A method for constrained energy-maximising control of heaving wave-energy converters via a nonlinear frequency response

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***Abstract*—A theoretical grounding is presented for justifying how frequency domain methods may be applied in the determination of constrained extracted-energy maximising controls in wave-energy conversion applications subject to nonlinear models. A computational method is subsequently outlined. The theory applies to forced Lur'e systems, an important class of nonlinear control systems, including nonlinear models of simple heaving point-absorber wave-energy converters, and which facilitates a well-defined and tractable frequency response for such systems.**

***Index Terms*—Frequency domain, Nonlinear control, Wave-energy conversion**

I. Introduction

The present paper continues the line of enquiry of designing extracted-energy maximizing controls for wave-energy converters (WECs), focussing on the well-studied case of point absorbers (PAs) moving in the heave direction only. These devices are viewed as amenable to control [1], and appear frequently as a test bed for potential control strategies. The motivation for research into wave-energy conversion is the timely, societal requirement to decarbonise global electrical energy generation to reduce emissions and tackle the climate crisis. Increased renewable energy conversion is recognised as one key mechanism to achieve this aim [2, p. 40]. There is broad consensus from researchers that the potential for wave energy globally is vast, and is as yet mostly untapped. However, despite now considerable research over the past decades, *"wave energy has not yet reached commercial viability"* [3, Abstract] and *"In comparison to other sources of renewable energy, wave energy is still too expensive."* [4, Section 4].

There are numerous WEC technologies, with the paper [5] reporting nearly 150 WECs in development in 2013, and reviews of the field include [3], [4], [6], [7]. Moreover, the reviews [8], [1] and [9] focus on oscillating water columns, point absorbers, and power take-off systems, respectively. The control of WECs has generated much interest, with numerous papers and reviews including [3], [10]–[12].

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Here we report results from the recent work [13] which, briefly, provide a rigorous theoretical underpinning of "frequency domain" methods for forced Lur'e (also Lurie or Lurye) systems; an important and ubiquitous class of nonlinear control systems comprising the feedback connection of a linear control system and a nonlinear output term. Lur'e systems are well-studied objects from both state-space [14] and input-output [15] perspectives. The approach of [13] is to establish so-called *incremental input-to-state stability* properties, as in [16] and related to contraction-type concepts [17], which are then known to give rise to desirable state- and output-response to (almost) periodic forcing terms, which we describe.

The relevance and novelty presently is that nonlinear models for heaving PA WECs may be expressed as forced Lur'e systems, as we demonstrate, with forcing terms which are likely to be almost periodic corresponding to the wave excitation force [18]. The results from [13] are applied to the design of extracted-energy maximising controls for such devices, at least under some assumptions which we describe, and may be designed to (approximately) accommodate various state and input constraints associated with safe operation of WECs. The ability to move beyond linear systems and treat nonlinear models for WECs is important for the development of the field, as argued and evidenced in [19]. The present method seeks to provide a rigorous underpinning for frequency domain methods in nonlinear control, beyond that of the harmonic balance method [20], and is also motivated by the prevalence of frequency-domain methods in wave-energy control; see, the comments in [21], for instance.

The present conference paper is organised as follows. The archetypal wave-energy converter is recalled in Section II. The main theoretical results are reported in Sections III and IV. The consequences of these results for the analysis and design of optimal controls is discussed in Section V. There is some overlap between the ideas in this section, and those of [22] and [23]. Summarising comments appear in Section VI.

A. Notation

Mathematical notation is standard, and kept to a minimum. We mention only a few items. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ denote the real and nonnegative real numbers, respectively. We let $\mathcal{K}$ denote the class of continuous, strictly increasing functions $\mathbb{R}_+ \to \mathbb{R}_+$ which are zero at zero. The subset of $\mathcal{K}$ of unbounded functions is denoted $\mathcal{K}_\infty$. The symbol $\mathcal{KL}$ denotes functions which are $\mathcal{K}$ in their first variable, and continuous and decreasing to zero in their second. The sets $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ are all common in nonlinear control theory, and are examples of so-called comparison functions, see [24].

II. A WAVE-ENERGY CONVERTER MODEL

Consider the following model for the vertical displacement z of a PA WEC moving in the heave direction:

$$m\ddot{z} + f_{\rm rad} + f_{\rm vis} + f_{\rm buoy} - f_{\rm ex} + f_{\rm PTO} + f_{\rm moor} = 0\,. \quad \text{(II.1a)}$$

The terms in (II.1a) are described in Table II.1.

Term	Description	Term	Description
m	mass of device	$f_{\rm ex}$	external excitation force
$f_{\rm moor}$	mooring force	$f_{\rm PTO}$	power take-off force
$f_{\rm rad}$	radiation force	$f_{\rm vis}$	viscous (damping) force
$f_{\rm buoy}$	buoyancy force		

TABLE II.1: Terms appearing in equation (II.1a)

Despite included in (II.1a) for correctness, we shall in fact for simplicity assume throughout that the mooring force is equal to zero, and so it shall play no role. We assume that the buoyancy term is a linear function of displacement, that is,

$$f_{\rm buoy} = kz\,, \quad \text{(II.1b)}$$

for positive constant $k = g\rho S_w$, the product of acceleration due to gravity g, the water density ρ, and the surface area of the body cut at the mean water level S_w.

In wave-energy conversion applications, the power take-off force is a control variable, which we label $u := f_{\rm PTO}$. The excitation force $f_{\rm ex}$ in (II.1a) is an external forcing term.

We wish to allow for the situation that the viscous drag term $f_{\rm vis}$ is nonlinear. Our theoretical development places only qualitative assumptions on $f_{\rm vis}$, which we shall impose as required. As a concrete example, functions which fall within the scope of our results include

$$d_0 v + d_1 v|v|^r \quad \forall\, v \in \mathbb{R}\,,$$

for $r \in \mathbb{N}$, $d_0, d_1 \geq 0$ with not both zero. Functions of this form have been considered as models of nonlinear viscous damping, such as in [25, equation (6)], which itself is based on the experimental law proposed in [26].

The radiation force $f_{\rm rad}$ is usually described mathematically by the Cummins' [27] equation

$$f_{\rm rad}(t) = m_\infty \ddot{z} + (h_r * \dot{z})(t)\,, \quad \text{(II.1c)}$$

where $*$ denotes convolution, the positive constant m_∞ is the so-called "added mass", and h_r for some non-parametric kernel (or impulse response). This kernel is typically determined numerically using hydrodynamic software such as the WAMIT [28] or FOAMM [29] toolboxes; see [30].

For notational convenience, we absorb m_∞ into the mass of the device by setting $M := m + m_\infty$. We assume a finite-dimensional state-space approximation of order $n_{\rm r}$ of the convolution term in (II.1c), yielding

$$\dot{x_{\rm r}} = A_{\rm r} x_{\rm r} + B_{\rm r}\dot{z}, \quad y_{\rm r} = B_{\rm r}^\top x_{\rm r} \,\Big(= G_{\rm rad}\dot{z}\Big)\,, \quad \text{(II.1d)}$$

and, consequently,

$$f_{\rm rad}(t) = m_\infty \ddot{z}(t) + y_{\rm r}(t)\,. \quad \text{(II.1e)}$$

The state-space system (II.1d) is assumed to be stable and passive, here meaning that $A_{\rm r}$ is Hurwitz and that

$$A_{\rm r}^\top + A_{\rm r} \leq 0\,. \quad \text{(II.2)}$$

A. An extracted-energy optimal control problem

We consider an extracted-energy optimal control problem associated with the PA WEC model (II.1), in the spirit of the *energy-balance method* of [31]. In full generality, the energy-balance method applies to control systems represented as linear graphs (see, for example [32, Section 2.1]). This level of generality is not required here, and instead we appeal to the principles of the energy-balance method. The first ingredient is the energy of the device, equal to the sum of the kinetic and potential energy of the device, and so given by

$$E(z(t)) := \frac{1}{2}M\dot{z}^2(t) + \frac{1}{2}kz^2(t)\,. \quad \text{(II.3)}$$

The *energy-balance equation* for the PA WEC model (II.1) with energy (II.3) is given by

$$E(z(t_1)) - E(z(t_0)) + \int_{t_0}^{t_1} \dot{z}u \,{\rm d}t = \int_{t_0}^{t_1} -f_{\rm vis}\dot{z} - y_{\rm r}\dot{z} + \dot{z}f_{\rm ex}\,{\rm d}t\,. \quad \text{(II.4)}$$

Equation (II.4) is an expression of conservation of energy. The left-hand side of (II.4) equals the change in internal energy plus the extracted energy, both over the given time horizon $[t_0, t_1]$. The three terms on the right-hand side of (II.4) correspond to the energy dissipated by the viscous damping, the energy dissipated by the radiation system, and the supplied energy, respectively, again each over the time horizon $[t_0, t_1]$.

Energy-balance equations are valid under quite general assumptions, see [31], but here (II.4) is most simply derived by differentiating the energy (II.3), using the equations (II.1), then integrating both sides from t_0 to t_1, and rearranging.

The left-hand side of (II.4) is the desired quantity to be maximised which, for convenience, we record as

$$E(z(t_1)) - E(z(t_0)) + \int_{t_0}^{t_1} \dot{z}u \,{\rm d}t\,. \quad \text{(II.5)}$$

We comment that in much (if not all) of the optimal control of WEC literature, the cost functional of extracted energy

$$\int_{t_0}^{t_1} \dot{z}u \,{\rm d}t\,, \quad \text{(II.6)}$$

is considered instead. However, we contend that maximising (II.5) is, in fact, much simpler than maximising (II.6). Roughly speaking, this is because (II.5) removes unwanted boundary effects associated with optimal control (namely, as currently formulated, that there is no time beyond t_1). Furthermore, the time horizon $[t_0, t_1]$ is determined practically by the horizon over which the excitation force f_{ex} may be predicted. For long-term WEC deployment, the cost-functional (II.5) shall be maximized over a series of successive time intervals $[t_k, t_{k+1}]$, leading to maximising

$$\sum_{k=0}^{N} \Big(E(z(t_{k+1})) - E(z(t_k)) + \int_{t_k}^{t_{k+1}} \dot{z} u \, \mathrm{d}t \Big) = E(z(t_{N+1})) - E(z(t_0)) + \int_{t_0}^{t_{N+1}} \dot{z} u \, \mathrm{d}t \,. \tag{II.7}$$

Over many time intervals, we expect that the difference between (II.6) and (II.7) is "small".

The essence of the energy-balance method is to maximise (II.5) — the left-hand side of (II.4) — by, equivalently, maximising the right-hand side of (II.4). The right-hand side is independent of u and, as discussed extensively in [31], is often simpler to maximise than (II.5) directly. To do so we view (in this case), $\dot{z}$ appearing in the right-hand side of (II.4) as the independent variable for the optimisation, and use the dynamic equation (II.1a) to *define* the corresponding optimal u.

To provide "ground truths" against which our later numerical method may be compared, we maximise (II.5) in the cases of linear and non-linear viscous damping separately.

B. Linear viscous damping

Here we assume that the viscous damping force f_{vis} is linear in $\dot{z}$, that is, $f_{\text{vis}}(\dot{z}) = d\dot{z}$ for $d > 0$. With this assumption the right-hand side of (II.4) is quadratic in $\dot{z}$, and we may solve the optimisation problem exactly by completing the square in the L^2-inner product. For which purpose, set $v := \dot{z}$ and $G := dI + G_{\text{rad}}$ which is strongly passive, and hence invertible with passive inverse. We express the right-hand side of (II.4) in terms of the L^2-inner product, namely,

$$- \big\langle Gv - f_{\text{ex}}, v \big\rangle_{L^2(t_0,t_1)} = -\Big\langle Gv - \frac{f_{\text{ex}}}{2}, G^{-1}\big(Gv - \frac{f_{\text{ex}}}{2}\big) \Big\rangle_{L^2} + \Big\langle \frac{f_{\text{ex}}}{2}, G^{-1}\frac{f_{\text{ex}}}{2} \Big\rangle_{L^2}$$

where we have written L^2 instead of $L^2(t_0, t_1)$ for brevity. Observe that the first term on the right-hand side of the above equality is non-positive as G^{-1} is passive, and the second term is independent of v. Consequently, both sides of the above are maximised by setting

$$v_* := G^{-1}(f_{\text{ex}}/2) \,, \tag{II.8}$$

and the maximum is equal to

$$\frac{1}{4}\big\langle f_{\text{ex}}, G^{-1} f_{\text{ex}} \big\rangle_{L^2(t_0,t_1)} = \frac{1}{4}\int_{t_0}^{t_1} f_{\text{ex}}(t)(G^{-1} f_{\text{ex}})(t) \, \mathrm{d}t \,.$$

Recall that (II.8) corresponds to the optimal velocity profile of the WEC device, and the dynamic equation (II.1a) is used to *define* the corresponding optimal control u_*. As a common sense check, in the case that $G_{\text{rad}} = 0$ (meaning no radiation system is present), then G is a static input-output operator $Gv = dv$, so that $G^{-1}v = v/d$. Then, the expression (II.8) the optimal velocity profile simplifies to

$$\dot{z}_* = f_{\text{ex}}/(2d) \,.$$

The above formula is well known; see, for example [33, equation (6.44), p. 206] for a frequency domain expression, or [34, equation (2.12)]. The resulting optimal control goes by a number of terms, as discussed in [33, p. 206], such as *phase and amplitude control* or *complex-conjugate control*.

In general, computing $\dot{z} = v$ from (II.8) requires both inverting $G = dI + G_{\text{rad}}$ and knowledge of the excitation force f_{ex}. Since here we are assuming that G_{rad} has a finite-dimensional state-space realisation $(A_{\text{r}}, B_{\text{r}}, B_{\text{r}}^\top)$, then clearly $(A_{\text{r}}, B_{\text{r}}, B_{\text{r}}^\top, dI)$ realises G, and it is well known that $(A_{\text{r}} - B_{\text{r}}(1/d)B_{\text{r}}^\top, -B_{\text{r}}/d, B_{\text{r}}^\top/d, 1/d)$ realises G^{-1}; see, for example [35, p.279]. We comment further that here the right-hand side of (II.4) could be maximised by completing the square, essentially exploiting the quadratic structure of the optimisation problem. Thus, we avoided the use of more involved direct or indirect optimization tools, such as the Pontryagin Principle (see, for instance [36, Theorem 2, p. 85]).

C. Nonlinear viscous damping

Here we consider a general nonlinear viscous damping term f_{vis} which is assumed to be differentiable and satisfy:

$$f_{\text{vis}}(v)v \geq 0 \quad \text{and} \quad g(v) := \big(f_{\text{vis}}(v)v\big)' \quad \text{invertible.} \tag{II.9}$$

The right-hand side of the energy-balance equation (II.4) reads

$$\int_{t_0}^{t_1} -f_{\text{vis}}(\dot{z})\dot{z} - (G_{\text{rad}}\dot{z})\dot{z} + \dot{z} f_{\text{ex}} \, \mathrm{d}t \,, \tag{II.10}$$

which we aim to maximise by again setting $v = \dot{z}$ as the independent variable. However, when $G_{\text{rad}} \neq 0$ then the integrand in (II.10) contains both a nonlinear term, which need not be quadratic, and a term with memory. As such, completing the square in the L^2-inner product will generally not be applicable, and neither will pointwise maximisation of the integrand — another method is required. We shall in fact present a numerical method in Section V which exploits the expected (almost) periodic nature of the problem.

Again for the purpose of later comparison, in the simplifying case that $G_{\text{rad}} = 0$ (that is, no radiation system is present), then the optimization problem may be solved by pointwise maximisation of the integrand in (II.10), that is, by maximising

$$v \mapsto -f_{\text{vis}}(v)v + v f_{\text{ex}}(t) \,,$$

at every $t \in [t_0, t_1]$. Routine calculus gives that a necessary condition for a maximum is

$$g(v) = f_{\text{ex}}(t) \tag{II.11}$$

which has a unique solution $g^{-1}(f_{\text{ex}}(t))$ by hypothesis (II.9).

As an example, in the case that $f_{\rm vis}(v) := dv|v|$ for all $v \in \mathbb{R}$, where $d > 0$ is a positive constant, we compute that

$$g(v) = 3dv^2 \mathrm{sign}\,(v) \quad \forall\, v \in \mathbb{R}\,,$$

which is invertible. In this case, the solution of (II.11) in the case that $G_{\rm rad} = 0$ is given by

$$v_*(t) = g^{-1}(f_{\rm ex}(t)) = \mathrm{sign}(f_{\rm ex}(t))\sqrt{\frac{|f_{\rm ex}(t)|}{3d}}\,. \qquad \text{(II.12)}$$

We conclude this section by noting that, upon setting

$$x := \begin{pmatrix} z \\ \dot{z} \\ x_{\rm r} \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 1 & 0 \\ -k/M & 0 & -B_{\rm r}^\top/M \\ 0 & B_{\rm r} & A_{\rm r} \end{pmatrix}$$
$$B := \begin{pmatrix} 0 & 1/M & 0 \end{pmatrix}^\top, \quad C := \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}\,,$$

the PA WEC model (II.1) is of the form

$$\dot{x} = Ax - Bf_{\rm vis}(Cx) + B(f_{\rm ex} + u)\,,$$

that is, a so-called forced Lur'e equation which comprises the focus of the next section.

III. A semi-global incremental passivity theorem for Lur'e systems

Here we report so-called incremental stability results for the following system of forced nonlinear differential equations

$$\dot{x}(t) = Ax(t) - Bf(Cx(t)) + Bw(t) \quad t \geq 0\,, \qquad \text{(III.1)}$$

called a Lur'e system for brevity. The terms A, B and C in (III.1) are $n \times n$, $n \times m$ and $m \times n$ matrices, respectively, for fixed integers m, n. The terms x and w, taking values in $\mathbb{R}^n$ and $\mathbb{R}^m$, are the state- and forcing-variables, respectively. In the subsequent wave-energy conversion applications, w shall comprise a forcing term plus a control term. Furthermore, the nonlinearity $f : \mathbb{R}^m \to \mathbb{R}^m$ is assumed to be locally Lipschitz.

Let $w : \mathbb{R}_+ \to \mathbb{R}^m$ be measurable and locally essentially bounded. A locally absolutely continuous function $x : \mathbb{R}_+ \to \mathbb{R}^n$ which satisfies (III.1) almost everywhere on $\mathbb{R}_+$ is called a *global solution* of (III.1), and we call the corresponding pair (w, x) a *trajectory* of (III.1), the set of which is denoted $\mathcal{B}$. In the usual absolute stability framework, the function f is uncertain, and rather only expressed in terms of qualitative properties. Under the typical assumption that $f(0) = 0$, then $(0, 0)$ is a constant (equilibrium) trajectory of (III.1). In this case, the Lur'e system (III.1) is called

- Input-to-state Stable (ISS) if there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all $(w, x) \in \mathcal{B}$,

$$\|x(t)\| \leq \psi(\|x(0)\|, t) + \phi(\|w\|_{L^\infty(0,t)}) \quad \forall\, t \geq 0\,.$$

- Exponentially Input-to-State Stable if there exist $L, \gamma > 0$ such that, for all $(w, x) \in \mathcal{B}$,

$$\|x(t)\| \leq L\big(\mathrm{e}^{-\gamma t}\|x(0)\| + \|w\|_{L^\infty(0,t)}\big) \quad \forall\, t \geq 0\,.$$

That is, the exponential ISS property is an ISS bound with

$$\phi(s, t) := L\mathrm{e}^{-\mu t}s \quad \text{and} \quad \psi(s) := Ls \quad \forall\, (s, t) \in \mathbb{R}_+ \times \mathbb{R}_+\,.$$

Presently, the relevant incremental stability concept is the following, we say that (III.1) is *semi-globally incrementally ISS* if, for all $Q > 0$ there exist $\psi \in \mathcal{KL}$ and $\phi \in \mathcal{K}$ such that, for all $(v_i, x_i) \in \mathcal{B}$, $i = 1, 2$, with $\|x_i(0)\| + \|w_i\|_{L^\infty} < Q$,

$$\|x_1(t) - x_2(t)\| \leq \psi(\|x_1(0) - x_2(0)\|, t) + \phi(\|w_1 - w_2\|_{L^\infty(0,t)})\,,$$

for all $t \geq 0$. The concept of *semi-global incremental exponential ISS* is formulated analogously. The above bound estimates the difference of solutions of (III.1) in terms of (in general nonlinear functions of) the difference of their initial states and difference of forcing terms. In particular, the functions ψ and ϕ may depend on Q. Since Q is fixed, but arbitrary, semi-global stability concepts seem no less useful in practical settings. Evidently, incremental stability concepts are equivalent to their non-incremental versions for linear control systems, but *not* for nonlinear control systems.

We record the following assumptions.

(A1) The pair (C, A) is detectable, and there exists a symmetric positive semi-definite $P \in \mathbb{R}^{n \times n}$ such that

$$\begin{pmatrix} A^\top P + PA & PB - C^\top \\ B^\top P - C & 0 \end{pmatrix} \leq 0\,. \qquad \text{(III.2)}$$

(A2) For every compact set $\Gamma \subset \mathbb{R}^m$, there exists $\theta_\Gamma \in \mathcal{K}_\infty$ such that

$$\|f(y + z) - f(z)\| \leq \theta_\Gamma(\|y\|) \quad \forall\, y \in \mathbb{R}^m,\ \forall\, z \in \Gamma\,.$$

(A3) For every compact set $\Gamma \subset \mathbb{R}^m$, there exists $\alpha_\Gamma \in \mathcal{K}_\infty$ such that

$$\|y\|\alpha_\Gamma(\|y\|) \leq \langle y, f(y+z) - f(z)\rangle \quad \forall y \in \mathbb{R}^m,\ \forall z \in \Gamma\,.$$

(A4) For every compact set $\Gamma \subset \mathbb{R}^m$, there exist $\mu_\Gamma, c_\gamma \geq 0$, $\mu_\Gamma c_\Gamma \geq 1$, such that

$$\|f(y + z) - f(z)\| \leq c_\Gamma\langle y, f(y + z) - f(z)\rangle$$
$$\forall\, y \in \mathbb{R}^m,\ \|y\| \geq \mu_\Gamma,\ \forall\, z \in \Gamma\,.$$

Commentary on the above assumptions is given below. Here $\langle\cdot,\cdot\rangle$ denotes the usual inner-product on $\mathbb{R}^m$. The following result contains sufficient conditions for two semi-global incremental stability notions for the Lur'e system (III.1), and is an incremental generalisation of [37, Theorem 1]. An expanded version of this result, with proof, appears in [13].

Theorem III.1. *Consider the Lur'e system* (III.1)*. If* **(A1)–(A4)** *hold, then* (III.1) *is semi-globally incrementally ISS. If, additionally,* $\alpha_\Gamma(s) = \varepsilon_\Gamma s$ *in* **(A3)** *for some* $\varepsilon_\Gamma > 0$*, then* (III.1) *is semi-globally incrementally exponentially ISS.*

Some remarks are in order. If the triple (A, B, C) is controllable and observable, with positive real (as in [38]) transfer function $\mathbf{G}(s) := C(sI - A)^{-1}B$, then a positive definite solution $P = P^\top$ to the LMI (III.2) follows from the Positive Real Lemma; see, for example [39, Corollary 5.6].

The hypotheses **(A2)–(A4)** relate to the nonlinearity f. They are, in essence, incremental versions of the hypotheses which appear in the ISS result [37, Theorem 1]. It can be shown that hypothesis **(A3)** implies **(A4)** in the case that $m = 1$, paralleling the situation noted in [37, Remark 2]. To provide further insight into **(A3)**, first recall that the condition

$$\langle z_1 - z_2, f(z_1) - f(z_2)\rangle \geq 0 \quad \forall\, z_1, z_2 \in \mathbb{R}\,,$$

which, in the scalar ($m = 1$) case, is equivalent to

$$0 \leq \frac{f(z_1) - f(z_2)}{z_1 - z_2} \quad \forall\, z_1, z_2 \in \mathbb{R},\ z_1 \neq z_2\,,$$

is simply monotonicity of f. In the scalar case, this is also termed a lower slope restriction, somewhat common in nonlinear control theory (e.g. [40]). Roughly speaking, condition **(A3)** with $\alpha_\Gamma \in \mathcal{K}_\infty$ "strengthens" the monotonicity of f, capturing a lower bound for its rate of increase.

Example III.2. (1) For $r \in \mathbb{N}$, the function

$$f(v) := d_0 v + d_1 v|v|^r \quad \forall\, v \in \mathbb{R}\,, \tag{III.3}$$

for $d_0, d_1 \geq 0$, not both zero, satisfies **(A2)–(A4)**. In fact, $\alpha_\Gamma(s) = \varepsilon_\Gamma s$ is valid if $d_0 > 0$. The verification is a somewhat tedious sequence of calculations which are presented in [13].

(2) Consider the so-called diagonal (or decoupled) nonlinearity $f : \mathbb{R}^m \to \mathbb{R}^m$ for $m > 1$, that is, f satisfies

$$\big(f(v)\big)_i = f_i(v_i) \quad \forall\, v \in \mathbb{R}^m,\ \forall\, i \in \{1, 2, \ldots, m\}\,, \tag{III.4}$$

for given component functions $f_i : \mathbb{R} \to \mathbb{R}$. It can be shown that f satisfies **(A2)–(A4)** if every f_i does.

IV. Consequences of incremental stability — response to almost periodic forcing terms

Here we present consequences of the incremental stability results of Section III, namely, investigating the response of such Lur'e systems to (almost) periodic functions. The subsequent results provide a theoretical underpinning for "frequency response" techniques for such nonlinear control systems.

We recall a few concepts of almost periodicity (in the sense of Bohr), and refer to the text [41] for further background. Let $R = \mathbb{R}_+$ or $\mathbb{R}$, and let $C(R, \mathbb{R}^n)$ denote the space of continuous functions $R \to \mathbb{R}^n$. A set $S \subseteq R$ is said to be *relatively dense* (in R) if there exists $l > 0$ such that

$$[a, a + l] \cap S \neq \emptyset \quad \forall\, a \in R\,.$$

The integers $\mathbb{Z}$ are relatively dense in $\mathbb{R}$, for instance. For $\varepsilon > 0$, we say that $\tau \in R$ is an *ε-period* of $v \in C(R, \mathbb{R}^n)$ if

$$\|v(t) - v(t + \tau)\| \leq \varepsilon \quad \forall\, t \in R\,.$$

We denote by $P(v, \varepsilon) \subseteq R$ the set of ε-periods of v and we say that $v \in C(R, \mathbb{R}^n)$ is *almost periodic* if $P(v, \varepsilon)$ is relatively dense in R for every $\varepsilon > 0$. We denote the set of almost periodic functions $v \in C(R, \mathbb{R}^n)$ by $AP(R, \mathbb{R}^n)$ which is a vector space (note, unlike the set of periodic functions). It is clear that any continuous periodic function is almost periodic.

The archetypal almost periodic functions are trigonometric polynomials, which are of the form

$$p(t) = \sum_{k=0}^{N} a_k \mathrm{e}^{i\omega_k t} \quad \forall\, t \in R\,, \tag{IV.1}$$

for (in this case) $a_k \in \mathbb{C}^n$ and $\omega_k \in \mathbb{R}$. It is well-known that

$$AP(R, \mathbb{R}^n) = \mathrm{clo}(TP(R, \mathbb{R}^n)), \quad R = \mathbb{R} \text{ or } \mathbb{R}_+\,,$$

providing a somewhat concrete description of almost periodic functions, and where the closure is taken in the space of bounded, uniformly continuous functions $R \to \mathbb{R}^n$ with the supremum norm.

The *characteristic exponents* of a function $f \in AP(\mathbb{R}, \mathbb{R}^n)$ are the $\lambda \in \mathbb{R}$ such that the following mean values

$$M(f_\lambda) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\mathrm{e}^{-i\lambda t}\,\mathrm{d}t\,,$$

are non-zero, that is, the set $\Lambda(f)$ given by

$$\Lambda(f) := \big\{\lambda \in \mathbb{R} \,:\, M(f_\lambda) \neq 0\big\}\,.$$

It is known that the above limit exists for all almost periodic functions and, for such functions, the set of characteristic exponents is countable (see [41, II, p.21] and [41, III, p.22], respectively). The *module* associated with $f \in AP(\mathbb{R}, \mathbb{R}^n)$ is the set of finite linear integer combinations of elements in $M(f_\lambda)$, that is,

$$\mathrm{mod}\ (f) = \Big\{\sum_{k=0}^{M} \alpha_k \lambda_k \,:\, \lambda_k \in \Lambda(f),\ \alpha_k \in \mathbb{Z},\ M \in \mathbb{N}\Big\}\,.$$

Example IV.1. Let the trigonometric polynomial p be as in (IV.1). The characteristic exponents of p are readily computed to be $\Lambda(p) = \{\omega_1, \omega_2, \ldots, \omega_N\}$, and consequently,

$$\mathrm{mod}\ (p) = \Big\{\sum_{k=0}^{N} \alpha_k \omega_k \,:\, \alpha_k \in \mathbb{Z}\Big\}\,.$$

We remark that there is a close relationship between $AP(\mathbb{R}, \mathbb{R}^n)$ and $AP(\mathbb{R}_+, \mathbb{R}^n)$, roughly by restriction or extension, (these spaces are isometrically isomorphic), and the modules of functions in $AP(\mathbb{R}_+, \mathbb{R}^n)$ are equal to those of the corresponding extension in $AP(\mathbb{R}, \mathbb{R}^n)$; see [42, Appendix C].

The following proposition is an abridged version of [13, Propositions 5.1, 5.2], and is the main result of this section.

Proposition IV.2. *Consider the Lur'e system* (III.1) *with $f(0) = 0$, $w \in AP(\mathbb{R}_+, \mathbb{R}^m)$, and assume that* **(A1)–(A4)** *hold. Then there exists a unique $z^{\mathrm{ap}} \in AP(\mathbb{R}_+, \mathbb{R}^n)$ such that $(w, z^{\mathrm{ap}}) \in \mathcal{B}$ and, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $P_1(w, \delta) \subset P(z^{\mathrm{ap}}, \varepsilon)$.*

Let $(\nu, x) \in \mathcal{B}$. The following further statements hold.

(1) *If $\nu \in L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ and $\|\nu - w\|_{L^\infty(t,\infty)} \to 0$ as $t \to \infty$, then*
$$\lim_{t\to\infty} \big(x(t) - z^{\mathrm{ap}}(t)\big) = 0\,.$$

(2) *If w is periodic with period τ, then z^{ap} is τ-periodic.*

(3) $\mathrm{mod}(z^{\mathrm{ap}}) \subseteq \mathrm{mod}(w)$.

(4) *There exists $\theta \in \mathcal{K}$ such that, for $(\nu, x^{\mathrm{ap}}) \in \mathcal{B}$ with $\nu \in AP(\mathbb{R}_+, \mathbb{R}^m)$ and $x^{\mathrm{ap}} \in AP(\mathbb{R}_+, \mathbb{R}^m)$*
$$\|x^{\mathrm{ap}} - z^{\mathrm{ap}}\|_{L^\infty} \leq \theta(\|\nu - w\|_{L^\infty})\,.$$

Loosely summarising, Proposition IV.2 gives the existence of an unique (almost) periodic trajectory of (III.1) when subject to an (almost) periodic input, which attracts all other state trajectories subject to the same input and has continuous dependence on the input data. Moreover, the module of the resulting almost periodic state is contained in that of the input. In particular, the "frequencies" of the resulting state are necessarily integer linear combinations of those of the input.

There is some overlap with the above result, and [43, Theorem 4.3]. The similarities and differences are discussed in more detail in [13]. Briefly, these results have different underlying hypotheses, and [43, Theorem 4.3] does not apply to superlinear functions, such as those of the form (III.3).

One powerful consequence of Proposition IV.2 is that the following Fourier series arguments may be rigorously deployed. Suppose that the forcing term w to (III.1) is τ-periodic, so that w admits a Fourier series

$$w(t) = \sum_{k=-\infty}^{\infty} w_k \mathrm{e}^{2\pi i k t/\tau} \,,$$

(which, note, is real when $w_{-k} := \overline{w_k}$). Then a real, periodic solution z^{ap} of (III.1) is ensured, so that,

$$z^{\mathrm{ap}}(t) = \sum_{k=-\infty}^{\infty} z_k \mathrm{e}^{2\pi i k t/\tau} \,,$$

for some $z_k \in \mathbb{C}^n$. In practice, we would approximate these series by large partial sums. Set $\lambda_k := 2\pi k/\tau$. The trajectory property of (w, z^{ap}) yields that

$$\sum_{k=-\infty}^{\infty} (\lambda_k i I - A)\mathrm{e}^{\lambda_k i t} z_k = -Bf(Cz^{\mathrm{ap}}) + B \sum_{k=-\infty}^{\infty} w_k \mathrm{e}^{\lambda_k i t} \,.$$

Multiplying both sides by $\mathrm{e}^{\lambda_j i}$ for $j \in \mathbb{Z}$, integrating over t from 0 to τ, and simplifying, gives

$$(\lambda_j i I - A) z_j = -\frac{1}{\tau}\int_0^\tau \mathrm{e}^{\lambda_j i t} Bf(Cz^{\mathrm{ap}})\,\mathrm{d}t + Bw_j \,, \quad \text{(IV.2)}$$

and so

$$Cz_j = \mathbf{G}(\lambda_j i)\Big(-\frac{1}{\tau}\int_0^\tau \mathrm{e}^{\lambda_j i t} f\Big(\sum_{k=-\infty}^{\infty} Cz_k \mathrm{e}^{\lambda_k i t}\Big)\,\mathrm{d}t + w_j\Big) \,.$$

This is a nonlinear system of equations for the Cz_j, which may be solved numerically, for instance by approximating the integral via quadrature. Once Cz_j are (approximately) determined, then z_j (and hence z^{ap}) may be recovered from (IV.2).

V. Application to constrained energy maximisation of a wave-energy converter model

Here, we apply the results of Section IV to the wave-energy optimisation problem of Section II-A. Recall, we seek to maximise (II.5) subject to the WEC model (II.1), and that (II.5) may be expressed via the energy-balance equation (II.10). Furthermore, we have already noted that (II.1) may be expressed as a Lur'e system (III.1). These models naturally satisfy the hypotheses of Sections III and IV.

We make the following conjecture:

(C) When (II.1) is subject to an (almost) periodic forcing term f_{ex}, then the optimal control is also (almost) periodic.

Assuming that (C) is true, then the input (optimal control plus forcing term) to model (II.1) is almost periodic, which yields (at least asymptotically) an almost periodic state response in accordance with Proposition IV.2. For simplicity, here we focus on the periodic case using Fourier series, and could treat the almost periodic case via trigonometric polynomials. Thus, our approach is to express the $\dot{z}$ in (II.10) as a Fourier series,

$$\dot{z}(t) = \sum_{k=0}^{N} a_k \cos(2\pi i k t/\tau) + b_k \sin(2\pi i k t/\tau) \quad \text{(V.1)}$$

with unknown $2N+1$ Fourier coefficients $V := \begin{pmatrix} a_0 & \dots & a_N & b_1 & \dots & b_N \end{pmatrix}^\top$, leading to a static optimisation problem, which may be solved numerically, particularly if the integral is approximated by quadrature. Constraints are approximately included as functions of V. Indeed, the velocity condition

$$|\dot{z}(t)| \le v_{\max} \quad \forall\, t \in [t_0, t_1] \,, \quad \text{(V.2)}$$

is enforced at the fixed points $s_j \in [t_0, t_1]$, and encoded as

$$\begin{pmatrix} W(s_j) \\ -W(s_j) \end{pmatrix} V \le \begin{pmatrix} v_{\max} \\ v_{\max} \end{pmatrix} ,$$

where $W(t)$ is the row-vector with components

$$W(t)_k := \begin{cases} 1 & k = 1 \\ \cos(2\pi(k-1)t/\tau) & k = 2, \dots, N+1 \\ \sin(2\pi(k-1-N)t/\tau) & k = N+2, \dots, 2N+1 \,. \end{cases}$$

The displacement condition $|z(t)| \le x_{\max}$ for all $t \in [t_0, t_1]$ is derived similarly, only now to the integral of $\dot{z}$ in (V.1). Recall that in the present framework, the input (here denoting the PTO force) $f_{\mathrm{PTO}} = u$ is determined via (II.1a) once $\dot{z}$ and z are determined. Therefore, pointwise input constraints on the magnitude of the input (maximum force of the PTO)

$$|u(t)| \le u_{\max} \quad \forall\, t \in [t_0, t_1] \,,$$

or the level of reactive power the PTO is able to supply

$$u(t)\dot{z}(t) \le p_{\max} \quad \forall\, t \in [t_0, t_1]$$

(cf. [44, equation (9)], up to sign convention) are also handled as, in general nonlinear, constraints on V.

We claim that the model (II.1) satisfies the hypotheses of our main results. The pair (C, A) is detectable, as can be shown by a routine argument and, by hypothesis, I is a solution of the LMI (III.2) for the triple $(A_{\mathrm{r}}, B_{\mathrm{r}}, B_{\mathrm{r}}^\top)$. Therefore, in light of (II.2), a straightforward calculation gives that

$$P := \operatorname{diag}\begin{pmatrix} k & M & I \end{pmatrix} ,$$

is a symmetric, positive-definite solution of (III.2), showing that **(A1)** holds. It has already been established in Example III.2 that f in (III.3) satisfies hypotheses **(A2)–(A4)**.

As a numerical simulation, we maximize (II.10) as a function of $\dot{z}$ — via determination of V — in three cases:

(a) f_{vis} is linear, and $G_{\mathrm{rad}} \neq 0$

(b) $f_{\rm vis}$ is nonlinear, and $G_{\rm rad} = 0$
(c) $f_{\rm vis}$ is nonlinear and $G_{\rm rad} \neq 0$

Observe that in cases (a) and (b) the exact optimum (in the absence of constraints) is known, given by (II.8) and (II.12), respectively. In each case we take $t_0 = 0$, $t_1 = 2$,

$$N = 4,\ \lambda = 2\pi\frac{3}{2},\ f_{\rm ex}(t) := \cos(2\lambda t) + \sin(\lambda t)\,,$$

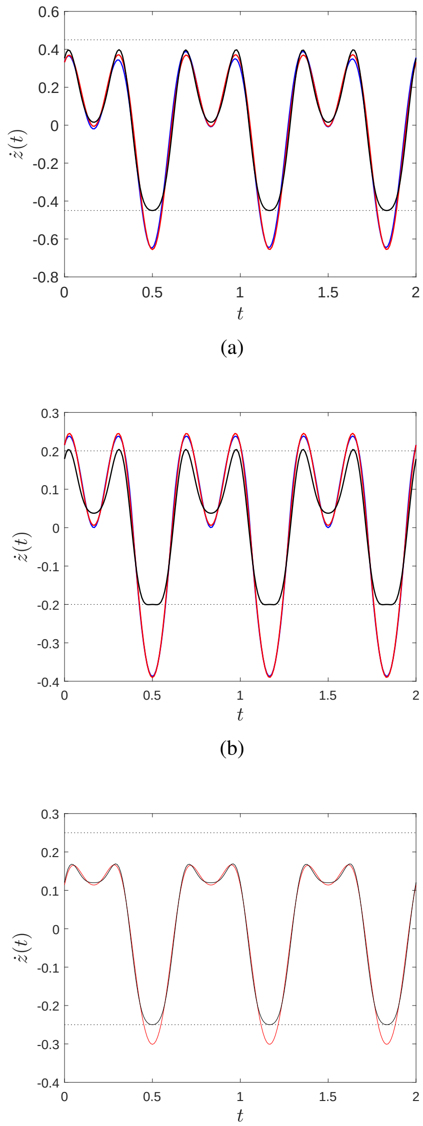
and approximate all integrals by the trapezoidal rule with 30 intervals. As an illustrative example, for simplicity we take

$$h_{\rm rad}(t) := 2{\rm e}^{-2t} - {\rm e}^{-t}\,.$$

For case (a), $f_{\rm vis}(v) := 1.5v$. For cases (b) and (c), we set $f_{\rm vis}(v) := 2v + v|v|$ (of the form (III.3) with $d = 1$).

Our results are plotted in Figure V.1. The optimization problems were solving using `fmincon` in MATLAB. Panels (a), (b) and (c) correspond to the cases above. In (a) and (b) the exact optimal velocity profile v_* is plotted in blue, the numerically-computed optimum in red, and the optimum subject to the velocity constraint (V.2) with $v_{\max} = 0.45$ (a) or $= 0.2$ (b) in black, enforced at the 21 points $s_k := 0.1k$, $k = 0, 1, ..., 20$. We see that the red and blue curves are close, where presumably the difference is a consequence of numerical approximation of the integrals. The black curve qualitatively follows the two other curves, yet respects the constraint. In panel (c) the numerical optimum is plotted in red, as here there is no theoretical maximum to compare to, and the constrained version plotted in black, with $v_{\max} = 0.25$.

We comment that the above example was solved essentially instantaneously on a standard laptop PC using one of the most basic quadrature rules and a single initial condition in the optimisation routine. Indeed, the numerical implementation of the proposed method could undoubtedly be refined and improved via NLP solvers, such as IPOPT [45]. The purpose of the present example is to illustrate the underlying method.



(c)

Fig. V.1: Numerical maximization of (II.10). See main text.

VI. Summary

Recent semi-global incremental stability results from [13] for a class of forced Lur'e systems have been reported and, novelly, deployed in the context of constrained extracted-energy maximising control for heaving point-absorber wave-energy converters. The trail of ideas is that incremental stability results provide a theoretical underpinning of what may be termed a "frequency response" for Lur'e systems. We contend that this is highly-relevant for wave-energy conversion given that the underlying problem is essentially one of safely amplifying periodic or almost periodic motion of the device, subject to the (almost) periodic motion of the sea. Another motivation for the study is to help overcome the linear systems barrier associated with frequency-domain methods.

The proposed framework is model-based, and device agnostic so may incorporate other, more realistic, WEC models, which is a strength. It relies on the underlying model admitting the form of a forced Lur'e system. The framework permits the inclusion of models of downstream components (such as PTOs, energy storage, grid connection). Here we have assumed that only the viscous damping force is nonlinear, which could be extended to incorporate other terms. However, nonlinear energy-dissipating terms shall appear in the resulting optimisation problem obtained via the energy-balance method, and hence will need estimating. Similarly, the challenge of estimating the excitation force remains, as both ours and known results show that this plays a key role in determining optimal controls. Furthermore, the results of Section V produce an open-loop control in terms of determination of unknown Fourier coefficients. Practically, it may be desirable to close the loop via feedback controllers to improve robustness, and this shall comprise future study, as shall comparisons with existing wave-energy conversion techniques.

Finally, we comment that the key intellectual step currently taken to apply the frequency response ideas of Section IV in Section V is the conjecture (C). If (C) is false, then the approach of Section V basically maximises (II.5) subject to (II.1) over the subset of (almost) periodic controls. However, the expressions (II.8) and (II.12) for the optimal velocity profile in those respective cases, when substituted back into (II.1a), give (almost) periodic $u = f_{\rm PTO}$ when $f_{\rm ex}$ is. Consequently, we have hope that such a conjecture may be true, at least quite generally. An answer to validity of (C) may be available from the theory of periodic optimal control; see, for instance, [46].

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