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Discrete Restriction for (x, x^3) and Related Topics

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Defining the truncated extension operator E for a sequence $a(n)$ with $n \in \mathbb{Z}$ by putting

$$Ea(\alpha, \beta) := \sum_{|n| \leq N} a(n) e(\alpha n^3 + \beta n),$$

we obtain the conjectured tenth moment estimate

$$\|Ea\|_{L^{10}(\mathbb{T}^2)} \lesssim_\epsilon N^{\frac{1}{10}+\epsilon} \|a\|_{\ell^2(\mathbb{Z})}.$$

We obtain related conclusions when the curve (x, x^3) is replaced by $(\phi_1(x), \phi_2(x))$ for suitably independent polynomials $\phi_1(x), \phi_2(x)$ having integer coefficients.

1 Introduction

We begin by recalling the discrete restriction conjecture for the curve (x, x^3). Define the truncated extension operator E for a sequence $a(n)$ with $n \in \mathbb{Z}$ by putting

$$Ea(\alpha, \beta) := \sum_{|n| \leq N} a(n) e(\alpha n^3 + \beta n)$$

†Dedicated to the memory of Jean Bourgain.

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for $\alpha, \beta \in \mathbb{R}$. Here and elsewhere, we write $e(t)$ in place of $e^{2\pi it}$. Since $e(\cdot)$ is $\mathbb{Z}$-periodic, we may regard α and β as elements of $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ or of any interval I in $\mathbb{R}$ of length 1 without any confusion. Based on the usual heuristics in the circle method, it is natural to make the following conjecture.

Conjecture 1.1. For each $p \in [1, \infty]$ there exists a constant $C_p > 0$ such that, for all $N \in \mathbb{N}$ and all sequences $a \in \ell^2(\mathbb{Z})$, one has the discrete restriction bounds

$$\|Ea\|_{L^p(\mathbb{T}^2)} \leq C_p \big(1 + N^{\frac{1}{2} - \frac{4}{p}}\big) \|a\|_{\ell^2(\mathbb{Z})}. \tag{1.1}$$

Bourgain initiated the study of this restriction estimate in [3], wherein he proved the bound $\|Ea\|_{L^6(\mathbb{T}^2)} \lesssim_\epsilon N^\epsilon \|a\|_{\ell^2(\mathbb{Z})}$ (see [3, equation (8.37) on page 227]). To facilitate further discussion we introduce a cruder version of the conjecture (1.1), to the effect that for each $\epsilon > 0$, there exists a constant $C_{p,\epsilon}$ having the property that, for all $N \in \mathbb{N}$ and all sequences $a \in \ell^2(\mathbb{Z})$, one has

$$\|Ea\|_{L^p(\mathbb{T}^2)} \leq C_{p,\epsilon} N^\epsilon \left(1 + N^{\frac{1}{2} - \frac{4}{p}}\right) \|a\|_{\ell^2(\mathbb{Z})}. \tag{1.2}$$

In colloquial terms the estimate (1.2) is the estimate (1.1) with an "ϵ-loss." Bourgain's work establishes this weaker conjecture for $1 \leq p \leq 6$. The problem of proving Conjecture 1.1 lay dormant for some time until Hu and Li [10] established (1.2) for $p = 14$. We remark that Hu and Li conjectured (1.1) for $2 \leq p \leq 8$ and (1.2) for all $8 \leq p \leq \infty$. Our conjecture here is a more optimistic version of [10, equation (1.2)] motivated by the observation that the underlying singular series does not diverge as it does in the quadratic case. Recently, Lai and Ding [12] proved (1.2) for $p = 12$ using the recent resolution of the main conjecture in the discrete restriction analogue of the cubic case of Vinogradov's mean value theorem. The latter was noted first in [19] as a consequence of the methods of [18], and was subsequently obtained by decoupling technology in [5] and via efficient congruencing in [20].

It is important to note that we are discussing discrete restriction estimates. Though the discrete restriction estimates (1.1) or (1.2) are referred to as a "discrete ℓ^2 decoupling inequality for L^p" in [8], one should not mistake these discrete restriction estimates for a decoupling or efficient congruencing inequality. Decoupling estimates and efficient congruencing estimates are stronger than discrete restriction estimates and therefore more difficult to obtain. Indeed, the analogous, putative decoupling estimate for the curve (x, x^3) fails in the range of exponents $6 < p < 12$ (see [7], Proposition 12.22 on page 283). Despite this, we improve the range of exponents p in which the conjectured estimate (1.2) is known to $p \geq 10$ in Section 2.

Theorem 1.2. The estimate (1.2) is true for $p = 10$, and (1.1) is true for all $p > 10$.

Our method of proof is motivated by corresponding techniques applied in the analogous number theoretic problem where $a(n)$ is identically 1. In this situation (where $a(n) = 1$ for all $n \in \mathbb{Z}$), the sixth moment estimate satisfies

$$\|a\|_{\ell^2(\mathbb{Z})} \lesssim \|Ea\|_{L^6(\mathbb{T}^2)} \lesssim \|a\|_{\ell^2(\mathbb{Z})}$$

and the ninth moment estimate satisfies

$$\|Ea\|_{L^9(\mathbb{T}^2)} \lesssim_\epsilon N^{\frac{1}{18}+\epsilon} \|a\|_{\ell^2(\mathbb{Z})}$$

for all $\epsilon > 0$; see [6, 15] and [17] respectively. In this case where $a(n) = 1$ for all $n \in \mathbb{Z}$, earlier work of Hua [11, Lemma 5.2 and Theorem 8] delivers the bounds

$$\|Ea\|_{L^6(\mathbb{T}^2)} \lesssim N^\epsilon \|a\|_{\ell^2(\mathbb{Z})} \quad \text{and} \quad \|Ea\|_{L^{10}(\mathbb{T}^2)} \lesssim_\epsilon N^{\frac{1}{10}+\epsilon} \|a\|_{\ell^2(\mathbb{Z})}.$$

The first of these bounds matches in strength that obtained by Bourgain, though in the special case $a(n) = 1$. Here the methods applied by Hua (in work whose origins lie at least as far back as 1947) would already have sufficed to establish the estimate of Bourgain. The interested reader will find the necessary ideas in the discussion following (2.6) below. The second of these bounds, meanwhile, matches in strength that obtained in Theorem 1.2, though again only in the special case $a(n) = 1$.

Hua's idea is to interpret the 10-th moment of the exponential sum $Ea(\alpha, \beta)$ in terms of an underlying pair of Diophantine equations. A second order Weyl-differencing argument replaces four generating functions by an expression amenable to divisor sum estimates, and thereby one is left to count solutions of a pair of simultaneous equations of the shape

$$y_1 y_2 w = x_1^3 + x_2^3 + x_3^3 - x_4^3 - x_5^3 - x_6^3$$
$$0 = x_1 + x_2 + x_3 - x_4 - x_5 - x_6.$$

What has obstructed the use of Hua's ideas in previous work on Conjecture 1.1 in the case $p = 10$ is the observation that, *a priori*, the difference variables y_1 and y_2 lose any information concerning the density of the set in which the variables x_i are constrained to lie. Thus, for all we are able to say, the variables y_1 and y_2 might well inhabit a complete interval of integers, and in the situation wherein $w = 0$ we are forced into an unacceptable loss in the ensuing estimates. We surmount this barrier in the proof of Theorem 2.1 by avoiding the second order Weyl-differencing operation in favour of a purely Diophantine operation. Little information is lost concerning the density of the sets underlying the variables that substitute for the difference variables, and thereby the situation corresponding to the difficult case $w = 0$ becomes a diagonal problem that is simple to control. In Sections 3 and 4 we extend our methods to give new restriction estimates for related extension operators. Many of these estimates are not expected to be sharp.

In this paper we write $f(n) \lesssim g(n)$ to mean that there exists a constant $C > 0$ with the property that $|f(n)| \le Cg(n)$ for all n. This is equivalent to Vinogradov's notation $\ll$. Also, when $k \ge 2$, we write $\tau_k(n)$ for the k-fold divisor function defined via the relation

$$\tau_k(n) = \sum_{\substack{d_1,\ldots,d_k \in \mathbb{N} \\ d_1 \ldots d_k = n}} 1.$$

2 The proof of Theorem 1.2

It transpires that the full restriction estimate reported in Theorem 1.2 is a consequence of the special case in which the sequence $a(n)$ is the characteristic function $1_{\mathcal{A}}$ of a subset $\mathcal{A}$ of the truncated integers $\mathbb{Z} \cap [-N, N]$. We write A for the cardinality of the set $\mathcal{A}$. Furthermore, in this context our extension operator is

$$E1_{\mathcal{A}}(\alpha, \beta) := \sum_{n \in \mathcal{A}} e\left(\alpha n^3 + \beta n\right)$$

for $\alpha, \beta \in \mathbb{T}$. Our goal is the upper bound contained in the following theorem.

Theorem 2.1. There is a positive constant κ such that, for each subset $\mathcal{A} \subset \mathbb{Z} \cap [-N, N]$ of cardinality A, one has

$$\int_{\mathbb{T}^2} \left|E1_{\mathcal{A}}(\alpha, \beta)\right|^{10} \mathrm{d}\alpha \, \mathrm{d}\beta \lesssim N \exp\left(\kappa \frac{\log N}{\log \log N}\right) \cdot A^5.$$

Proof. Fix the interval $[-N,N]$ and subset $\mathcal{A}\subset\mathbb{Z}\cap[-N,N]$, and let $a=1_{\mathcal{A}}$. The tenth moment $\|Ea\|_{10}^{10}$ counts the number of solutions to the system of equations

$$\sum_{i=1}^{3}(x_i^3-y_i^3)=\sum_{i=4}^{5}(x_i^3-y_i^3)$$
$$\sum_{i=1}^{3}(x_i-y_i)=\sum_{i=4}^{5}(x_i-y_i),$$

with each $\boldsymbol{x},\boldsymbol{y}\in\mathcal{A}^5$. We will foliate over the possible common values

$$x_1-y_1+x_2-y_2+x_3-y_3=h=x_4-y_4+x_5-y_5, \tag{2.1}$$

as h varies over $\mathbb{Z}$. Since the set $\mathcal{A}$ is contained in $[-N,N]$, we find that solutions are possible only when $h\in[-4N,4N]$. Fourier analytically, we may then write $\|Ea\|_{10}^{10}$ as

$$\sum_{|h|\le 4N}\int_{\mathbb{T}}\int_{\mathbb{T}}|Ea(\alpha_1,\alpha_2)|^4e(-\alpha_2h)\,\mathrm{d}\alpha_2\int_{\mathbb{T}}|Ea(\alpha_1,\alpha_3)|^6e(-\alpha_3h)\,\mathrm{d}\alpha_3\,\mathrm{d}\alpha_1.$$

Taking absolute values and applying the triangle inequality we deduce that

$$\|Ea\|_{10}^{10}\le(8N+1)\int_{\mathbb{T}}\int_{\mathbb{T}}\int_{\mathbb{T}}|Ea(\alpha_1,\alpha_2)|^4\,|Ea(\alpha_1,\alpha_3)|^6\,\mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\,\mathrm{d}\alpha_3. \tag{2.2}$$

Note here that we have thrown away potential oscillation in order to impose the restriction that $h=0$ in (2.1).

We next foliate over common values in the cubic equation. When $t\in\mathbb{N}$ and $l\in\mathbb{Z}$, write $c_t(l)$ for the number of solutions of the simultaneous equations

$$\sum_{i=1}^{t}(x_i^3-y_i^3)=l\quad\text{and}\quad\sum_{i=1}^{t}(x_i-y_i)=0,$$

with $\boldsymbol{x},\boldsymbol{y}\in\mathcal{A}^t$. Then, in a manner similar to that underlying our earlier discussion regarding the linear equation, it follows via orthogonality that

$$\|Ea\|_{10}^{10}\le(8N+1)\sum_{|l|\le 4N^3}c_2(l)c_3(l). \tag{2.3}$$

Our argument now divides into two parts according to whether the summand l is zero or non-zero.

In order to treat the contribution in (2.3) from the summand with $l = 0$, we begin by observing that $c_2(0)$ counts the number of solutions of the simultaneous equations

$$x_1^3 + x_2^3 = y_1^3 + y_2^3 \quad \text{and} \quad x_1 + x_2 = y_1 + y_2,$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^2$. The contribution arising from those solutions with $x_1+x_2 = 0 = y_1+y_2$ is plainly at most A^2. When $x_1 + x_2 \neq 0$, meanwhile, one may divide the respective left and right hand sides of these equations to deduce that $x_1^2 - x_1x_2 + x_2^2 = y_1^2 - y_1y_2 + y_2^2$, whence $x_1x_2 = y_1y_2$. Thus $\{x_1, x_2\} = \{y_1, y_2\}$, and there are at most $2A^2$ solutions of this type. We thus have $c_2(0) \leq 3A^2$. Moreover, it is a consequence of the discussion surrounding [3, equation (8.37)] that for a suitable positive number κ, one has

$$c_3(0) \lesssim \exp\left(\kappa \log N / \log\log N\right) \cdot A^3. \tag{2.4}$$

Since the argument of the latter source is more complicated than would be available via earlier methods (see [11, Lemma 5.2 of Chapter V]), and further fails to address the case $b = a^3$ of [3, equation (8.44)], we presently make a detour to justify the estimate (2.4). For now, it suffices to combine our estimates for $c_2(0)$ and $c_3(0)$ to obtain the bound

$$c_2(0)c_3(0) \lesssim \exp(\kappa \log N / \log\log N) \cdot A^5. \tag{2.5}$$

We now give an alternate argument to give the claimed bound on $c_3(0)$. Observe that $c_3(0)$ counts the number of solutions of the simultaneous equations

$$x_1^3 + x_2^3 + x_3^3 = y_1^3 + y_2^3 + y_3^3 \quad \text{and} \quad x_1 + x_2 + x_3 = y_1 + y_2 + y_3, \tag{2.6}$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^3$. Since

$$(x_1 + x_2 + x_3)^3 - (x_1^3 + x_2^3 + x_3^3) = 3(x_1 + x_2)(x_2 + x_3)(x_3 + x_1),$$

we see that

$$(x_1 + x_2)(x_2 + x_3)(x_3 + x_1) = (y_1 + y_2)(y_2 + y_3)(y_3 + y_1). \tag{2.7}$$

Thus, in particular, if $x_i + x_j = 0$ for some distinct indices i and j in $\{1, 2, 3\}$, then $y_{i'} + y_{j'} = 0$ for some distinct indices i' and j' in $\{1, 2, 3\}$, and one has also $x_k = y_{k'}$ for some indices k and k' in $\{1, 2, 3\}$. In this way we see that there are $O(A^3)$ choices for

$\boldsymbol{x}$ and $\boldsymbol{y}$ satisfying (2.6) for which the left hand side of (2.7) is 0. Given any fixed one of the $O(A^3)$ choices for $\boldsymbol{x} \in \mathcal{A}^3$ in which the left hand side of (2.7) is equal to a nonzero integer L, meanwhile, each factor on the right hand side of (2.7) is equal to a divisor of L. It consequently follows that there are at most $8\max_{1\le n\le 8N^3}\tau_3(n)$ choices for (positive or negative) integers d_1, d_2, d_3 with $d_1d_2d_3 = L$ having the property that

$$y_1 + y_2 = d_1, \quad y_2 + y_3 = d_2, \quad y_3 + y_1 = d_3.$$

Writing M for the fixed integer $x_1 + x_2 + x_3$, we see that for a fixed choice of $\boldsymbol{d}$, one has

$$y_1 = M - d_2, \quad y_2 = M - d_3, \quad y_3 = M - d_1,$$

so that $\boldsymbol{y}$ is also fixed. Making use of standard estimates for $\tau_3(n)$, we may thus conclude that there is a positive number κ for which

$$c_3(0) \lesssim A^3 + A^3 \max_{1\le n\le 8N^3} \tau_3(n) \lesssim \exp(\kappa \log N/\log\log N)\cdot A^3,$$

justifying our earlier assertion.

We next turn to consider the contribution in (2.3) of the non-zero summands l. When l is a fixed integer with $1 \le |l| \le 4N^3$, we see that $c_2(l)$ is equal to the number of solutions of the simultaneous equations

$$x_1^3 + x_2^3 - y_1^3 - y_2^3 = l \quad \text{and} \quad y_2 = x_1 + x_2 - y_1.$$

Substituting from the latter of these equations into the former, we obtain the equation

$$(x_1 + x_2 - y_1)^3 - (x_1^3 + x_2^3 - y_1^3) = -l,$$

whence

$$(x_1 + x_2)(x_1 - y_1)(x_2 - y_1) = -l/3.$$

We therefore deduce that $3|l$ and, as in the previous paragraph, there are at most $8\tau_3(|l/3|)$ possible choices for integers e_1, e_2, e_3 with $e_1e_2e_3 = -l/3$ and

$$x_1 + x_2 = e_1, \quad x_1 - y_1 = e_2, \quad x_2 - y_1 = e_3.$$

For any fixed such choice of $\boldsymbol{e}$, one sees that

$$e_1 - e_2 - e_3 = 2y_1, \quad e_2 - e_3 - e_1 = -2x_2, \quad e_3 - e_1 - e_2 = -2x_1,$$

so that x_1, x_2, y_1 are fixed. Since $y_2 = x_1 + x_2 - y_1$, it follows that y_2 is also fixed. Thus we have

$$\max_{1\leq |l| \leq 4N^3} c_2(l) \lesssim \max_{1\leq |n| \leq 2N^3} \tau_3(n) \lesssim \exp\left(\kappa \log N / \log\log N\right).$$

Making use of our estimate for $c_2(l)$, we find that

$$\sum_{1\leq |l| \leq 4N^3} c_2(l) c_3(l) \lesssim \exp\left(\kappa \log N / \log\log N\right) \sum_{|l| \leq 6N^3} c_3(l).$$

The last sum counts the number of solutions of the equation

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3,$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^3$, which is plainly $O(A^5)$. Thus we infer that

$$\sum_{1\leq |l| \leq 4N^3} c_2(l) c_3(l) \lesssim \exp\left(\kappa \log N / \log\log N\right) \cdot A^5.$$

The conclusion of the theorem follows by substituting this estimate and (2.5) into the relation (2.3). ■

Proof of Theorem 1.2. We now deduce Theorem 1.2 from Theorem 2.1. The argument to do so is a standard "vertical layer cake decomposition" argument in the theory of Lorentz spaces. Although an elementary dyadic decomposition argument suffices for our purposes, for the sake of concision it is expedient to make reference to [8, Lemma 3.1]. Thus, we recall the special case $p = 2$ of the latter for the reader's convenience.

Lemma 2.2. Let $T : \mathbb{C}^N \to [0, \infty)$ be a sublinear function such that $T(\mathbf{1}_{\mathcal{A}}) \leq C \|\mathbf{1}_{\mathcal{A}}\|_{\ell^2(\mathbb{Z})}$ for all subsets $\mathcal{A} \subset \{1, \ldots, N\}$. Then for all $a \in \mathbb{C}^N$, one has

$$T(a) \leq 2^{1/2} C (2 + (\log N)^{1/2}) \|a\|_{\ell^2(\mathbb{Z})}.$$

We apply this lemma by taking $T(\cdot)$ to be $\|E(\cdot)\|_{10}$ with

$$C = N^{1/10} \exp\left(\kappa \log N / \log\log N\right).$$

Theorem 2.1 now implies that

$$\|Ea\|_{10} \lesssim N^{1/10} \exp\left(\kappa \frac{\log N}{\log\log N}\right)(1 + (\log N)^{1/2})\|a\|_{\ell^2(\mathbb{Z})}.$$

The conclusion of Theorem 1.2 follows on noting that for all $\epsilon > 0$, there exists a constant C_ϵ such that for all sufficiently large N, we have

$$\exp\left(\kappa \frac{\log N}{\log\log N}\right)\left(1 + (\log N)^{1/2}\right) \le C_\epsilon N^\epsilon.$$

Our final task in the proof of Theorem 1.2 is to prove (1.1) for $p > 10$. For this we use the "ϵ-removal lemmas" [9, Theorem 1.4 and Lemma 3.1], which were adapted from [2]. To be precise, one fixes $d = 1$ and $k = 3$, and in the statement of [9, Lemma 3.1], one takes $C = 0$, $p = 10$, $q > 10$ and $\zeta = 1/16$. ■

3 Generalizations

We consider now the extension operator associated with two polynomials ϕ_1 and ϕ_2 with integral coefficients defined by

$$Ea(\alpha_1, \alpha_2) := \sum_{|n| \le N} a(n) e(\alpha_1 \phi_1(n) + \alpha_2 \phi_2(n))$$

for $\alpha_1, \alpha_2 \in \mathbb{R}$. Since $e(\cdot)$ is $\mathbb{Z}$-periodic and the polynomials ϕ_1, ϕ_2 have integral coefficients, we may regard α_1 and α_2 as elements of $\mathbb{T}$ without any confusion. By making use of recent progress on decoupling and efficient congruencing, one may obtain the estimates contained in the following theorem.

Theorem 3.1. Let ϕ_1, ϕ_2 be polynomials with integer coefficients and respective degrees k_1 and k_2 with $1 \le k_1 \le k_2$. If ϕ_1' and ϕ_2' are linearly independent over $\mathbb{Q}$, then we have

$$\|Ea\|_{L^6(\mathbb{T}^2)} \lesssim_\epsilon N^\epsilon \|a\|_{\ell^2(\mathbb{Z})} \tag{3.1}$$

and

$$\|Ea\|_{L^{k_2(k_2+1)}(\mathbb{T}^2)} \lesssim_\epsilon N^{\frac{1}{2}-\frac{k_1+k_2}{k_2(k_2+1)}+\epsilon}\|a\|_{\ell^2(\mathbb{Z})} \tag{3.2}$$

for each $\epsilon > 0$ as $N \to \infty$.

Regarding estimate (3.1), see [4, Corollary 1.3] or the case $k = 2$ and $s = 3$ of [20, Theorem 1.1]. Meanwhile, when ϕ_1 and ϕ_2 are two distinct monomials, the estimate (3.2) is a special case of [12, Theorem 1.1]. The reader will have no difficulty in verifying that one may adapt the arguments of [12] in a straightforward manner to handle the situation in which ϕ_1' and ϕ_2' are linearly independent over $\mathbb{Q}$.

A further consequence of the efficient congruencing/decoupling machinery is the following theorem.

Theorem 3.2. Let ϕ_1, ϕ_2 be polynomials with integer coefficients and respective degrees k_1 and k_2 with $\min\{k_1, k_2\} > 1$. If ϕ_1'' and ϕ_2'' are linearly independent over $\mathbb{Q}$, then we have the estimate

$$\|Ea\|_{L^{12}(\mathbb{T}^2)} \lesssim_\epsilon N^{\frac{1}{12}+\epsilon}\|a\|_{\ell^2(\mathbb{Z})} \tag{3.3}$$

for each $\epsilon > 0$ as $N \to \infty$.

To derive this conclusion, one considers the auxiliary extension operator

$$Fa(\alpha_1, \alpha_2, \alpha_3) := \sum_{|n|\le N} a(n)e(\alpha_1\phi_1(n) + \alpha_2\phi_2(n) + \alpha_3 n),$$

for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. It is a consequence of the triangle inequality that

$$\|Ea\|_{L^{12}(\mathbb{T}^2)} \le (24N+1)^{\frac{1}{12}}\|Fa\|_{L^{12}(\mathbb{T}^3)}.$$

Thus, the conclusion of Theorem 3.2 follows from the estimate

$$\|Fa\|_{L^{12}(\mathbb{T}^3)} \lesssim_\epsilon N^\epsilon\|a\|_{\ell^2(\mathbb{Z})}.$$

This bound is immediate from the case $k = 3$ and $s = 6$ of [20, Theorem 1.1], on checking that the Wronskian of first derivatives of the polynomials $\phi_1(t)$, $\phi_2(t)$ and t is nonzero.

We expect the following sharp bound to hold in general.

Conjecture 3.3. Let ϕ_1, ϕ_2 be polynomials with integer coefficients having respective degrees k_1 and k_2 satisfying $\max\{k_1, k_2\} \geq 3$. If ϕ_1' and ϕ_2' are linearly independent over $\mathbb{Q}$, then for each $p \in [1, \infty]$, we have

$$\|Ea\|_{L^p(\mathbb{T}^2)} \lesssim \left(1 + N^{\frac{1}{2} - \frac{k_1+k_2}{p}}\right) \|a\|_{\ell^2(\mathbb{Z})}$$

as $N \to \infty$.

Note that the analogue of Conjecture 3.3 corresponding to the case $(k_1, k_2) = (1, 2)$ cannot hold in the sharp form stated here, for an additional factor at least as large as $(\log N)^{1/6}$ is required when $p = 6$ (see the discussion around [2, equation (2.51)], wherein Bourgain obtained nearly optimal bounds for all $p \in [1, \infty]$). We will prove the following new bound towards this conjecture.

Theorem 3.4. Let ϕ_1, ϕ_2 be polynomials with integer coefficients and respective degrees k_1 and k_2 with $1 \leq k_1 < k_2$ and $k_2 \geq 3$. Then one has the estimate

$$\|Ea\|_{L^{10}(\mathbb{T}^2)} \lesssim_\epsilon N^{\frac{1}{10}+\epsilon} \|a\|_{\ell^2(\mathbb{Z})} \tag{3.4}$$

for each $\epsilon > 0$ as $N \to \infty$.

In situations in which ϕ_1 is not linear, it follows by interpolating between the 6-th moment estimate (3.1) and the 12th moment estimate (3.3) that one has the bound

$$\|Ea\|_{L^{10}(\mathbb{T}^2)} \lesssim_\epsilon N^{\frac{1}{15}+\epsilon} \|a\|_{\ell^2(\mathbb{Z})}$$

for all $\epsilon > 0$. Consequently, in the proof below we may assume that ϕ_1 is linear. Indeed it suffices to take $\phi_1(x) = x$.

We will need a simple variant of [16, Lemma 2] in the proof of Theorem 3.4. We include a proof for the sake of completeness.

Lemma 3.5. Let $\psi(x_1, \ldots, x_s)$ be a non-zero multivariate polynomial with integer coefficients of total degree k. If $\mathcal{A} \subset \mathbb{Z}$ is a finite set of cardinality A, then the number of integer solutions to the equation $\psi(\mathbf{x}) = 0$ with $x_i \in \mathcal{A}$ for $i = 1, \ldots, s$ is at most kA^{s-1}.

Proof. We proceed by induction on s. The desired conclusion plainly holds when $s = 1$. Suppose that the conclusion of the lemma holds for each s with $1 \leq s < t$, and let

$\Psi \in \mathbb{Z}[x_1, \ldots, x_t]$ be a nonzero polynomial of total degree k. By rearranging variables, if necessary, we may suppose that $\Psi(x_1, \ldots, x_t)$ is a polynomial in x_t with at least one non-zero coefficient. Let the degree of Ψ with respect to x_t be r, and suppose that the coefficient of x_t^r is the polynomial $\Phi(x_1, \ldots, x_{t-1})$. Then Φ is a nonzero polynomial in $t-1$ variables of degree at most $k-r$. By the inductive hypothesis, the number of solutions of the equation $\Phi(x_1, \ldots, x_{t-1}) = 0$ with $x_i \in \mathcal{A}$ $(1 \le i \le t-1)$ is at most $(k-r)A^{t-2}$. Then the number of solutions $(x_1, \ldots, x_t)$ of $\Psi(x_1, \ldots, x_t) = 0$ satisfying $\Phi(x_1, \ldots, x_{t-1}) = 0$ and with $x_i \in \mathcal{A}$ $(1 \le i \le t)$ is at most $(k-r)A^{t-1}$. Meanwhile, if $\Phi(x_1, \ldots, x_{t-1})$ is nonzero then x_t satisfies a nontrivial polynomial of degree r. So there are at most rA^{t-1} solutions with $\Phi(x_1, \ldots, x_{t-1})$ non-zero. We therefore conclude that there are at most kA^{t-1} solutions altogether, and the inductive hypothesis holds with $t+1$ replacing t. This completes the proof of the lemma. ■

Proof of Theorem 3.4. By the remark above we may assume that $\phi_1(x) = x$. As such, we write ϕ in place of ϕ_2 and k in place of k_2 in the proof. By Lemma 2.2 we only need to prove (3.4) for sequences a which are the characteristic function of some subset $\mathcal{A} \subset \mathbb{Z} \cap [-N, N]$. Therefore, we want to bound the number of solutions to the system of equations

$$\sum_{i=1}^{3}(\phi(x_i) - \phi(y_i)) = \sum_{i=4}^{5}(\phi(x_i) - \phi(y_i)),$$
$$\sum_{i=1}^{3}(x_i - y_i) = \sum_{i=4}^{5}(x_i - y_i),$$

with $\mathbf{x}, \mathbf{y} \in \mathcal{A}^5$. As in the argument employed above to deliver the relation (2.2), we find that at the expense of a factor of $8N+1$ we only need to bound the number of solutions to the system of equations

$$\begin{aligned} \sum_{i=1}^{3}(\phi(x_i) - \phi(y_i)) &= \sum_{i=4}^{5}(\phi(x_i) - \phi(y_i)), \\ \sum_{i=1}^{3}(x_i - y_i) = 0 &= \sum_{i=4}^{5}(x_i - y_i), \end{aligned} \tag{3.5}$$

with $\mathbf{x}, \mathbf{y} \in \mathcal{A}^5$.

When $t \in \mathbb{N}$ and $l \in \mathbb{Z}$, we now write $c_t(l)$ for the number of solutions of the simultaneous equations

$$\sum_{i=1}^{t}(\phi(x_i) - \phi(y_i)) = l \quad \text{and} \quad \sum_{i=1}^{t}(x_i - y_i) = 0,$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^t$. Then, by foliating over common values in the equation (3.5) involving ϕ, just as in our proof of Theorem 2.1, we find that a bound analogous to (2.3) holds in our present situation. That is, it follows via orthogonality that there exists a positive constant C depending on the coefficients of ϕ such that

$$\|Ea\|_{10}^{10} \leq (8N+1) \sum_{|l| \leq CN^k} c_2(l)c_3(l). \tag{3.6}$$

Our argument again divides into two parts according to whether the summand l is zero or non-zero.

In the present circumstances, one sees that $c_2(0)$ counts the number of solutions of the simultaneous equations

$$\phi(x_1) - \phi(y_1) + \phi(x_2) = \phi(y_2) \quad \text{and} \quad x_1 - y_1 + x_2 = y_2.$$

Upon substitution of the latter equation into the former, one finds that

$$\phi(x_1) - \phi(y_1) + \phi(x_2) - \phi(x_1 - y_1 + x_2) = 0.$$

The polynomial on the left hand side has factors $x_1 - y_1$ and $y_1 - x_2$, whence there is a quotient polynomial $\psi(x_1, y_1, x_2)$ having integer coefficients with the property that

$$(x_1 - y_1)(x_2 - y_1)\psi(x_1, y_1, x_2) = 0.$$

The solutions with $x_1 = y_1$ or $x_2 = y_1$ contribute at most $2A^2$ solutions to the count $c_2(0)$. If, on the other hand, neither $x_1 = y_1$ nor $y_1 = x_2$, then $\psi(x_1, y_1, x_2) = 0$. By Lemma 3.5, the number of solutions of $\psi(x_1, y_1, x_2) = 0$ with $x_1, y_1, x_2 \in \mathcal{A}$ is $O(A^2)$. Since y_2 is fixed by a choice for x_1, y_1, x_2, one infers that

$$c_2(0) = O(A^2). \tag{3.7}$$

The estimate $c_3(0) \lesssim N^{\epsilon} A^3$ is immediate from (3.1), and thus we conclude that

$$c_2(0)c_3(0) \lesssim N^{\epsilon} A^5. \tag{3.8}$$

We turn next to the contribution in (3.6) from the nonzero summands l. We begin by observing that $c_2(l)$ counts the number of solutions of the simultaneous equations

$$\phi(x_1) - \phi(y_1) + \phi(x_2) - \phi(y_2) = l \quad \text{and} \quad x_1 - y_1 + x_2 = y_2,$$

with $x_1, y_1, x_2, y_2 \in \mathcal{A}$. As above, these equations imply that

$$(x_1 - y_1)(x_2 - y_1)\psi(x_1, y_1, x_2) = l.$$

There are at most $8\tau_3(|l|)$ possible choices for nonzero integers e_1, e_2, e_3 with $e_1e_2e_3 = l$,

$$x_1 - y_1 = e_1, \quad x_2 - y_1 = e_2 \quad \text{and} \quad \psi(x_1, y_1, x_2) = e_3. \tag{3.9}$$

For any fixed such choice of $\boldsymbol{e}$, one has $\psi(y_1 + e_1, y_1, y_1 + e_2) = e_3$. One has

$$-e_1e_2\psi(y_1 + e_1, y_1, y_1 + e_2) = \phi(y_1 + e_1 + e_2) - \phi(y_1 + e_2) - \phi(y_1 + e_1) + \phi(y_1).$$

The right hand side here is the second order difference polynomial associated with ϕ, which is nonconstant as a polynomial in y_1 because $\deg(\phi) = k \geq 3$. Thus the number of solutions for $y_1 \in \mathcal{A}$ to the equation $\psi(y_1 + e_1, y_1, y_1 + e_2) = e_3$ is $O(1)$. Any fixed choice of y_1 determines x_1 and x_2 via (3.9), and then $y_2 = x_1 - y_1 + x_2$ is also determined. In this way we deduce that

$$\max_{1 \leq |l| \leq CN^k} c_2(l) \lesssim \max_{1 \leq n \leq CN^k} \tau_3(n) \lesssim_{\epsilon} N^{\epsilon}. \tag{3.10}$$

Applying our newly obtained bound for $c_2(l)$ we find that

$$\sum_{1 \leq |l| \leq CN^k} c_2(l)c_3(l) \lesssim_{\epsilon} N^{\epsilon} \sum_{1 \leq |l| \leq CN^k} c_3(l).$$

The last sum is bounded above by the number of solutions of the equation

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^3$, which is $O(A^5)$. Thus,

$$\sum_{1 \le |l| \le CN^k} c_2(l) c_3(l) \lesssim_\epsilon N^\epsilon A^5,$$

and we infer from (3.8) and (3.6) that

$$\|Ea\|_{10}^{10} \lesssim_\epsilon N^{1+\epsilon} A^5.$$

The conclusion of the theorem now follows by invoking Lemma 2.2. ∎

4 Discrete restriction for univariate polynomials

For ϕ, a polynomial with integer coefficients of degree at least 3, we (re-)define our extension operator as

$$Ea(\alpha) := \sum_{|n| \le N} a(n) e(\alpha \phi(n)),$$

and we also make use of the auxiliary extension operator

$$Fa(\alpha, \beta) := \sum_{|n| \le N} a(n) e(\alpha \phi(n) + \beta n).$$

These operators for a quadratic polynomial ϕ were studied by Bourgain in [1]. The main goal of this section is the proof of the following theorem.

Theorem 4.1. Suppose that ϕ is a polynomial with integer coefficients of degree $k \ge 3$. For all $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\|Ea\|_{L^4(\mathbb{T})}^4 \le C_\epsilon N^\epsilon \|a\|_{\ell^2(\mathbb{Z})}^4 \tag{4.1}$$

and

$$\|Ea\|_{L^8(\mathbb{T})}^8 \le C_\epsilon N^{1+\epsilon} \|a\|_{\ell^2(\mathbb{Z})}^8. \tag{4.2}$$

When $k = 3$ and $p > 8$, we have the sharp bound

$$\|Ea\|_{L^p(\mathbb{T})} \lesssim_p N^{\frac{1}{2} - \frac{3}{p}} \|a\|_{\ell^2(\mathbb{Z})}. \tag{4.3}$$

When $\phi(n)$ has degree 3, the bound (4.2) is essentially sharp, up to the factor of N^{ϵ}. Furthermore, when $a(n)$ is identically 1, it follows from [14, Theorem 2] that there exists a positive constant C such that

$$\|Ea\|_{L^8(\mathbb{T})}^8 \leq CN\|a\|_{\ell^2(\mathbb{Z})}^8.$$

Estimate (4.2) is not sharp in general. When $k \geq 27$, standard arguments lead from [13, Theorem 1.1] to the conclusion that in the special case $\phi(n) = n^k$, there exists a positive constant δ depending on k such that

$$\|Ea\|_{L^8(\mathbb{T})}^8 \lesssim_\epsilon N^{1-\delta+\epsilon}\|a\|_{\ell^2(\mathbb{Z})}^8$$

for all $\epsilon > 0$. Indeed, one may take

$$1-\delta = \frac{16}{3\sqrt{3k}} + \max\left\{\frac{2}{\sqrt{k}}, \frac{1}{\sqrt{k}} + \frac{6}{k+3}\right\}.$$

Note that $1-\delta \to 0$ as $k \to \infty$. We expect the following sharp bound to hold in general.

Conjecture 4.2. Let ϕ be a polynomial with integer coefficients of degree $k \geq 3$. Then for each $p \in [1, \infty]$, we have

$$\|Ea\|_{L^p(\mathbb{T}^2)} \lesssim \left(1 + N^{\frac{1}{2}-\frac{k}{p}}\right)\|a\|_{\ell^2(\mathbb{Z})},$$

as $N \to \infty$.

Proof of Theorem 4.1. We begin with a proof of the fourth moment estimate (4.1). Applying Lemma 2.2, we reduce to proving (4.1) for sequences given by the characteristic function of some subset of the integers. As such, fix $[-N, N]$ and our subset $\mathcal{A} \subset \mathbb{Z} \cap [-N, N]$. Let $a = 1_{\mathcal{A}}$. The fourth moment counts the number of solutions to the equation

$$\phi(x_1) - \phi(y_1) = \phi(x_2) - \phi(y_2),$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^2$. There exists a polynomial $\psi(x, y)$ with integer coefficients such that

$$\phi(x) - \phi(y) = (x-y)\psi(x, y).$$

On writing $x = y + e$, one sees that

$$\phi(x) - \phi(y) = \phi(y+e) - \phi(y)$$

is the first order difference polynomial associated with ϕ. Since the degree of ϕ is at least 2, one has that $\psi(y+e, y)$ is not constant as a polynomial in y.

We distinguish between two cases. The first case is when $\phi(x_1) - \phi(y_1) = 0$. In this case we have two further cases to consider: either $x_1 = y_1$ or $\psi(x_1, y_1) = 0$. By Lemma 3.5, there are at most $O(A)$ putative solutions of $\psi(x_1, y_1) = 0$ with $x_1, y_1 \in \mathcal{A}$, and the same is self-evidently the case when $x_1 = y_1$. It follows that there are at most $O(A)$ solutions to the equation $\phi(x_1) - \phi(y_1) = 0$. By symmetry, there are also at most $O(A)$ solutions to the equation $\phi(x_2) - \phi(y_2) = 0$. Hence these solutions contribute at most $O(A^2)$ solutions to the fourth moment.

The second case is when $\phi(x_1) - \phi(y_1) \neq 0$. There are at most A^2 choices for x_1, y_1 in the set $\mathcal{A}$ with this property. Fixing any one such choice of x_1, y_1, we may assume that $\phi(x_1) - \phi(y_1) = l$ where $1 \leq |l| \leq CN^k$ for an appropriate constant C depending on the coefficients of ϕ. There are at most $4\tau_2(|l|)$ possible choices for non-zero integers e_1, e_2 with $e_1 e_2 = l$,

$$x_2 - y_2 = e_1 \quad \text{and} \quad \psi(x_2, y_2) = e_2.$$

For any fixed choice of e_1 and e_2, one has $\psi(y_2 + e_1, y_2) = e_2$. Since this polynomial equation is non-constant in y_2, there are at most $O(1)$ possible solutions for y_2. Consequently, there are at most $O(1)$ possible solutions for x_2. Thus, the contribution of the solutions of this second type to the fourth moment is

$$O\left(A^2 \max_{1 \leq |l| \leq CN^k} \tau_2(|l|)\right) \lesssim_\epsilon N^\epsilon A^2.$$

This completes the proof of the fourth moment estimate.

We proceed now to examine the 8th moment. By applying Lemma 2.2, it suffices to prove (4.2) for sequences given by characteristic functions of subsets of the integers. With this observation in mind, we again fix $[-N, N]$ and our subset $\mathcal{A} \subset \mathbb{Z} \cap [-N, N]$. Also, let $a = 1_{\mathcal{A}}$. The eighth moment $\|Ea\|_8^8$ counts the number of solutions to the equation

$$\sum_{i=1}^{2}(\phi(x_i) - \phi(y_i)) = \sum_{i=3}^{4}(\phi(x_i) - \phi(y_i)),$$

with each $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^4$. We foliate our set of solutions over the solutions to the equation $h = x_1 - y_2 + x_2 - y_2$ as h ranges in $[-4N, 4N]$. Writing this Fourier analytically, we thus deduce that

$$\int_{\mathbb{T}} |Ea(\alpha)|^8 \,\mathrm{d}\alpha = \sum_{|h|\le 4N} \int_{\mathbb{T}}\int_{\mathbb{T}} |Fa(\alpha,\beta)|^4 |Ea(\alpha)|^4 e(-\beta h)\,\mathrm{d}\beta\,\mathrm{d}\alpha.$$

Taking absolute values, we may impose the restriction that $h = 0$ and obtain the bound

$$\|Ea\|_8^8 \le (8N+1)\int_{\mathbb{T}}\int_{\mathbb{T}} |Fa(\alpha,\beta)|^4 |Ea(\alpha)|^4\,\mathrm{d}\beta\,\mathrm{d}\alpha.$$

The mean value on the right hand side here counts the number of solutions to the system of equations

$$\begin{aligned}&\sum_{i=1}^{2}(\phi(x_i) - \phi(y_i)) = \sum_{i=3}^{4}(\phi(x_i) - \phi(y_i))\\ &x_1 - y_1 + x_2 - y_2 = 0,\end{aligned} \tag{4.4}$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^4$.

Recall from Section 3 that when $l \in \mathbb{Z}$, we write $c_2(l)$ for the number of solutions of the simultaneous equations

$$\phi(x_1) + \phi(x_2) - \phi(y_1) - \phi(y_2) = l \quad \text{and} \quad x_1 - y_1 + x_2 = y_2,$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^2$. Also, when $l \in \mathbb{Z}$, write $c_2'(l)$ for the number of solutions of the equation

$$\phi(x_1) + \phi(x_2) - \phi(y_1) - \phi(y_2) = l,$$

with $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^2$. By foliating over common values in the equation involving ϕ in (4.4), we find that a bound analogous to (2.3) holds in our present situation. That is,

$$\|Ea\|_8^8 \le (8N+1)\sum_{|l|\le CN^k} c_2(l)c_2'(l).$$

We have the trivial bound $\sum_{l\in\mathbb{Z}} c_2'(l) \le A^4$ so that

$$\|Ea\|_8^8 \le (8N+1)\Big(c_2(0)c_2'(0) + A^4 \max_{1\le|l|\le CN^k} c_2(l)\Big).$$

Observe that, in view of the fourth moment estimate already derived, one has

$$c_2'(0) = \|Ea\|_4^4 \lesssim_\epsilon N^\epsilon A^2.$$

Thus, on recalling also (3.7) and (3.10), we deduce that

$$\|Ea\|_8^8 \lesssim_\epsilon N\big(N^\epsilon A^2 \cdot A^2 + A^4 N^\epsilon\big) \lesssim_\epsilon N^{1+\epsilon} A^4.$$

From here, as we have already explained, the proof of the eighth moment estimate follows by appealing to Lemma 2.2.

Finally, by applying [9, Theorem 4.1 and Lemma 3.1], the estimate (4.3) follows from (4.2) when $\phi(n) = n^3$. The keen reader may verify that one may adapt the arguments of [9, Section 4] to deduce (4.3) for an arbitrary cubic polynomial having integer coefficients. To be precise, in the statement of [9, Lemma 3.1], one takes $C = 0$, $p = 8$, $q > 8$ and $\zeta = 2^{-3}$, and in the statement of [9, Theorem 4.1], one takes $\tau = 1/4$. ■

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