbert space, let $h \in L^p(\mathbb{R}_+, \mathcal{B}(U))$ with $1 \leq p < \infty$ and assume that $h(t) \in \mathcal{K}(U)$ for almost all $t \geq 0$. Then the linear map $T : U \to L^p(\mathbb{R}_+, U)$ given by $Tu := h(\cdot)u$ for all $u \in U$ is compact.*

Proof It suffices to assume that $h \in L^p(\mathbb{R}, \mathcal{K}(U))$, else choose $g \in L^p(\mathbb{R}, \mathcal{K}(U))$ with $\|h - g\|_{L^p(\mathbb{R}_+, \mathcal{B}(U))} = 0$ and argue with g below.

The proof leverages that $L^p(\mathbb{R}_+, \mathbb{C}) \otimes \mathcal{K}(U) \subseteq L^p(\mathbb{R}, \mathcal{K}(U))$ is dense in $L^p(\mathbb{R}_+, \mathcal{K}(U))$, as follows from [2, Theorem 1.3.6, (viii), p. 388]. Let $h_n \in L^p(\mathbb{R}_+, \mathbb{C}) \otimes \mathcal{K}(U)$ be such that $h_n \to h$ in $L^p(\mathbb{R}_+, \mathcal{K}(U))$ as $n \to \infty$. It is clear that T_n defined by $T_n u := h_n(\cdot)u$ for all $u \in U$ converges uniformly to T. Thus, it suffices to prove that T_n is compact for each n. However, each T_n is of the form

$$T_n = \sum_{k=1}^{n} f_{k,n} \otimes M_{k,n} \quad f_{k,n} \in L^p(\mathbb{R}, \mathbb{C}), \; M_{k,n} \in \mathcal{K}(U)$$

and $u \mapsto Su := f \otimes Mu$ with $f \in L^p(\mathbb{R}, \mathbb{C})$ and $M \in \mathcal{K}(U)$, shown in the commutative diagram below, is compact as the composition of a compact and bounded operator.



Hence, T_n is compact, as required. □

Proof of Theorem 2.3 Statement (1) is known when U is finite dimensional, see [8, Appendix 1, p.895]. In the case that U is infinite dimensional we essentially use density arguments, particularly that

$$L^1(\mathbb{R}_+, \mathbb{C}) \widehat{\otimes}_\pi U = L^1(\mathbb{R}_+, U) \quad \text{with equal norms,} \tag{2.4}$$

as follows from [25, Example 2.19, p. 29]. Here the tensor product on the left-hand side of (2.4) is the projective tensor product, cf. [25, Chapter 2]. It can be shown that $\Gamma = \Gamma_h$ is the uniform limit of compact operators Γ_{h_n}, using that $\Gamma_{f_j \otimes M_j} = \Gamma_{f_j} \otimes M_j \in \mathcal{K}(L^q(\mathbb{R}_+)) \otimes \mathcal{K}(U)$ is compact for $f \in L^1(\mathbb{R}_+)$ and $M_j \in \mathcal{K}(U)$, along the lines of [14, Theorem 2]. For brevity, we do not give the details.

The proof of statement (2) is inspired by that of [11, Lemma 4.9]. In particular, the equality (2.1) yields that Γ defined by (1.2) is bounded on $W^{1,1}(\mathbb{R}_+, U)$. Here we use that $|u(0)|_U \lesssim \|u\|_{W^{1,1}(\mathbb{R}_+)}$. To show compactness, let $(u_n)_n$ denote a bounded sequence in $W^{1,1}(\mathbb{R}_+, U)$. We may extract subsequences $(\Gamma u_{n_j})_j$ and $(\Gamma(u'_{n_j}))_j$ which converge in $L^1(\mathbb{R}_+, U)$ by compactness of $\Gamma \in \mathcal{B}(L^1(\mathbb{R}_+, U))$ from statement (1). An application of Lemma 2.4 with $p = 1$ ensures that there exists a subsequence of $(hu_{n_j}(0))_j$, also converging in $L^1(\mathbb{R}_+, U)$. Extracting a single common subsequence, in light of (2.1), it follows that $(\Gamma u_{n_j})_j$ is Cauchy in $W^{1,1}(\mathbb{R}_+, U)$, and hence convergent.

Statement (3) follows from an application of the results of Lax [15, Theorems I–III] applied to $T := \Gamma^*\Gamma$ or $\Gamma\Gamma^*$ (here Γ^* is the L^2-adjoint of Γ) with, in the former case, Banach space $B := W^{1,1}(\mathbb{R}_+, U)$, Hilbert space $H := L^2(\mathbb{R}_+, U)$, and bilinear form the usual inner-product on $L^2(\mathbb{R}_+, U)$. Note that the completion of $W^{1,1}(\mathbb{R}_+, U)$ in $L^2(\mathbb{R}_+, U)$ is simply $L^2(\mathbb{R}_+, U)$. Here we are additionally using that a short calculation shows that $\Gamma^* : L^2(\mathbb{R}_+, U) \to L^2(\mathbb{R}_+, U)$ is given by (1.2) with h replaced by h^*, that is, $h^* \in L^1(\mathbb{R}_+, \mathcal{K}(U))$ given by $h^*(t) = (h(t))^*$ which takes compact values by Schauder's Theorem. Consequently, $\Gamma^*\Gamma$ is compact $B \to B$ and $H \to H$ as the composition of a bounded and compact operator.

In light of statement (4), for statement (5) it suffices to prove the compactness property only. We use induction again. That the claim is true with $m = 1$ follows along the lines of the proof of statement (2), now using that $\Gamma : L^2(\mathbb{R}_+, U) \to L^2(\mathbb{R}_+, U)$ is compact. For the inductive step, if $(u_n)_n$ is a bounded sequence in $H^m(\mathbb{R}_+, U)$, then $(\Gamma u_n)_n$ has a Cauchy subsequence in $H^{m-1}(\mathbb{R}_+, U)$ and so, *a fortiori*, in $L^2(\mathbb{R}_+, U)$. Moreover, the induction hypothesis also yields that $(\Gamma(u'_n))_n$ has a Cauchy subsequence in $H^{m-1}(\mathbb{R}_+, U)$ by inductive hypothesis. Finally, the regularity of h ensures that $(h^{(m-1)}u_n(0))$ has a Cauchy subsequence in $L^2(\mathbb{R}_+, U)$ by Lemma 2.4 with $p = 2$. Therefore, in light of (2.2) with $k = m$, it follows that $\big((\Gamma u_n)^{(m)}\big)_n$ has a Cauchy subsequence in $L^2(\mathbb{R}_+, U)$, the upshot being that $(\Gamma u_n)_n$ has a Cauchy subsequence in $H^m(\mathbb{R}_+, U)$, as required.

The proof of statement (5) is the same as that of (3), *mutatis mutandis*. □

3 Examples

For brevity in the sequel we write $L^2(R)$ for $L^2(R, \mathbb{C})$ where $R = \mathbb{R}_+$ or $\mathbb{R}$, and similarly for Sobolev spaces. The first two examples are based on [12, Example 4.3] and [19, Example 18], respectively.

Example 3.1 Consider the ubiquitous finite-dimensional controlled and observed system of linear ordinary differential equations

$$\dot{x} = Ax + Bu, \quad x(0) = x^0, \quad y = Cx + Du, \tag{3.1}$$

with input, state and output denoted u, x and y, respectively. The input, state and output spaces are $U = \mathbb{C}^p$, $X = \mathbb{C}^n$ and U, respectively, and A, B, C and D may be identified with compatibly-sized complex matrices. Furthermore, $\mathcal{K}(U) = \mathcal{B}(U)$.

Let $h : \mathbb{R}_+ \to \mathbb{C}^{p\times p}$ be given by $t \mapsto h(t) := C\mathrm{e}^{At}B$, and consider the Hankel operator Γ as in (1.2a). If every eigenvalue of A has negative real part, then, evidently $h \in L^1(\mathbb{R}_+, \mathcal{K}(U))$ and, in light of

$$t \mapsto h^{(k)}(t) = C\mathrm{e}^{At}A^kB \in L^2(\mathbb{R}_+, \mathcal{K}(U)) \quad \forall\, k \in \mathbb{Z}_+,$$

it follows from Proposition 2.1 and Theorem 2.3 that Γ restricts to a compact operator $H^m(\mathbb{R}_+, U) \to H^m(\mathbb{R}_+, U)$ and the singular vectors of Γ belong to $H^m(\mathbb{R}_+, U)$, both for every $m \in \mathbb{N}$.

Consider now the case that (3.1) denotes (at least formally) an infinite-dimensional linear control system, where $A : X \supseteq D(A) \to X$ generates an exponentially stable C_0-semigroup $(\mathbb{T}(t))_{t\geq 0} = (\mathrm{e}^{At})_{t\geq 0}$. If C is bounded, meaning $C \in \mathcal{B}(X, U)$, and $A^kB \in \mathcal{B}(U, X)$ for $k \in \mathbb{Z}_+$, then the restriction of Γ to $H^m(\mathbb{R}_+, U)$ maps continuously into $H^m(\mathbb{R}_+, U)$ for $m = k + 1$. We refer the reader to [20] for a number of examples of controlled and observed partial differential equations where the condition $A^kB \in \mathcal{B}(U, X)$ is satisfied. □

Example 3.2 Consider the following first-order damped hyperbolic PDE on the unit interval (0, 1) with dynamic boundary control u and observation y:

$$\left.\begin{aligned} w_t &= -w_\xi - \varepsilon w \\ w_t(t,0) + w(t,0) &= u(t) \\ z_t(t) + z(t) &= w(t,1) \\ y(t) &= z(t) \end{aligned}\right\} \quad t > 0,$$

where $\varepsilon > 0$. Here $U = \mathbb{C}$. Routine calculations give that the associated integral Hankel operator Γ has impulse response

$$h(t) = \chi_{[1,\infty)}(t)(t-1)\mathrm{e}^{-(t-1)}\mathrm{e}^{-\varepsilon} \quad \forall\, t \in \mathbb{R}_+,$$

where χ_J is the indicator function on interval $J \subseteq \mathbb{R}$. Evidently, $h \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Moreover, $h \in H^1(\mathbb{R}_+)$, but the discontinuity in h' at $t = 1$ gives that $h \notin H^k(\mathbb{R}_+)$ for $k \in \mathbb{N}$, $k \geq 2$. Consequently, Theorem 2.3 yields that Γ restricts to a compact operator $H^1(\mathbb{R}_+) \to H^1(\mathbb{R}_+)$ (and not to $H^k(\mathbb{R}_+)$ for $k \geq 2$) with Schmidt vectors in $H^1(\mathbb{R}_+)$. □

In the context of integral Hankel operators arising from controlled and observed partial differential equations, infinite smoothness of the singular vectors typically arises from parabolic problems, as indicated in the next example.

Example 3.3 Consider the controlled and observed damped heat equation on the real line:

$$\left.\begin{aligned} & w_t(t,\xi) = -\varepsilon w(t,\xi) + w_{\xi\xi}(t,\xi) + \delta_0(\xi)u(t), \\ & \lim_{\xi\to\infty} w(t,\xi) = 0, \\ & y(t) = w(t,\xi_0) \end{aligned}\right\} \quad t > 0,\ \xi \in \mathbb{R}, \qquad (3.2)$$

where $\varepsilon \geq 0$ is a constant, δ_0 is a Dirac-delta corresponding to point control at $\xi = 0$, and $\xi_0 \neq 0$ is a measurement location. Here $U = \mathbb{C}$.

This problem is naturally formulated on the state-space $L^2(\mathbb{R})$ and the PDE in (3.2) can be viewed as an equality in the dual space $H^{-1}(\mathbb{R}) = (H^1(\mathbb{R}))'$ via the associated weak form

$$\langle w_t, \phi\rangle_{L^2(\mathbb{R})} = -\varepsilon\langle w, \phi\rangle_{L^2(\mathbb{R})} - \langle w_\xi, \phi_\xi\rangle_{L^2(\mathbb{R})} + \phi(0)u(t) \quad \forall\, \phi \in H^1(\mathbb{R}).$$

Here, we indicate that the associated impulse response satisfies the hypotheses of our main result. To that end, impose the initial condition $w(\xi, 0) = 0$ on $\mathbb{R}$, fix $u(t) \equiv 1$ and let $K(t, \xi)$ denote the heat kernel in one dimension. Inspired by Duhamel's principle, the solution to (3.2) is given by

$$w(t, \xi) = \int_0^t \mathrm{e}^{-\varepsilon(t-s)} K(t-s, \xi)\,\mathrm{d}s = \int_0^t \mathrm{e}^{-\varepsilon s} K(s, \xi)\,\mathrm{d}s \quad \xi \in \mathbb{R},\ t > 0.$$

Therefore, the impulse response is given by

$$h(t) = \frac{\mathrm{d}}{\mathrm{d}t} y(t) = w_t(t, \xi_0) = \mathrm{e}^{-\varepsilon t} K(t, \xi_0) \quad \forall\, t > 0.$$

Since $\xi_0 \neq 0$, it follows from properties of K that:

(i) $\lim_{t \searrow 0} h^{(k)}(t) = 0$ for all $k \in \mathbb{Z}_+$
(ii) $h \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$
(iii) If $\varepsilon > 0$, then $h \in L^1(\mathbb{R}_+) \cap H^k(\mathbb{R}_+)$ for all $k \in \mathbb{Z}_+$.

Note the requirement that $\varepsilon > 0$ for (iii) above. In particular, when $\varepsilon > 0$ it follows that h satisfies the hypotheses of Proposition 2.1 and Theorem 2.3 for all $m \in \mathbb{N}$. □

Finally, again in the context of controlled and observed partial differential equations, infinite-dimensional U arises in the setting of boundary control and observation, such as the situations studied in [3]. Calculations involving multi-dimensional concrete examples tend to be rather involved. Thus, our fourth example provides sufficient conditions under which the compactness and integrability properties of h as in (1.2a) are satisfied in the case that U is infinite dimensional. We comment that the hypotheses imposed below are typically satisfied by parabolic partial differential equations.

Example 3.4 Let X and U be separable Hilbert spaces. Suppose that A generates an exponentially stable, analytic semigroup $(\mathbb{T}(t))_{t\geq 0}$ on X with compact resolvent.

Let $B \in \mathcal{B}(U, X_{-1})$ and $C \in \mathcal{B}(X_1, U)$, where X_1 and X_{-1} are the usual interpolation and extrapolation spaces associated with A and X, see for example [18] or [28, Section 3.6]. Consider

$$h(t) := C\mathbb{T}(t)B \quad t > 0.$$

We claim that

(i) $h(t)$ is well-defined for all $t > 0$ and $h \in C^\infty((0, \infty), \mathcal{B}(U))$;
(ii) $h(t) \in \mathcal{K}(U)$ for all $t > 0$.

Further, if $B \in \mathcal{B}(U, X_\beta)$ and $C \in \mathcal{B}(X_\alpha, U)$ with $\alpha, \beta \in \mathbb{R}$ and such that $\alpha - \beta < 1$, then

(iii) $h \in L^1(\mathbb{R}_+, \mathcal{K}(U))$

Here X_γ are the fractional spaces associated with A and X, as in [28, Section 3.9, p. 147].

Before establishing the above claims, observe that in the simplifying case that $U = X$ and $C = B = I$, we have $h(t) = \mathbb{T}(t) = I$ when $t = 0$, which is compact if, and only if, X is finite dimensional. In particular, the requirement that h takes compact values only almost everywhere in Lemma 2.4 is essential.

To establish (i), let $t > 0$ and choose $\varepsilon > 0$ such that $t - 2\varepsilon > 0$. As $(\mathbb{T}(t))_{t \geq 0}$ is analytic, it is differentiable and, therefore, it follows that $\mathbb{T}(\varepsilon) \in \mathcal{B}(X_1, X) \cap \mathcal{B}(X, X_{-1})$ (see, for example [7, p. 110]). We conclude that

$$C\mathbb{T}(\varepsilon) : X \to X_1 \to U \quad \text{and} \quad \mathbb{T}(\varepsilon)B : U \to X_{-1} \to X,$$

are bounded operators. Consequently, by the semigroup property

$$h(t) = C\mathbb{T}(t)B = C\mathbb{T}(\varepsilon)\mathbb{T}(t - 2\varepsilon)\mathbb{T}(\varepsilon)B, \tag{3.3}$$

which is well-defined as the composition of bounded operators, and infinitely-differentiable as $\mathbb{T}$ is analytic.

Compactness of $h(t)$ follows from (3.3) once $\mathbb{T}(t - 2\varepsilon)$ is compact, as then $h(t)$ is expressed as the composition of compact and bounded operators. However, $\mathbb{T}(t - 2\varepsilon)$ is compact from [7, Lemma 4.28] as $\mathbb{T}(t - 2\varepsilon)$ is norm continuous (as analytic) and $(\mathbb{T}(t))_{t \geq 0}$ has compact resolvent by hypothesis.

The third claim follows from [28, Theorem 5.7.3]. The conclusion there is that $h \in L^1_{\text{loc}}(\mathbb{R}_+, \mathcal{B}(U))$, but the assumed exponential stability yields that $h \in L^1(\mathbb{R}_+, \mathcal{B}(U))$.

□

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Data availability No datasets were generated or analysed during the current study.

Declarations

Conflicts of interest There are no conflict of interest. No generative AI was used in the preparation of this work.

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