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Exponential input-to-state stability for Lur'e systems via Integral Quadratic Constraints and Zames–Falb multipliers

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Absolute stability criteria that are sufficient for global exponential stability are shown, under a Lipschitz assumption, to be sufficient for the *a priori* stronger exponential input-to-state stability property. Important corollaries of this result are as follows: (i) absolute stability results obtained using Zames–Falb multipliers for systems containing slope-restricted nonlinearities provide exponential input-to-state-stability under a mild detectability assumption; and (ii) more generally, many absolute stability results obtained via Integral Quadratic Constraint methods provide, with the additional Lipschitz assumption, this stronger property.

Keywords: absolute stability; exponential input-to-state stability; Integral Quadratic Constraint; Lur'e system; Zames–Falb multiplier.

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1. Introduction

Consider the forced Lur'e (also Lurie or Lurye) system of nonlinear differential equations:

$$\dot{x} = Ax + Bf(Cx + v_2) + v_1 \,, \tag{1.1}$$

where, as usual, x is the state variable and $v = (v_1, v_2)$ is an external forcing (or control or disturbance) term. Here A, B and C are compatibly-sized matrices and f is a (in general nonlinear) function. Lur'e systems are the feedback connection of the linear control system

$$\dot{x} = Ax + Bu + v_1, \quad y = Cx + v_2 \,, \tag{1.2}$$

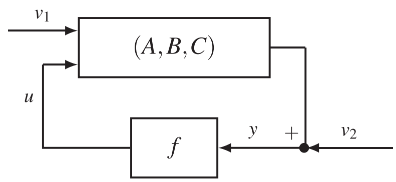

FIG. 1. Block diagram of forced Lur'e system (1.1).

with control input, state and output u, x and y, respectively, and the static nonlinearity $u = f(y)$. They comprise a common and important class of nonlinear control systems, and are consequently well-studied objects. A diagram of the feedback connection is shown in Figure 1.

In this paper, we establish the following result for Lur'e systems.

Main result: Restricting to the subclass of Lur'e systems (1.1) where f is globally Lipschitz, whenever an absolute stability criterion guarantees global exponential stability of the unforced equation, or a linear $\mathscr{L}_2$-input-output stability estimate (combined with the detectability property for the underlying linear system), then the same absolute stability criterion in fact guarantees the stronger stability property of *exponential input-to-state stability* (exponential ISS).

Recall that the Lur'e system (1.1) is called exponentially ISS if there exist positive constants L_i, λ such that every trajectory (v, x) of (1.1) satisfies

$$\|x(t+\tau; v)\| \le L_1 e^{-\lambda t}\|x(\tau)\| + L_2\|v\|_{\mathscr{L}_\infty(\tau, t+\tau)} \quad \forall\, t, \tau \ge 0\,. \tag{1.3}$$

The above upper bound (also called an estimate) is of course valid for the special case of stable linear control systems, that is, the first equation in (1.2) with $v = Bu + v_1$ and asymptotically stable matrix A, meaning all the eigenvalues of A have negative real part. The exponential ISS property extends this notion to include forced nonlinear control systems.

An absolute stability criterion is a sufficient condition for stability, meaning a number of possible notions, in terms of a condition on the linear control system (1.2) and sector- or boundedness-data on the nonlinear term f. The term *absolute* refers to the property that stability is ensured for all nonlinear terms within a given class, a key robustness requirement. Absolute stability theory has garnered much academic interest (see, for instance, the extensive literature reviewed in (Liberzon, 2006)) and traces its roots back to the 1940s and the Aizerman Conjecture (Aizerman, 1949). Classical absolute stability criteria include the complex (also known as the complexified or generalised) Aizerman Conjecture, which despite its name, is true, as established in Hinrichsen & Pritchard (1992) (see also Hinrichsen & Pritchard (2011, Section 5.6.3)), as well as the celebrated circle and Popov criteria (Popov, 1962); see, for example, Khalil (2002) and Haddad & Chellaboina (2008). The Circle Criterion generalises the sufficiency part of the Nyquist Criterion in the single-input single-output setting to Lur'e systems, allowing the tools of the linear systems to be applied to this class of nonlinear system. Yet another approach to absolute stability is via so-called Zames–Falb multipliers, initiated by O'Shea (1966, 1967), and further developed by Zames & Falb (1968), with a recent tutorial and perspectives given in Carrasco *et al.* (2016). Further criteria are afforded by the Integral Quadratic Constraints (IQCs) framework, pioneered by the work of Megretski (1993) and Megretski & Rantzer (1997), although the term IQC dates back to the work of Yakubovich in the 1970s; see Fu *et al.* (2005) and the references therein.

ISS is a stability concept for controlled (or forced) systems of nonlinear differential equations, initiated in the 1989 work of Sontag (1989), with subsequent developments in the 1990s by Sontag and others across, for example, Jiang *et al.* (1994); Sontag & Wang (1995) and Sontag & Wang (1996). There is now a vast literature on the subject, and a number of variations of the ISS property have since been proposed, such as so-called *integral* ISS (Sontag, 1998; Angeli *et al*.,, 2000), *strong* ISS (Chaillet *et al*.,, 2014) and *incremental* ISS (Angeli, 2002). The ISS property has been developed in discrete time as well, from Jiang & Wang (2001) onwards. For more background on ISS, we refer the reader to the survey papers of Dashkovskiy *et al.* (2011) and Sontag (2008). One strength of the input-to-state stability paradigm is that it both encompasses and unifies asymptotic and input-output approaches to stability, the latter initiated by Sandberg and Zames in the 1960s; see Desoer & Vidyasagar (1975). The comparison functions ensured by the ISS property are somewhat general and, consequently, one advantage of exponential ISS over the usual version is that it, at least qualitatively, estimates the form of the comparison functions involved. Practically, ISS is important as it shows that 'small' external noise, disturbance or unmodelled dynamics result in a correspondingly 'small' effect on the resulting state.

Much attention has been dedicated to establishing ISS properties for Lur'e systems, originating in the work of Arcak & Teel (2002). Arguably, these authors initiated a line of enquiry investigating the extent to which classical absolute stability criteria may be strengthened to ensure certain ISS properties of the corresponding forced Lur'e system. Indeed, the result by Arcak & Teel (2002, Theorem 1) is reminiscent of the classical positivity theorem (in the 'infinite sector case') for absolute stability; see, for example, Khalil (2002, Theorem 7.1, p. 265)—there with $K_1 = 0$. As is well known, ISS ensures asymptotic stability properties, and the converse is false in general. Whilst there are some subtle pathological cases, such as the examples considered in Teel & Hespanha (2004), broadly speaking, it *is* the case that suitably strengthened absolute stability criteria *do* ensure various ISS notions. That the complex Aizerman Conjecture and Circle Criterion may be strengthened to ensure various ISS-type properties has been established by the work of Logemann, and his students and collaborators, dating back to Jayawardhana *et al.* (2009) and including Jayawardhana *et al.* (2011); Sarkans & Logemann (2015); Bill *et al.* (2016); Sarkans & Logemann (2016); Guiver *et al.* (2019); Gilmore *et al.* (2020) and Guiver & Logemann (2020). Roughly, and as evidenced in the cited papers, the extension from an absolute stability criterion to a sufficient condition for ISS often involves detailed and lengthy technical arguments.

Here we continue the line of enquiry of strengthening classical absolute stability criteria to ensure the ISS property. We invoke the recent characterisations of the exponential ISS property from Guiver & Logemann (2023), which, in turn, builds on an observation from Khalil (2002, Lemma 4.6, p. 176). Namely, we show that, under the assumption that the function f in (1.1) is globally Lipschitz, then global exponential stability of the unforced version of (1.1), or a *linear* $\mathscr{L}_2$-input-output gain (combined with a detectability assumption), are in fact necessary and sufficient for exponential ISS of (1.1).

Whilst the assumption that f is globally Lipschitz is somewhat restrictive, and by no means necessary for exponential ISS of (1.1), it appears as a hypothesis in many absolute stability criteria such as Zames–Falb multiplier theorems or the Kalman conjecture (Kalman, 1957; Barabanov, 1988), in the guise of slope-restricted nonlinearities. The upshot is that a number of, to the best of our knowledge as-yet-untreated, absolute stability criteria including IQCs, Zames–Falb multipliers and Popov criteria, ensure exponential ISS with little additional effort, at least under a global Lipschitz assumption. We comment that whilst the observation that 'global exponential stability and globally Lipschitz implies (or should imply) ISS' is perhaps control folklore, or at least unsurprising, to the best of our knowledge, this observation has not been previously applied to IQC and Zames–Falb multiplier methods in the literature.

The note is organised as follows. Our main results appear across Sections 2 and 3, and concluding remarks appear in Section 4.

1.1. *Notation*

Mathematical notation is mainly standard and follows Guiver & Logemann (2023), but the following is listed for convenience. For a measurable function z defined on $\mathbb{R}_+$ and taking values in Euclidean space $\mathbb{R}^n$ (a signal), the $\mathscr{L}_2$-norm of z over the horizon $(\tau, t+\tau)$ is defined as

$$\|z\|_{\mathscr{L}_2(\tau,t+\tau)} = \left(\int_\tau^{t+\tau} \|z(\theta)\|^2 \, \mathrm{d}\theta\right)^{\frac{1}{2}} \quad t, \tau \geq 0\,,$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$. Similarly, the $\mathscr{L}_\infty$-norm of z over the horizon $(\tau, t+\tau)$ is defined as

$$\|z\|_{\mathscr{L}_\infty(\tau,t+\tau)} = \text{ess sup}_{\theta\in(\tau,t+\tau)}\|z(\theta)\|_\infty \quad t, \tau \geq 0\,,$$

where $\|\cdot\|_\infty$ is the maximum norm on $\mathbb{R}^n$. When the above quantities are finite for all $0 \leq t, \tau < \infty$, then we say that z belongs to $\mathscr{L}_{2,\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ and $\mathscr{L}_{\infty,\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, respectively, that is, z is locally square integrable or z is locally essentially bounded, as a function on $\mathbb{R}_+$ taking values in $\mathbb{R}^n$.

Let $\mathbf{G}(s)$ denote a transfer function—a proper rational function of the complex variable s defined on some open right-half complex plane. When every pole of $\mathbf{G}$ has negative real part, then the $\mathscr{H}_\infty$ norm of $\mathbf{G}$ is defined as

$$\|\mathbf{G}\|_{\mathscr{H}_\infty} := \sup_{\omega\in\mathbb{R}} \bar{\sigma}\left(\mathbf{G}(j\omega)\right),$$

where $\bar{\sigma}(\cdot)$ denotes maximum singular value.

Finally, recall that a square matrix is called Hurwitz if every eigenvalue has negative real part.

2. Absolute stability criteria for exponential ISS

2.1. *Preliminaries*

Consider the Lur'e system of forced nonlinear differential equations (1.1), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $f : \mathbb{R}^p \to \mathbb{R}^m$ for positive integers m, n and p. We assume throughout that f is locally Lipschitz and linearly bounded, meaning there exists $k > 0$ such that

$$\|f(z)\| \leq k\|z\| \quad \forall\, z \in \mathbb{R}^p\,. \tag{2.1}$$

Under these assumptions, it follows routinely from the theory of ordinary differential equations that for each $v = (v_1, v_2) \in \mathscr{L}_{\infty,\text{loc}}(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^p)$, there exists a locally absolutely continuous function $x : \mathbb{R}_+ \to \mathbb{R}^n$ such that (1.1) holds almost everywhere. We call the pair (v, x) a *trajectory* of (1.1). Moreover, for each $v \in \mathscr{L}_{\infty,\text{loc}}(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^p)$ and $\xi \in \mathbb{R}^n$, there exists a unique trajectory (v, x) of (1.1) with $x(0) = \xi$. Evidently, the sector condition (2.1) yields that $f(0) = 0$ and so $(0, 0)$ is a constant trajectory of (1.1). For clarity, we shall always assume that v in (1.1) belongs to $\mathscr{L}_{\infty,\text{loc}}(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^p)$.

Lur'e system (1.1) is called *linearly* $\mathscr{L}_2$*-input-output stable* if there exists a positive constant α such that every trajectory (v, x) of (1.1) with $x(0) = 0$ satisfies

$$\|y\|^2_{\mathscr{L}_2(0,t)} + \|u\|^2_{\mathscr{L}_2(0,t)} \leq \alpha\|v\|^2_{\mathscr{L}_2(0,t)} \quad \forall\, t \geq 0, \tag{2.2a}$$

where, here and throughout for (1.1), $y = Cx + v_2$ and $u = f(y)$. The above property is often simply called *stable* in the IQC literature, namely Megretski & Rantzer (1997, Definition, p.281). Strictly speaking, the feedback connection in Figure 1 is a slight generalisation of that in Megretski & Rantzer (1997, Figure 1) to which Megretski & Rantzer (1997, Definition, p. 281) applies, as here v_1 need not be of the form $v_1 = Bv_1'$, for some disturbance signal v_1'. Under our standing linear-bound assumption (2.1), it follows that if there exists a positive constant α_0 such that every trajectory (v, x) of (1.1) with $x(0) = 0$ satisfies

$$\|y\|^2_{\mathscr{L}_2(0,t)} \leq \alpha_0 \|v\|^2_{\mathscr{L}_2(0,t)} \quad \forall\, t \geq 0\,, \tag{2.2b}$$

then the inequality (2.2a) holds with $\alpha := \alpha_0 + k^2$ where k is as in (2.1). Observe that inequality (2.2b) clearly follows from (2.2a) with $\alpha_0 := \alpha$. Therefore, for the Lur'e systems considered presently, the inequalities (2.2a) and (2.2b) are equivalent, that is, one holds if, and only if, the other does.

Note that $v \in \mathscr{L}_{\infty,\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^p)$ implies that v is locally square integrable, that is, belongs to $\mathscr{L}_{2,\mathrm{loc}}$ and so the right-hand sides of (2.2) are finite. Obviously, for the right-hand sides of (2.2) to remain finite in the limit $t \to \infty$ requires that $v \in \mathscr{L}_2(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^p)$, which is *not* implied by local essential boundedness of v in general.

We say that (1.1) has the *linear* $\mathscr{L}_2$*-state/input-to-state gain* property if there exist positive constants β_1, β_2 such that, for all trajectories (v, x) of (1.1),

$$\|x\|_{\mathscr{L}_2(0,t)} \leq \beta_1 \|x(0)\| + \beta_2 \|v\|_{\mathscr{L}_2(0,t)} \quad \forall\, t \geq 0\,. \tag{2.3}$$

(Strictly, the property (2.3) should hold for all $t, \tau \geq 0$—see Guiver & Logemann (2023, Definition 3.3)—but this is equivalent to the displayed property (2.3) when f in (1.1) is time-invariant, as is the case here, by a standard shift- and causality-argument.)

Similarly, recall that (1.1) is said to be *exponentially input-to-state stable* (exponentially ISS) if there exist positive constants L_1, L_2, γ such that the estimate (1.3) holds for all trajectories (v, x) of (1.1).

Since it plays a key role presently, we record the main result of Guiver & Logemann (2023), namely Guiver & Logemann (2023, Theorem 3.4), in the context of the forced Lur'e system (1.1).

THEOREM 1. Consider the forced Lur'e system (1.1) and assume that f is globally Lipschitz. Consider also the unforced version

$$\dot{x} = Ax + Bf(Cx), \tag{2.4}$$

that is, equation (1.1) with $v = 0$. The following statements are equivalent:

(1) (2.4) is globally exponentially stable;

(2) (1.1) is exponentially ISS;

(3) (1.1) has the linear $\mathscr{L}_2$-state/input-to-state gain property;

(4) (1.1) admits an exponential ISS Lyapunov function, that is, there exist a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ and positive constants b_1, b_2, b_3 and b_4 such that

$$b_1 \|z\|^2 \leq V(z) \leq b_2 \|z\|^2 \quad \forall\, z \in \mathbb{R}^n, \tag{2.5a}$$

and

$$\langle \nabla V(z), Az + Bf(Cz + w_2) + w_1 \rangle \le -2b_3 V(z) + b_4 \|w\|^2 \quad \forall\, (z, w) \in \mathbb{R}^n \times \big(\mathbb{R}^n \times \mathbb{R}^p\big). \tag{2.5b}$$

Recall from, for example, Logemann & Ryan (2014, p. 286), that a function $V : \mathbb{R}^n \to \mathbb{R}_+$ is continuously differentiable if, and only if, every partial derivative of V exists and is continuous. In this case the gradient of V, denoted ∇V, and which is often defined as the derivative of V, may be identified with a vector of partial derivatives of V.

2.2. *Main result*

The following theorem is the main result of the present note.

THEOREM 2. Consider the forced Lur'e system (1.1) and assume that f is globally Lipschitz. The following statements hold:

(1) If an absolute stability criterion ensures that zero is a globally exponentially stable equilibrium of (2.4), then (1.1) is exponentially ISS.

(2) If the pair (C, A) is detectable and an absolute stability criterion ensures that (1.1) is linearly $\mathscr{L}_2$-input-output stable, then

(a) (1.1) has linear $\mathscr{L}_2$-state/input-to-state gain;

(b) (1.1) is exponentially ISS; and

(c) there exist positive constants Γ_1, Γ_2, γ such that every trajectory (v, x) of (1.1) satisfies

$$\|x(t+\tau)\|^2 \le \Gamma_1 e^{-2\gamma t} \|x(\tau)\|^2 + \Gamma_2 \|v\|^2_{\mathscr{L}_2(\tau, t+\tau)} \quad \forall\, t, \tau \ge 0. \tag{2.6}$$

Property (2.6) is somewhere between exponential ISS and the linear $\mathscr{L}_2$-state/input-to-state gain property, with the bound for $\|x(t+\tau)\|^2$ in terms of exponential decay in the state and linear growth in the $\mathscr{L}_2$-norm of the input. Observe that the inequality (2.6) with $v = 0$ trivially shows that zero is a globally exponentially stable equilibrium of the unforced Lur'e system (2.4), and so is equivalent to exponential ISS (and hence the linear $\mathscr{L}_2$-state/input-to-state gain property) of (1.1) by statement (1). In particular, for the Lur'e systems under consideration, the inequalities (1.3) and (2.6) are equivalent, which may be situationally advantageous when one or the other is easier to verify, or when either the $\mathscr{L}_2$- or $\mathscr{L}_\infty$-norm of the forcing term v is more relevant.

Statement (2) of Theorem 2 draws inspiration from Megretski & Rantzer (1997, Proposition 1) (which does not consider outputs or the exponential ISS property), where property (2.6) is shown to be equivalent to the linear $\mathscr{L}_2$-state/input-to-state gain property when $x(0) = 0$. In particular, we obtain a new proof of Megretski & Rantzer (1997, Proposition 1) for forced Lur'e systems with Lipschitz nonlinear term. We note that the detectability assumption in statement (2) is natural when seeking to infer stability properties of a state from those of an output.

Proof of Theorem 2. Statement (1) follows immediately from Theorem 1.

To prove statement (2), let $v \in \mathscr{L}_{\infty,\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n \times \mathbb{R}^p)$ and let (v, x) denote a trajectory of (1.1).

(a) Define $z := x - e_{-1}x(0)$ where $e_{-1}(t) := e^{-t}$ for all $t \geq 0$, and $y_z := Cz + v_2$. A routine calculation shows that

$$\dot{z} = Az + Bf(Cz + v_2) + w\,,$$

where

$$w := v_1 + B\big(f(Cz + Ce_{-1}x(0) + v_2) - f(Cz + v_2)\big) + (A + I)e_{-1}x(0)\,.$$

In particular, $\big((w, v_2), z\big)$ is a trajectory of (1.1) with $z(0) = 0$. Hence, the linear $\mathscr{L}_2$-input-output stability hypothesis (2.2b) here reads

$$\|y_z\|_{\mathscr{L}_2(0,t)} \leq \sqrt{\alpha_0}\big\|(w, v_2)\big\|_{\mathscr{L}_2(0,t)} \quad \forall\, t \geq 0\,.$$

Invoking that f is globally Lipschitz, we may majorise w by

$$\|w\| \leq c_1\big(\|v_1\| + e_{-1}\|x(0)\|\big)\,,$$

for some positive constant c_1 and which, therefore, yields that

$$\|y_z\|_{\mathscr{L}_2(0,t)} \leq c_2\big(\|v\|_{\mathscr{L}_2(0,t)} + \|x(0)\|\big) \quad \forall\, t \geq 0\,, \tag{2.7}$$

for some positive constant c_2.

Revisiting x, the detectability assumption of the pair (C, A) guarantees the existence of $H \in \mathbb{R}^{n\times p}$ such that $A - HC$ is Hurwitz. The Lur'e system (1.1) may be rewritten as

$$\begin{aligned}\dot{x} &= (A - HC)x + Bf(Cx + v_2) + HCx + v_1\\ &= (A - HC)x + Bf(y_z + Ce_{-1}x(0)) + Hy_z + HC(e_{-1}x(0) - v_2) + v_1\,.\end{aligned}$$

The variation of parameters formula for x yields that, for all $t \geq 0$,

$$\begin{aligned}x(t) &= e_{(A-HC)}(t)x(0)\\ &\quad + \Big(e_{(A-HC)} \star \big(Bf(y_z + Ce_{-1}x(0)) + Hy_z + HC(e_{-1}x(0) - v_2) + v_1\big)\Big)(t)\,,\end{aligned} \tag{2.8}$$

where $\star$ denotes convolution, and

$$e_{(A-HC)}(t) := e^{(A-HC)t} \quad \forall\, t \geq 0\,.$$

From the Hurwitz property of $A - HC$, there exists $c_3 > 0$ such that

$$\max\big\{\|e_{(A-HC)}\|_{\mathscr{L}_2(0,t)}, \|e_{(A-HC)}\|_{\mathscr{L}_1(0,t)}\big\} \leq c_3 \quad \forall\, t \geq 0\,. \tag{2.9}$$

Taking $\mathscr{L}_2(0, t)$ norms in (2.8) for $t \geq 0$ and invoking that f is globally Lipschitz, and the upper bounds (2.7) and (2.9), gives (2.3), for some $\beta_1, \beta_2 > 0$, as required.

Part (b) follows from (a) and Theorem 1.

Finally, to prove part (c), we invoke statement (b) and Theorem 1 to ensure the existence of an exponential ISS Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ as in (2.5). The dissipation inequality (2.5b) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq -2b_3 V(x(t)) + b_4 \|v(t)\|^2 \quad \text{for almost all } t \geq 0\,.$$

From an application of the variation of parameters formula, and a suitably modified version of Logemann & Ryan (2014, Lemma 5.43), we deduce the inequality

$$V(x(\tau + s)) \leq e^{-2b_3 s} V(x(\tau)) + b_4 \int_{\tau}^{s+\tau} e^{-2b_3(s+\tau-p)} \|v(p)\|^2 \,\mathrm{d}p \quad \forall\, s, \tau \geq 0\,. \tag{2.10}$$

Therefore, estimating both sides of the above using (2.5a) gives

$$b_1 \|x(t+\tau)\|^2 \leq b_2 e^{-2b_3 t} \|x(\tau)\|^2 + b_4 \int_{\tau}^{t+\tau} \|v(s)\|^2 \,\mathrm{d}s \quad \forall\, t, \tau \geq 0\,,$$

which is (2.6) with $\Gamma_1 := b_2/b_1$, $\gamma := -b_3$ and $\Gamma_2 := b_4/b_1$. □

In the remainder of the note, we comment on how and where Theorem 2 applies to a selection of results across the literature. Given the breadth and depth of study on absolute stability criteria, the following description is by no means exhaustive.

For which purpose, we set $\mathbf{G}(s) := C(sI - A)^{-1}B$, which is the transfer function from input u to output y in the linear control system in (1.2). In much control theoretic literature, the terminology *slope-restricted* is used in the single-input single-output setting. Recall that the scalar-valued function f is called *slope-restricted* if there exist $a < b$ such that

$$a \leq \frac{f(z_1) - f(z_2)}{z_1 - z_2} \leq b \quad \forall\, z_1, z_2 \in \mathbb{R},\; z_1 \neq z_2\,. \tag{2.11}$$

Slope-restricted functions are globally Lipschitz and, when $a \geq 0$, are also monotone nondecreasing, in the sense that

$$0 \leq (z_1 - z_2)\big(f(z_1) - f(z_2)\big) \quad \forall\, z_1, z_2 \in \mathbb{R}\,.$$

In the multivariate setting, a function $f : \mathbb{R}^p \to \mathbb{R}^p$ is called *diagonal* (or *decoupled*) if

$$\big(f(z)\big)_i = f_i(z_i) \quad \forall\, z = \begin{pmatrix} z_1 & \dots & z_p \end{pmatrix}^\top \in \mathbb{R}^p,\; \forall\, i \in \{1, 2, \dots, p\}\,. \tag{2.12}$$

Such a function is called slope-restricted if every scalar-valued component $f_i : \mathbb{R} \to \mathbb{R}$ is.

A slope-restricted condition on f and the usual assumption in control systems that $f(0) = 0$ are sufficient for the linear-bound condition (2.1) to hold.

2.3. *Zames–Falb multipliers*

Of particular interest to the present authors are absolute stability criteria in terms of Zames–Falb multipliers. Zames–Falb multipliers treat absolute stability problems for nonlinearities that are either monotonic or slope restricted; results obtained using them are often less conservative than ones obtained using Circle/Popov Criteria and the more usual sector bounds. Roughly speaking, the approach seeks multipliers which guarantee passivity of a certain transfer function (Carrasco *et al.* 2016; Turner, 2021) and the existence of such a multiplier then guarantees that the system (1.1) is $\mathscr{L}_2$-input-output stable. The classical result of Carrasco *et al.* (2016, Theorem 1) does not explicitly mention *linear* $\mathscr{L}_2$-input-output stability as a conclusion. However, an inspection of the proof of Zames & Falb (1968, Lemma 1), the key stability result used in proving Zames & Falb (1968, Theorem 1), shows that this property is in fact an unmentioned conclusion.

Recently, the control systems literature has seen a surge of papers on Zames–Falb multipliers, such as Mancera & Safonov (2005); Chang *et al.* (2012); Fetzer & Scherer (2017) and Turner & Drummond (2020), with the main interest being the application of the method to systems to slope-restricted nonlinearities, that is systems for which f satisfies inequality (2.11). As observed in the previous section, such nonlinearities are globally Lipschitz and, therefore, by Theorem 2 stability results obtained via Zames–Falb multipliers guarantee exponential ISS with *no further assumptions*, apart from the mild assumption of detectability of the linear part of the system. The same conclusion is true of the various multivariable extensions of Zames–Falb's result (Mancera & Safonov, 2005).

2.4. *Popov criteria*

The eponymously-named Popov criterion dates back to Popov (1962). The result is a classical absolute stability criterion and appears in a number of different forms across the literature, including in terms of an IQC interpretation and via a Lyapunov approach in terms of a so-called Lur'e-Postnikov Lyapunov function as appearing in the monographs by Khalil (2002, Theorem 7.3), Haddad & Chellaboina (2008, Theorem 5.20) and Vidyasagar (2002, Theorem 46, p. 231). There are distinct formulations of the Popov criterion, but all essentially combine a (possibly infinite) sector condition with a positive-realness assumption of an auxiliary function (the transfer function multiplied by a so-called Popov multiplier). Popov criteria require that the dimensions of the input and output spaces coincide, meaning $m = p$ in (1.1) and (2.4).

We show how Theorem 2 applies to two distinct Popov criteria:

1. The result Haddad & Chellaboina (2008, Theorem 5.20) assumes that (A, B, C) is minimal and that the nonlinear term $\phi = -f$ is diagonal (or decoupled) as in (2.12). If ϕ and positive definite diagonal $M \in \mathbb{R}^{m\times m}$ are such that

$$\langle \phi(z), \phi(z) - Mz \rangle = \sum_{i=1}^{m} \phi_i(z_i)(\phi_i(z_i) - M_{ii}z_i) \le 0 \quad \forall\, z \in \mathbb{R}^m ,$$

   and $s \mapsto I + M(I + Ns)\mathbf{G}(s)$ is strictly positive real for some positive definite diagonal $N \in \mathbb{R}^{m\times m}$, then the zero equilibrium of (2.4) with $f = -\phi$ is globally exponentially stable. The result (Haddad & Chellaboina, 2008, Theorem 5.20) only claims global asymptotic stability, but in fact global exponential stability follows from their argument—integrating the inequality (Haddad & Chellaboina, 2008, (5.234), p. 385) between 0 and t gives that (using the notation of

Haddad & Chellaboina (2008))

$$p_1 \varepsilon \|x\|^2_{\mathscr{L}_2(0,t)} \le \varepsilon \langle x, Px \rangle_{\mathscr{L}_2(0,t)} \le V(x(0)) - V(x(t)) \le p_2 \|x(0)\|^2 \quad \forall\, t \ge 0\,,$$

for some positive constants p_1 and p_2, where we have used that P is positive definite and ϕ is sector bounded. Global exponential stability of (2.4) with $f = -\phi$ now follows from Guiver & Logemann (2023, Theorem 2.1).

2. The result (Fliegner *et al.*, 2006, Theorem 2) considers the single-input single-output ($m = 1$) case and assumes that $\lim_{s \to 0} s\mathbf{G}(s) > 0$, that $\alpha \in (0, \infty)$ and $q \ge 0$ are such that

$$s \mapsto 1 + \alpha(qs + 1)\mathbf{G}(s) \quad \text{is strongly positive real,}$$

and that $\phi = -f$ satisfies

$$\alpha z \phi(z) \ge \phi^2(z) \quad \text{and} \quad z\phi(z) \ge \alpha_0 z^2 \quad \forall\, z \in \mathbb{R}\,,$$

for some $\alpha_0 > 0$. In this case, it follows from Fliegner *et al.* (2006, Theorem 2, statement (i)) that zero is globally exponentially stable. Note that Fliegner *et al.* (2006, Theorem 2) do not impose that the linear system is controllable and observable, and the matrix A in (1.1) has a simple eigenvalue at zero.

In both cases, an application of Theorem 2 gives that if ϕ is globally Lipschitz, then (1.1) with $f = -\phi$ is exponentially ISS. For clarity and the avoidance of doubt, we comment that Popov criteria, such as the above two results, *do not require* that f is globally Lipschitz. Rather, the present novel findings are that, under an additional global Lipschitz assumption on f, these Popov criteria in fact ensure additional stability properties.

2.5. *Integral Quadratic Constraints*

IQCs are a powerful tool for determining input-output stability of rather general feedback connections—namely a stable linear system (with transfer function $\mathbf{G}$) and a causal operator Δ with bounded gain. The classical IQC result (Megretski & Rantzer, 1997, Theorem 1) provides sufficient conditions for when this feedback connection is linearly $\mathscr{L}_2$-input-output stable. Both the classical Zames–Falb multiplier theorem (Zames & Falb, 1968, Theorem 1, Corollary 1) and the Popov Criterion (Jönsson, 1997) can be interpreted in an IQC context.

However, IQC analysis for Lur'e systems is *not limited* to classical approaches and if (i) *any* IQC stability analysis concludes stability in the sense of Megretski & Rantzer (1997, Theorem 1); and (ii) the operator Δ is a static, globally Lipschitz nonlinearity, then the system fits the form of (1.1) and satisfies the hypotheses of Theorem 2. The upshot is that exponential ISS may be concluded, providing again that the mild condition on detectability of (C, A) is satisfied. In a sense, our work here complements that in Seiler (2015) where a time-domain interpretation of the IQC result is given, enabling one to see, through a time-domain dissipation argument, both asymptotic (exponential) stability and linear $\mathscr{L}_2$-input-output stability of system (1.1). The results in Seiler (2015) stop short of proving exponential ISS, however.

2.6. *Further absolute stability criteria*

There are dozens, if not hundreds, of papers that introduce, develop and refine absolute stability criteria. All results which ensure global exponential stability of (2.4), or a linear $\mathscr{L}_2$-input-output gain of (1.1) and detectable linear part, and impose that f is globally Lipschitz, satisfy the hypotheses of Theorem 2. As specific examples, we highlight the papers Park (2002); Turner & Kerr (2012) and Valmorbida *et al.* (2018), noting that the function f is assumed slope-restricted in each of these works.

3. Comments on stability rates and gains

Theorem 2 relates various stability notions qualitatively. Here, we briefly describe certain quantitative connections between these concepts. We comment upfront that state and input-output stability concepts are, in isolation and in general, quantitatively independent. Indeed, this may be seen through two scalar examples, namely, the two control systems

$$\dot{x}_1 = h(x_1) + a_1 u_1,\ y_1 = x_1 \quad \text{and} \quad \dot{x}_2 = -2a_2 x_2 + 2a_2 u_2,\ y_2 = x_2\,,$$

for some nonlinear function $h : \mathbb{R} \to \mathbb{R}$. On the one hand, global exponential stability of the first system, when unforced (so that $u_1 = 0$), is independent of $a_1 > 0$, yet a_1 will occur in any input-output estimate. On the other hand, the second system is linearly $\mathscr{L}_2$-input/output stable with gain one, as

$$\|y_2\|_{\mathscr{L}_2(0,t)} = \|x_2\|_{\mathscr{L}_2(0,t)} \le \|\mathbf{G}\|_{\mathscr{H}_\infty} \|u_2\|_{\mathscr{L}_2(0,t)} = \|u_2\|_{\mathscr{L}_2(0,t)} \quad \forall\, t \ge 0\,,$$

since $\mathbf{G}(s) = 2a_2/(s + 2a_2)$ satisfies

$$\|\mathbf{G}\|_{\mathscr{H}_\infty} = \mathbf{G}(0) = 1\,,$$

independently of $a_2 > 0$. However, in terminology to come, $a_2 > 0$ is a rate of exponential convergence for this control system.

One strength of the exponential ISS property in this context is that it enables both quantitative estimates for state- and input-state stability notions. Quantitative estimates for input-output stability notions may then be determined from output equations. For which purpose, we require additional terminology. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz with $g(0) = 0$. Recall that the zero trajectory of the differential equation

$$\dot{x} = g(x)\,, \tag{3.1}$$

is called globally exponentially stable (GES) if every trajectory (here just a solution) of (3.1) is global, and there exist $k, r > 0$ such that every trajectory x of (3.1) satisfies

$$\|x(t+\tau)\| \le k e^{-rt} \|x(\tau)\| \quad \forall\, t, \tau \ge 0\,. \tag{3.2}$$

In this case, we simply write that (3.1) is GES. (We note that since (3.1) is autonomous, the estimate (3.2) is satisfied if the inequality holds for $\tau = 0$ only.) Following Corless & Glielmo (1998, Section 2), the terms k and r in (3.2) are called a *gain of exponential convergence* and *rate of exponential convergence*, respectively.

A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a *GES Lyapunov function (for (3.1))* if there exist positive constants a_1, a_2 and a_3 such that

$$a_1\|z\|^2 \le V(z) \le a_2\|z\|^2 \quad \forall\, z \in \mathbb{R}^n\,, \tag{3.3a}$$

$$\langle \nabla V(z), g(z)\rangle \le -2a_3 V(z) \quad \forall\, z \in \mathbb{R}^n\,. \tag{3.3b}$$

It is well known that existence of such a V is sufficient for GES of (3.1) and, from Corless & Glielmo (1998, Fact 3), with gain $\beta := \sqrt{a_2/a_1}$ and rate $\alpha := a_3$.

Consider now the general forced system of nonlinear differential equations

$$\dot{x} = h(x, d)\,, \tag{3.4}$$

for locally Lipschitz h with $h(0,0) = 0$. The input d takes values in $\mathbb{R}^q$. The exponential ISS property and an exponential ISS Lyapunov function for (3.4) are analogous to those for the special case of the forced Lur'e system (1.1). Existence of an exponential ISS Lyapunov function is sufficient for the exponential ISS property (1.3) and, up to some additional assumptions on h, necessary as well; see Guiver & Logemann (2023, Theorem 3.4). We call the constants L_i and γ in (1.3) *exponential ISS gains* and an *exponential ISS rate*, respectively. Evidently, an exponential ISS Lyapunov function for (3.4) is a GES Lyapunov function for the unforced version (3.1) with $g(z) = h(z, 0)$, so that L_1 and γ in (1.3) are also a gain and rate of exponential convergence, respectively.

The following lemma extracts various rates and gains from an exponential ISS Lyapunov function. The result is valid for control systems of the form (3.4), which includes forced Lur'e systems (1.1) as a special case.

LEMMA 3. Suppose that $V : \mathbb{R}^n \to \mathbb{R}_+$ as in (2.5) is an exponential ISS Lyapunov function for (3.4). The following statements hold:

(a) (3.4) has the linear $\mathscr{L}_2$-state/input-to-state gain property (2.3) with gains $\beta_1 := \sqrt{b_2/(b_1 b_3)}$ and $\beta_2 := \sqrt{b_4/(b_1 b_3)}$.

(b) (3.4) is exponentially ISS with gains $L_1 := \sqrt{b_2/b_1}$, $L_2 := \sqrt{b_4/(2b_1 b_3)}$ and rate $\gamma := b_3$.

(c) If h is globally Lipschitz with Lipschitz constant L, and W is a GES Lyapunov function for (3.1) with $g(z) := h(z, 0)$ with linearly bounded gradient, meaning

$$\|(\nabla W)(z)\| \le a_4\|z\| \quad \forall\, z \in \mathbb{R}^n\,, \tag{3.5}$$

for some $a_4 > 0$, then W is an exponential ISS Lyapunov function for (3.4) with

$$b_1 = a_1, \quad b_2 = a_2, \quad b_3 = a_3 - \varepsilon, \quad b_4 = \frac{L^2 a_4^2}{8a_1\varepsilon}\,,$$

where $\varepsilon \in (0, a_3)$ is arbitrary.

The upshot of statements (a) and (b) of the above lemma is that linear $\mathscr{L}_2$-state/input-to-state gains and exponential ISS gains/rates may be computed from exponential ISS Lyapunov function bounds and the dissipation inequality. Combining statement (c) with these statements provides estimates for linear $\mathscr{L}_2$-state/input-to-state gains and exponential ISS gains/rates from from a GES Lyapunov function for the unforced system. Observe that the exponential ISS rate b_3 may be arbitrarily close to the exponential rate of convergence a_3 of the unforced system, at the price of a larger exponential ISS input gain b_4. However, this is likely an artefact of the proof and we suspect will lead to a conservative value of b_4 which, as argued at the start of the section, is not so surprising since GES of the unforced system alone provides no information in general on input-to-state or input-to-output gains.

Proof of Lemma 3. For the proof, it is convenient to record the elementary inequality

$$\sqrt{c^2+d^2} \le c+d \quad \forall\, c,d \ge 0\,. \tag{3.6}$$

Statement (a) follows from Guiver & Logemann (2023, Lemma 3.8). An inspection of that proof, particularly the displayed equation below Guiver & Logemann (2023, inequality (3.8)), combined with the above inequality, gives the claimed linear $\mathscr{L}_2$-state/input-to-state gain gains. Note that a_3 in the notation of Guiver & Logemann (2023, Lemma 3.8) is equal to $2b_3$ here.

To prove statement (b), let V be as described and let (d, x) be a trajectory of (3.4). The variation of parameters inequality (2.10) holds (with $v = d$) from which we estimate both sides using (2.5a) to give

$$\begin{aligned} b_1\|x(t+\tau)\|^2 &\le b_2 e^{-2b_3 t}\|x(\tau)\|^2 + b_4\|d\|^2_{\mathscr{L}_\infty(\tau,t+\tau)}\int_\tau^{t+\tau} e^{-2b_3 s}\,\mathrm{d}s \\ &\le b_2 e^{-2b_3 t}\|x(\tau)\|^2 + \frac{b_4}{2b_3}\|d\|^2_{\mathscr{L}_\infty(\tau,t+\tau)} \quad \forall\, t,\tau \ge 0\,. \end{aligned}$$

Dividing both sides of the above by b_1 and using the inequality (3.6) gives the desired exponential ISS rates.

The proof of statement (c) is inspired by the estimates in the proof of Khalil (2002, Lemma 4.6, p. 176). Let $L > 0$ be a Lipschitz constant for h. We use the dissipation inequality (3.3b), the gradient inequality (3.5), and the perturbation argument

$$\begin{aligned} \langle \nabla W(z), h(z,w)\rangle &= \langle \nabla W(z), h(z,0)\rangle + \langle \nabla W(z), h(z,w)-h(z,0)\rangle \\ &\le -2a_3 W(z) + \|\nabla W\|\cdot\|h(z,w)-h(z,0)\| \le -2a_3 W(z) + La_4\|z\|\|w\| \\ &\le -2(a_3-\varepsilon)W(z) + \frac{L^2 a_4^2}{8a_1\varepsilon}\|w\|^2 \quad \forall\,(z,w)\in\mathbb{R}^n\times\mathbb{R}^q\,, \end{aligned}$$

as required. Here we have also used the standard quadratic inequality

$$\begin{aligned} La_4\|z\|\|w\| &= 2\sqrt{2\varepsilon a_1}\|z\|\big(La_4\|w\|/(2\sqrt{2\varepsilon a_1})\big) \\ &\le 2\varepsilon a_1\|z\|^2 + \frac{L^2a_4^2}{8a_1\varepsilon}\|w\|^2 \quad \forall\,(z,w)\in\mathbb{R}^n\times\mathbb{R}^q\,. \end{aligned}$$

□

4. Summary

Absolute stability criteria that ensure either global exponential stability of an unforced Lur'e system, or guarantee a linear $\mathscr{L}_2$-input-output gain of the forced version, have been shown to in fact be sufficient for the *a priori* stronger exponential ISS property when the nonlinear term is globally Lipschitz (or slope-restricted). While the connection between exponential stability and the linear $\mathscr{L}_2$-input-output property is fairly well known, the fact that both of these properties implies exponential ISS is not widely understood. We have identified a selection of settings from across the literature where these hypotheses are satisfied, including results from the IQC and Zames–Falb multiplier frameworks. Given the vast absolute stability literature, one motivation for the current work is to prevent a proliferation of related ISS results which fall into the rather general framework considered presently. Another motivation for the present study is an attempt to better connect approaches from the IQC/multiplier and input-to-state stability communities by showing overlap between concepts from both.

We emphasise that our main result, Theorem 2, connects various stability notions *qualitatively*. As discussed in Section 3, in the absence of additional information, these stability notions may be *quantitatively* independent, so that relating them quantitatively from the qualitative connections alone is likely to be conservative. However, a number of rates and/or gains associated with these stability notions may be estimated from an exponential ISS Lyapunov function.

Two natural future lines of enquiry are to investigate the extent to which (a) quantitative rates and/or gains may be derived from Zames–Falb multipliers themselves, and; (b) the results generalise to time-varying Lur'e systems, where $f(Cx + v_2)$ in (1.1) is replaced by $f(t, Cx + v_2)$. In light of the above discussion, to make progress with (a) may require identifying additional structure or imposing further assumptions. With regards to (b), this would likely require time-varying versions of the results of Guiver & Logemann (2023), which, to the best of the authors' knowledge, are not currently available. However, and roughly speaking, it has been noted in Guiver & Logemann (2020, Remark 3.9) in the context of various ISS notions, that results for such time-varying Lur'e systems are often obtainable from the original result when the nonlinearity satisfies given norm- or sector-bounds uniformly in time. Presently, in the single-input single-output setting, the natural generalisation of the slope-restricted condition (2.11) is that the constants $a < b$ are such that

$$a \leq \frac{f(t, z_1) - f(t, z_2)}{z_1 - z_2} \leq b \quad \forall\, z_1, z_2 \in \mathbb{R},\; z_1 \neq z_2,\; \text{almost all } t \geq 0.$$

Finally, we comment that exponential ISS has been established for Lur'e systems via other arguments which do not require slope-restricted nonlinearities. For simplicity, assume that A in (1.1) is Hurwitz. It is known from, for example, Sarkans & Logemann (2015, Theorem 3.2, comments on p.451), that the simple small-gain condition

$$\|\mathbf{G}\|_{\mathscr{H}_\infty} \sup_{z \neq 0} \frac{\|f(z)\|}{\|z\|} < 1\,,$$

is sufficient for exponential ISS. Whilst the above condition is typically conservative, it does not enforce that f is globally Lipschitz and permits, for instance, the increasingly-rapidly-oscillating function

$$f(z) = c_0 z(1 + c_1 \sin(z^2)) \quad \forall\, z \in \mathbb{R}\,,$$

for suitable constants c_0 and c_1. Since f in (1.1) is typically assumed to be locally Lipschitz to ensure existence of unique solutions to (1.1), in light of the present work, future attention in determining sufficient (at least qualitative) conditions for exponential ISS should focus on the case where f is locally, but not globally, Lipschitz.

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