

Stabilisation by adaptive feedback control for positive difference equations with applications in pest management

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Abstract

An adaptive feedback control scheme is proposed for stabilising a class of forced nonlinear positive difference equations. The adaptive scheme is based on so-called high-gain adaptive controllers, and contains substantial robustness with respect to model uncertainty as well as with respect to persistent forcing signals, including measurement errors. Our results take advantage of the underlying positive systems structure and ideas from input-to-state stability from nonlinear control theory. Our motivating application is to pest or weed control, and in this context the present work substantially strengthens previous work by the authors. The theory is illustrated with examples.

Keywords: feedback control, mathematical ecology, Lur'e system, pest and weed management, positive system, simple adaptive control

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1 Introduction

We propose an adaptive feedback controller for stabilising the following class of systems of forced nonlinear positive difference equations

$$x(t+1) = Ax(t) + B\Theta(Ex(t)) + B_e v(t), \quad x(0) = x^0, \quad t = 0, 1, 2, \dots \quad (1.1)$$

The vectors x and v are a state and external forcing variable, respectively. Furthermore, A , B , B_e and E are appropriately-sized matrices, and Θ is a (nonlinear) function, all of which have certain positivity properties. The term positive refers to the property that the dynamics (1.1) leave invariant the non-negative orthant of n -dimensional Euclidean space, reflecting the property that (1.1) models necessarily nonnegative quantities. The dynamics in (1.1) are reasonably general as the right hand side contains both linear and nonlinear terms, and is occasionally called a semilinear difference equation [34] or a semilinear map [42]. The model (1.1) encompasses inhomogeneous linear systems of difference equations in the special case that $\Theta = 0$.

In a control theoretic setting, the model (1.1) is often called a system of Lur'e difference equations, after the Soviet scientist A.I. Lur'e, who made early contributions to their stability theory in continuous-time. Other Anglicisations of the Russian name include Lurie and Lurye. Lur'e systems arise in a variety of classical systems and control theoretic contexts, and are a well-studied and active area of research. Relevant works include, but are not limited, to [20, 26, 28, 32, 45] and, particularly for the discrete-time case, the papers [16, 41] and the references therein.

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Systems of positive Lur'e difference equations have recently been proposed and considered as models in ecology in, for example, [9, 11, 14, 15, 39, 42, 44]. Briefly, as Lur'e difference equations contain both linear and nonlinear components, they are often an appropriate framework for modelling density-independent (that is, linear) and density-dependent (that is, non-linear) vital- or transition-rates. The inclusion of density-dependence permits modelling Allee [8], competition or crowding effects. Consequently, (1.1) admits a range of realistic and non-trivial dynamic behaviour including boundedness of solutions [15], attractive zero and non-zero equilibria, as well as exhibiting fluctuating and even chaotic solutions. Another appealing facet is that Lur'e systems are reasonably well-understood mathematically and amenable to analysis. For example, “trichotomies of stability” (also known as “limit set trichotomies”) are presented in [39] and [44] for certain unforced ($v = 0$) models of the form (1.1) in terms of the model data.

The motivating context for the current study is the situation wherein x in (1.1) models a stage-structured local pest or weed population which we seek to manage via some control scheme, such as chemical pesticide application or release of bio-control agents [25]: a timely control engineering problem of significant societal, economical and environmental relevance. Pests include numerous organisms which cause damage in agriculture and horticulture. Invertebrate pests reduce crop quality, vector plant diseases and directly cause crop losses before and after harvest. Insect pests alone may account for 14–18% of losses in total yield [38]. Importantly, pest pressure also varies considerably with climate and crop species. Much greater losses to insect pests can occur in developing countries [38], while cosmetic damage to fruit and vegetable crops can mean that 30% of production remains un-harvested in the UK [13]. The food security challenge for the present century is to increase global levels of food production without placing additional stress on the environment. Rational and informed control interventions, therefore, can help improve crop yields, minimise impacts and reduce costs, which require continued research and development. Moreover, we believe that there is great utility and value in exploring the use of robust control strategies, such as adaptive controllers, in pest management owing to the considerable uncertainties present. Indeed, the exact effects of intervention strategies, be it chemical pesticide or bio-control application, are not likely to be known. Furthermore, pest species' ranges are changing in response to climate change [5], meaning that farmers and other end-users are likely to have to manage novel pests.

Adaptive control is arguably one of the two main pillars of robust control theory, the other being H^∞ -control which traces its roots back to [47]; and is now the subject of numerous textbooks including, for instance, [50]. Adaptive control is a broad term, with no one single agreed definition, and dates back to the control of aircraft in the 1950s. The early history is discussed in the reviews [2, 37], for a more recent review of some aspects see [4]. Monographs on the subject include [3, 27]. Roughly, the idea behind simple adaptive control is that the control effort is a dynamic variable, governed by some prescribed set of equations which are tied to an output (a measured variable) of the to-be-controlled system. The output need not be the whole state x — in practice it may just be a portion of the state, such as knowledge of one stage-class. The power of simple adaptive controllers is threefold and is: (i) their ease of computation and thus implementation; (ii) their ability to achieve dynamic control objectives with a paucity of information, such as just a measured output, and; (iii) their global robustness properties, meaning that their theoretical efficacy is ensured for all systems from a class or *universum*, typically prescribed by certain structural properties. In particular, local robustness arguments, such as “sufficiently accurate” nominal parameter estimates, are not required for simple adaptive control schemes. In the control engineering jargon the term “self-tuning” has also been used instead of “adaptive”, and the nomenclature “simple” refers to the non-identifier property of the controller, meaning that it does not seek to update the underlying dynamical model over time or estimate model parameters. In natural resource management the word “adaptive” is less specific and generally means a feedback, see [46]. Two drawbacks of simple adaptive controllers are: first, their lack of optimality (which one could argue has been traded off to ensure their strong robustness properties) — in fact, robust performance metrics need not be included; and second, that they typically only ensure desired asymptotic dynamic behaviour — transient behaviour is not addressed.

In [18] we proposed simple adaptive feedback control as a theoretical approach for pest management. There we considered the feedback interconnection of the system of linear positive difference equations

$$x(t+1) = Ax(t) + w(t), \quad x(0) = x^0, \quad t = 0, 1, 2, \dots,$$

and the simple adaptive controller

$$u(t+1) = u(t) + \Phi(\|y(t)\|), \quad u(0) = u^0, \quad t = 0, 1, 2, \dots, \quad (1.2)$$

where $w(t)$ is the per time-step control *effect* of the control *action* or *effort* $u(t)$, the function Φ determines the rate of adaptation of u , and $y(t)$ is a measured variable available for feedback purposes. In [18] we used irreducibility and positivity arguments, and avoided classical assumptions associated with high-gain control to establish stability properties of the state variable $x(t)$.

The underlying idea in [18] is that the stabilising effect of control increases with increasing control effort, and if an infinite control effort is assumed to be stabilising, then, under certain assumptions, the adaptive control scheme (1.2) “finds” a finite control effort which is also stabilising. One of the benefits of an adaptive feedback control approach is that a finite stabilising control effort may not be known in practice. Two drawbacks of [18] are that it considered only the somewhat limited case that the underlying model is linear, when in fact most realistic pest models are nonlinear (that is, density-dependent). Furthermore, simple adaptive controllers are known to be susceptible to persistent measurement error, meaning that $y(t)$ in (1.2) is replaced by $y(t) + \varepsilon(t)$ for some measurement error term $\varepsilon(t)$. The results of [18] do not apply in this setting.

The present work non-trivially extends [18] by considering stabilisation of (1.1), facilitating the much more realistic situation wherein pest models can be nonlinear. Moreover, here we augment the adaptation law (1.2) with a prescribed level $\lambda > 0$ of tolerance, reminiscent of a so-called λ -tracker; see [23] or [24]. The addition of robustness with respect to measurement errors is important owing to the difficulties in accurately measuring pest populations. As is typical for adaptive feedback controllers, we seek to make as few assumptions pertaining to knowledge of (1.1) as possible, including the exact effect of control actions, our emphasis being on designing controllers which are robust to such sources of uncertainty. There are strong arguments, such as optimising performance, as to why other controllers are more appropriate when the to-be-controlled system is well modelled and the effects of control are well understood.

Our main result is Theorem 2.1 which gives stability and convergence properties of the adaptive feedback control system we consider. The main ideas are the same as those outlined above, namely that under the assumption that an infinite control effort is stabilising, the adaptive feedback control system we propose “finds” a finite stabilising control effort, in a sense we describe. The difference equations for the state x are a controlled version of (1.1) and, as discussed above, the control variable is determined by a simple adaptive feedback controller with measurement error tolerance.

We comment that Lur’e systems, and their stability properties, are hugely well-studied objects. In addition to the literature already cited, they arise in, for example, models for gene regulation [31], networked control systems [48], and classes of nonlinear discrete-time ARMA(X) models [40]. Yet another line of enquiry is the stabilisation of chaotic Lur’e systems, see [49] and the references therein. There are numerous approaches to the stability of Lur’e systems, including the use of Linear Matrix Inequalities (LMIs) and subsequent Lyapunov analysis, as well as so-called multiplier-based methods [7]. Other recent works include the use of sensitivity-type tools in the study of positive Lur’e systems arising in ecology, see [10]. We are not aware of other papers which specifically consider adaptive control of positive Lur’e systems, or Lur’e systems in the context of pest management, and the overlap between these cited works and the present contribution is minimal. Indeed, our analysis is crucially underpinned by comparison and monotonicity arguments from the theory of positive dynamical systems, or just positive systems; see, for instance [6, 21, 29]. Adaptive control of continuous-time positive linear systems is considered in [21, Chapter 15], and differences between the results there and [18] are discussed in [18, Remark 2.15].

The paper is organised as follows. In Section 2 we fully describe our model and state our main results. Examples are presented in Section 3 and we make some summarizing comments in Section 4. All mathematical proofs and supporting material for our numerical examples appear in the Appendix.

Notation: We collect mathematical notation and terminology. As usual, let \mathbb{N} , \mathbb{Z} and \mathbb{R} denote the sets of positive integers (natural numbers), integers, and real numbers, respectively. Furthermore, we set

$$\mathbb{Z}_+ := \{m \in \mathbb{Z} : m \geq 0\} = \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}.$$

For $n, m \in \mathbb{N}$, we let $\underline{n} := \{1, 2, \dots, n\}$, and \mathbb{R}^n and $\mathbb{R}^{n \times m}$ denote usual n -dimensional Euclidean space and the space of $n \times m$ matrices with real entries, respectively. The superscript T denotes both matrix and vector transposition. The symbol I denotes the identity matrix, the size of which is consistent with the context. For $M, N \in \mathbb{R}^{n \times m}$ with entries m_{ij} and n_{ij} , respectively, we write

$$M \leq N \quad \text{if } m_{ij} \leq n_{ij} \text{ for all } i \in \underline{n}, j \in \underline{m},$$

and $M < N$ if $M \leq N$ and $M \neq N$. We use the corresponding conventions for \geq and $>$, respectively. We let $\mathbb{R}_+^{n \times m}$ denote the set of nonnegative matrices, that is, $M \in \mathbb{R}_+^{n \times m}$ if $0 \leq M$. We call M positive if $0 < M$ and strictly positive if every entry of M is positive, noting that there are different conventions present in the academic literature for the term *positive matrix*. We recall that a nonnegative square matrix $M \in \mathbb{R}_+^{n \times n}$ is irreducible if, and only if, for each $i, j \in \underline{n}$ there exists $k \in \mathbb{N}$ such that the (i, j) -th entry of M^k is positive. We let $\rho(M)$ denote the spectral radius of $M \in \mathbb{R}^{n \times n}$.

Following the terminology in, for example [30, p.37], a norm $\|\cdot\|$ on \mathbb{R}^n is called *monotonic* if $0 \leq x \leq y$ implies that $\|x\| \leq \|y\|$. Every Euclidean norm is monotonic. Here we let $\|\cdot\|$ denote a (any) monotonic norm on \mathbb{R}^n , and the corresponding induced operator norm on $\mathbb{R}^{m \times n}$, respectively.

The symbol \mathcal{K} denotes the set of so-called comparison functions — continuous functions $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ which are strictly increasing and zero at zero; see, for example [33, p. 172].

Given a nonempty interval $J \subseteq \mathbb{R}_+$ and matrix-valued function of a nonnegative scalar variable M (whose domain includes J), we call M non-increasing on J if, for all $w_1, w_2 \in J$,

$$w_1 \leq w_2 \quad \Rightarrow \quad M(w_1) \geq M(w_2), \quad (1.3)$$

and similarly for non-decreasing on J . When the above holds with $J = \mathbb{R}_+$, then we just call M non-increasing, and analogously for non-decreasing. We let $\mathcal{F}(\mathbb{Z}_+, \mathbb{R}^n)$ denote the set of functions (sequences) $\mathbb{Z}_+ \rightarrow \mathbb{R}^n$, and for $x \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^n)$ we let x^+ denote the image of x under the left-shift operator, so that $x^+(t) = x(t+1)$. Further, we set

$$\|v\|_{\ell^\infty(t_1, t_2)} := \max \{ \|v(\tau)\| : t_1 \leq \tau \leq t_2 \} \quad \forall t_1, t_2 \in \mathbb{Z}_+, t_1 \leq t_2.$$

If $v \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^n)$ is bounded, then we set $\|v\|_{\ell^\infty} := \sup_{t \in \mathbb{Z}_+} \|v(t)\|$.

2 Main results

Consider the system of nonlinear positive difference equations (1.1), where $A \in \mathbb{R}_+^{n \times n}$, $B \in \mathbb{R}_+^{n \times m}$, $B_e \in \mathbb{R}_+^{n \times s}$, $E \in \mathbb{R}_+^{q \times n}$ for some $m, n, q, s \in \mathbb{N}$ and function $\Theta : \mathbb{R}_+^q \rightarrow \mathbb{R}_+^n$. For the applications we have in mind, the state variable x in (1.1) shall denote a stage-structured pest or weed population with n discrete stage-classes. The variable v takes values in \mathbb{R}^s and $B_e v$ denotes a (structured) external disturbance term, which we shall call a forcing term. Throughout the present work, we shall assume that not all of x is necessarily known to the modeller, and thus not available to inform feedback control strategies. There are numerous reasons, such as practicality, feasibility or cost, as to why this may be the case. However, we assume that a measured variable

$$y = Cx + D_e v,$$

is known, which corresponds to some observed portion of the state Cx , possibly subject to measurement error denoted $D_e v$. We call y the output. The term C satisfies $C \in \mathbb{R}_+^{p \times n}$ for some $p \in \mathbb{N}$, so that $y \in \mathbb{R}^p$, and we shall always assume that C has no zero rows, as zero rows correspond to a trivial (zero) measurement, and are inappropriate.

We assume that the control action acts on the dynamics for x via the matrices A , B and E , and the function Θ , as a possibly nonlinear and unknown, but non-increasing, function of an applied control effort, denoted u . The control need not act on all of these terms, but it is assumed to act on at least one. We propose an adaptation law for u which, roughly, means that u increases so long as the norm of the output is no smaller than a prescribed tolerance. The larger the control effort u , the more efficacious the control effect, which in turn further reduces x , and so y as well.

To make the above overview concrete, we propose an adaptation law for u which results in the following closed-loop simple adaptive feedback control system:

$$\left. \begin{aligned} x^+ &= A(u)x + B(u)\Theta(u, E(u)x) + B_e v, & x(0) &= x^0, \\ y &= Cx + D_e v, \\ u^+ &= u + \Psi(\max\{0, \|\phi(y)\| - \lambda\}), & u(0) &= u^0, \end{aligned} \right\} \quad (2.1)$$

where now A , B and E are matrix-valued functions of the applied control effort u , and Θ may also depend on u . Here $x^+(t) = x(t+1)$ for all nonnegative integers t . It is assumed that the term B_e is independent of u . The function $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ belongs to \mathcal{K} , and $u^0 \in \mathbb{R}_+$ and $\lambda \geq 0$ are design parameters. These represent the rate of adaptation of u , its initial value and a forcing tolerance level, respectively. The function ϕ is discussed below.

The model (2.1) includes as a special case the situation $B_e v = v_1$ and $D_e v = v_2$ by taking

$$B_e = (I, 0), \quad D_e = (0, I), \quad \text{and} \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

For all $v \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^s)$ and all initial conditions $(x^0, u^0) \in \mathbb{R}_+^n \times \mathbb{R}_+$, throughout the work we let (x, u) denote the unique solution of (2.1). It is clear that such a unique solution exists. In the case that $\Theta(\cdot, 0) = 0$, $v = 0$ and $\phi(0) = 0$ (which shall be the case), then $(0, u^0)$ is clearly a constant solution of (2.1).

We shall assume throughout that $B_e v(t) \geq 0$, which ensures that $x(t) \geq 0$ for all $t \in \mathbb{Z}_+$. Some comments on the situation wherein $B_e v(t) \geq 0$ is relaxed appear on p. 6. As a (noisy) measurement of nonnegative state variables of a biological process, we would expect the output y to be nonnegative valued. It is rather restrictive to assume that the noise $D_e v$ is nonnegative valued, and thus as feedback we saturate the output y , that is, use $\phi(y)$ where $\phi : \mathbb{R}^p \rightarrow \mathbb{R}^p$ is defined componentwise by

$$(\phi(z))_i := \max\{0, z_i\} \quad \forall z \in \mathbb{R}^p \quad \forall i \in \underline{p}, \quad (2.2)$$

and where z_i denotes the i -th component of z . The rationale for including ϕ is that negative components of $y(t)$ have been caused by noise, as $Cx(t) \geq 0$ for all t , and are thus artefactual and should not lead to an increase in control effort u . Note that if $D_e v \geq 0$, then $\phi(y) = y$.

The tolerance λ seeks to add robustness of (2.1) with respect to the unknown and potentially persistent forcing v — Theorem 2.1 below states that (2.1) has certain stability properties for all sufficiently small $\|v\|_{\ell^\infty}$, as a function of λ . Hence, the larger λ is, the larger the set of potential forcing terms which can be accommodated (which is desirable), but the cost is that the resulting “true” output $y - D_e v$ may also be larger (which is undesirable).

We record the following structural and nonnegativity properties associated with the adaptive feedback control system (2.1):

- (M1) $A : \mathbb{R}_+ \rightarrow \mathbb{R}_+^{n \times n}$, $B : \mathbb{R}_+ \rightarrow \mathbb{R}_+^{n \times m}$, $E : \mathbb{R}_+ \rightarrow \mathbb{R}_+^{q \times n}$ for $m, n, q \in \mathbb{N}$ are non-increasing functions in the sense of (1.3) and A is continuous;
- (M2) $\Theta : \mathbb{R}_+ \times \mathbb{R}_+^q \rightarrow \mathbb{R}_+^m$ satisfies $\Theta(\cdot, 0) = 0$, is continuous in its second variable for each fixed first variable, and $\Theta(w_1, \cdot) \leq \Theta(w_2, \cdot)$ for all $w_1 \geq w_2$;
- (M3) for all $w \geq 0$, there exists $\Delta = \Delta(w) \in \mathbb{R}_+^{m \times q}$ such that

$$\Theta(w, z) \geq \Delta z \quad \forall z \in \mathbb{R}_+^q,$$

and $A(w) + B(w)\Delta E(w)$ is irreducible;

- (M4) $C \in \mathbb{R}_+^{p \times n}$ for some $p \in \mathbb{N}$ and C has no zero rows;
- (M5) $\Psi \in \mathcal{K}$.

We shall use the symbol **(M)** to denote the collected properties **(M1)**–**(M5)**. By way of commentary, assumptions **(M1)** and **(M2)** contain structural properties of the linear and nonlinear data in (2.1), respectively. Assumptions **(M4)** and **(M5)** relate to the measured variable y and the difference equation for the control variable u in (2.1), respectively. Finally, assumption **(M3)** is a growth condition for Θ and provides a coupling condition between the nonlinear and linear parts of the dynamics for x in (2.1). We shall use it to couple the unknown variable x to the known observations $y = Cx + D_e v$. It is trivially satisfied with $\Delta = 0$ if $A(w)$ itself is irreducible for every $w \geq 0$. In the case $m = q = 1$, a sufficient condition for the existence of the lower bound Δ is that

$$\Delta := \inf_{z > 0} \frac{\Theta(w, z)}{z} > 0 \quad \forall w \geq 0.$$

We next introduce our stabilisability assumption:

(S) there exist $\sigma \geq 0$ and $\Sigma \in \mathbb{R}_+^{m \times q}$ such that

$$\Theta(\sigma, z) \leq \Sigma z \quad \forall z \in \mathbb{R}_+^q \quad \text{and} \quad \rho(A(\sigma) + B(\sigma)\Sigma E(\sigma)) < 1.$$

Roughly speaking, this means that there is a large enough control effort σ so that the linear system with that control, $x^+ = (A(\sigma) + B(\sigma)\Sigma E(\sigma))x$, is exponentially stable and provides an upper bound for x given by the nonlinear difference equation in (2.1) but with fixed control $u = \sigma$.

Our main result is the following theorem which describes stability and convergence properties of the adaptive feedback control system (2.1).

Theorem 2.1. *Consider the adaptive feedback control system (2.1) with $\lambda > 0$, and assume that (M) and (S) hold. Then, there exists $\alpha = \alpha(\lambda) > 0$ such that, for all $(x^0, u^0) \in \mathbb{R}_+^n \times \mathbb{R}_+$ and all $v \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^s)$ with $B_e v \geq 0$ and*

$$\max \{ \|B_e v\|_{\ell^\infty}, \|D_e v\|_{\ell^\infty} \} < \alpha, \tag{2.3}$$

the following statements apply to the solution (x, u) of (2.1):

- (i) u is bounded, and hence convergent;
- (ii) y is bounded and satisfies $\max \{ 0, \|\phi(y(t))\| - \lambda \} \rightarrow 0$ as $t \rightarrow \infty$;
- (iii) x is bounded;
- (iv) if $\tau \in \mathbb{Z}_+$ is such that $u(\tau) \geq \sigma$, then there exist $M > 0$ and $\gamma \in (0, 1)$ such that

$$\|x(t + \tau)\| \leq M(\gamma^t \|x(\tau)\| + \|B_e v\|_{\ell^\infty(\tau, \tau+t-1)}) \quad \forall t \in \mathbb{N}. \tag{2.4}$$

The constants M and γ in (2.4) depend on the model data in (2.1) (including on the $u(\tau) \geq \sigma$), but are independent of x^0 , u^0 and v .

If instead we consider (2.1) with $\lambda = 0$ and $v = 0$, and replace (2.3) by

$$\sum_{j \in \mathbb{Z}_+} \Psi(\delta \varepsilon^j) < \infty \quad \forall \delta > 0 \quad \forall \varepsilon \in (0, 1), \tag{2.5}$$

then statements (i)–(iv) still hold and, additionally,

- (v) $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

We provide some commentary on the above theorem — its hypotheses and conclusions.

Model assumptions. The model assumptions (M) are structural and positivity properties imposed on the class of models to be controlled, and are arguably not physically restrictive. They are robust with respect to parametric (that is, the values of A , B , E and Θ) and structural (such as the various dimensions n , m and p) uncertainty in (1.1), which is desirable aim of the present work.

Assumption (M3) is a growth condition and the irreducibility requirement, roughly speaking, means that every state component experiences the same rate of growth or decline. Consequently, although Cx perhaps only measures a small portion of the state, that alone is sufficient to gauge the rate with which *all* the state variables are changing. Irreducibility of the closed-loop dynamics is a key structural assumption in [18], and has been argued as a reasonable assumption for empirically derived ecological models in [43].

Theorem 2.1 is still true without the assumption that $B_e v(t) \geq 0$, provided that $x(t) \geq 0$ for every $t \in \mathbb{Z}_+$. This is an admittedly unsatisfactory assumption, however, as describing the possible $B_e v(t)$ which maintain the physical requirement that $x(t) \geq 0$ for x determined by the nonlinear difference equation (2.1) seems subtle. A treatment when x is specified by a forced system of *linear* difference equations is given in [17].

The growth condition (2.5) in words states that Ψ respects the summability of convergent geometric series. It need not be satisfied for general $\Psi \in \mathcal{K}$. It is not an overly restrictive assumption in practical applications, as it is satisfied if Ψ is locally Lipschitz continuous at zero, or Hölder continuous (with any positive exponent), for instance. This assumption played a key role in [18], but can be obviated here when (2.1) contains a positive forcing tolerance $\lambda > 0$.

Stabilisability assumptions. Another key hypothesis of Theorem 2.1 is the stabilisability assumption **(S)** which, as our examples suggest, is a reasonable assumption in ecologically motivated scenarios. Possibly the simplest situation in which **(S)** holds is if $\rho(A(w)) < 1$ and Σ may be chosen sufficiently small. The condition **(S)** contains a linear bound and an eigenvalue condition, the latter of which grows in size with the size of A . The following lemma contains a sufficient condition for **(S)**.

Lemma 2.2. *Consider (2.1) and assume that **(M)** holds. If there exist $\sigma \geq 0$, $\Sigma \in \mathbb{R}_+^{m \times q}$, strictly positive $\zeta \in \mathbb{R}_+^q$ and $\kappa \in (0, 1)$ such that $\rho(A(\sigma)) < 1$ and*

$$\Theta(\sigma, z) \leq \Sigma z \quad \forall z \in \mathbb{R}_+^q \quad \text{and} \quad \zeta^T \mathbf{H}(\sigma) \Sigma \leq \kappa \zeta^T, \quad (2.6)$$

where

$$\mathbf{H}(w) := E(w)(I - A(w))^{-1}B(w) \quad \forall w \geq \sigma, \quad (2.7)$$

then assumption **(S)** holds with σ and Σ as above.

In control theory jargon, the matrix $\mathbf{H}(w)$ in (2.7) is the steady state gain of the linear control system specified by the triple $(A(w), B(w), E(w))$. The second inequality in (2.6) is a so-called weighted small-gain condition, and is used extensively in [19] in the stability analysis of systems of positive Lur'e difference equations. The matrix $\mathbf{H}(\sigma)\Sigma$ is order $q \times q$, and indeed is scalar if $q = 1$, in which case the second condition in (2.6) reduces to the scalar inequality $\mathbf{H}(\sigma)\Sigma < 1$. Therefore, in certain cases, such as when $m, q \ll n$, verifying (2.6) may be easier than verifying **(S)** directly.

Establishing whether the weighted small-gain condition (2.6) is satisfied evidently requires knowledge of the assumed unknown Θ and \mathbf{H} (the latter depending on A , B and E), although knowledge of suitable $\sigma \geq 0$, strictly positive $\zeta \in \mathbb{R}_+^q$, and $\kappa \in (0, 1)$ as in (2.6) is not required to implement (2.1) or for Theorem 2.1 to apply. Since \mathbf{H} is non-increasing (see Lemma C.1), if it is known, perhaps by other or *a fortiori* arguments, that $\rho(A(w)) < 1$ and $\mathbf{H}(w) \rightarrow 0$ as $w \rightarrow \infty$, and that Θ admits the linear bound in (2.6) uniformly in its first variable, then (2.6) holds.

External forcing. In the context of robustness of (2.1) with respect to the additive external forcing term v , which models disturbances into the state dynamics and measurement errors, the third key hypothesis of Theorem 2.1 is that $\|v\|_{\ell^\infty}$ is small enough. Specifically, the estimate (2.3) is a hypothesis of Theorem 2.1, where the threshold α is a function of the tolerance λ . This is admittedly a local robustness property. Thus, Theorem 2.1 contains some robustness in this sense — at least qualitatively if not quantitatively. Indeed, one drawback of Theorem 2.1 is that α is not explicitly constructed¹. That written, the proof of Theorem 2.1 does show that the condition (2.3) may be relaxed to

$$\max \left\{ \limsup_{t \rightarrow \infty} \|B_e v(t)\|, \limsup_{t \rightarrow \infty} \|D_e v(t)\| \right\} < \alpha, \quad (2.8)$$

and the conclusions of the theorem are still valid. Moreover, in the simpler case that $B_e v = 0$, that is, only measurement noise is present, then the proof of Theorem 2.1 shows that $\alpha(\lambda)$ can be chosen to equal λ . Moreover, the threshold $\alpha(\lambda) = \lambda$ is the largest possible threshold which ensures that the conclusions of Theorem 2.1 can be expected to hold. Indeed, if $D_e v \geq 0$ and

$$\limsup_{t \rightarrow \infty} \|D_e v(t)\| > \lambda,$$

then it is clear from (2.1) and the definition of y that u is unbounded, which is very undesirable.

Finally, straightforward adjustments to the proof of Theorem 2.1 demonstrate that statements (i)–(v) still hold if the saturation function ϕ is omitted from (2.1).

The next two examples seek to illustrate the role of the hypotheses and consequences of the conclusions of Theorem 2.1, respectively.

Example 2.3. We claim that the stabilisability assumption **(S)**, or coupling condition **(M3)**, cannot be dropped from the hypotheses of Theorem 2.1 in general. The simplest way in which **(S)** fails is if $\rho(A(w)) \geq 1$ for all $w \geq 0$. For the sake of simplicity, assume that $v = 0$, that

$$\Lambda := \lim_{w \rightarrow \infty} A(w),$$

¹For more information, see the estimate (B.11) and supporting discussion.

exists and is irreducible, and that $\lambda > 0$ is sufficiently small. Then with $\Theta = 0$, it follows that x admits the estimate $x^+ = A(u)x \geq \Lambda x$. The continuity of ρ gives $\rho(\Lambda) \geq 1$ which, when combined with the irreducibility of Λ , yields that

$$\limsup_{t \rightarrow \infty} (\|y(t)\| - \lambda) > 0,$$

and hence u diverges. If, in fact $\rho(\Lambda) > 1$, then x and $y = Cx$ diverge as well.

The irreducibility properties associated with (2.1), formulated in the coupling condition **(M3)**, must also hold, unless further assumptions are placed on the observation matrix C . As an example, consider (2.1) with $n = 2$, $v = 0$, $\lambda = 0$ and

$$A(u) := \begin{pmatrix} a_1(u) & 0 \\ 0 & a_2(u) \end{pmatrix}, \quad B(u) := \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad E(u) = C := (0 \quad 1), \quad \Theta := 0.$$

(The terms B_e and D_e are unimportant when $v = 0$.) Here $a_1, a_2 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous, non-increasing functions. We assume that there exists some large $u^\dagger > 0$ such that $a_1(u^\dagger), a_2(u^\dagger) < 1$, so that assumption **(S)** holds with $\sigma = u^\dagger$ and $\Sigma = 0$. Moreover, since $\Theta = 0$, assumption **(M3)** is not satisfied as $A(w)$ is reducible (when $\Theta = 0$, the only possible candidate for Δ in **(M3)** is $\Delta = 0$).

Under the above hypotheses, the adaptive feedback control system (2.1) reduces to the three scalar difference equations

$$x_k^+ = a_k(u)x_k \quad k \in \{1, 2\} \quad \text{and} \quad u^+ = u + \Psi(\|y\|) = u + \Psi(x_2). \quad (2.9)$$

In light of (2.9), by choosing $0 < w_2 < w_1 < u^\dagger$ and functions a_1, a_2 additionally such that

$$a_2(w_2) < 1 \quad \text{and} \quad a_1(w_1) > 1,$$

(informally, satisfied if a_2 decays more quickly than a_1), we can then choose x^0, u^0 and Ψ such that $x_2(t) \rightarrow 0$ and $u(t) \rightarrow w \in [w_2, w_1]$ as $t \rightarrow \infty$. However, since $a_1(w_1) > 1$, it follows that

$$x_1^+ = a_1(u)x_1 \geq a_1(w_1)x_1,$$

leading to $x_1(t) \rightarrow \infty$ as $t \rightarrow \infty$. □

Example 2.4. Theorem 2.1 does not enforce that $x(t) \rightarrow 0$ as $t \rightarrow \infty$ when $\lambda \neq 0$. Indeed, consider (2.1) with $B_e v = 0$, $\Theta = 0$, sufficiently large $\lambda > 0$ and bounded $D_e v$. If $u_* \geq 0$ and $x_* \in \mathbb{R}_+^n$ are such that

$$A(u_*)x_* = x_*, \quad (2.10)$$

(in particular, if $\rho(A(u_*)) = 1$ and $x_* \in \mathbb{R}_+^n$ is a corresponding eigenvector) and v, λ and x_* together have the property that

$$\sup_{t \in \mathbb{Z}_+} \|\phi(Cx_* + D_e v(t))\| - \lambda < 0,$$

then (x_*, u_*) is a constant solution of (2.1). In particular, there are constant solutions of (2.1) with non-zero state component. We see that statements (i)–(iii) of Theorem 2.1 hold, but that the hypotheses of statements (iv) and (v) are not satisfied. Indeed, necessarily it must be the case that $u_* < \sigma$. Further, the equality (2.10) does not violate assumption **(S)**. Instead, in other words, we conclude that u_* is not a stabilising control level in the sense of (2.4). We note that the choice that $\Theta = 0$ in this example is not restrictive, and is made to simplify the exposition — examples illustrating the same features of (2.1) can be constructed with non-zero Θ . □

As a corollary of Theorem 2.1, we obtain a result for linear positive systems by taking $\Theta = 0$ in (2.1). The special case of (2.1) with $\Theta = 0$, $D_e v = 0$ and $\lambda = 0$ was considered in [18].

Corollary 2.5. *Consider (2.1) in the special case that $\Theta = 0$ and $A(w)$ is irreducible for every $w \geq 0$. If **(M)** holds and $\rho(A(\sigma)) < 1$ for some $\sigma \geq 0$, then the conclusions of Theorem 2.1 apply.*

Returning to (2.1), we comment that unless $A(w)$ is itself irreducible for every $w \geq 0$, a non-zero Δ in assumption **(M3)** is required. This in turn necessitates that $\Theta(w, \cdot)$ is globally linearly bounded from below, and hence unbounded, which need not be the case in many ecological models. Therefore, roughly, our next result allows for Θ which satisfy a weaker linear lower bound condition, provided that Θ satisfies an additional growth condition.

Proposition 2.6. Consider the adaptive feedback control system (2.1), and assume that properties **(M)** and **(S)** hold, with **(M3)** replaced by

(M3)' There exist $\Gamma_1 \in \mathbb{R}_+^{m \times q}$ and $\Gamma_2 \in \mathbb{R}_+^m$ such that $\rho(A(0) + B(0)\Gamma_1 E(0)) < 1$ and

$$\Theta(0, z) \leq \Gamma_1 z + \Gamma_2 \quad \forall z \in \mathbb{R}_+^q.$$

Further, for every $w, k \geq 0$, there exists $\Delta = \Delta(w, k) \in \mathbb{R}_+^{m \times q}$, such that

$$\Theta(w, z) \geq \Delta z \quad \forall z \in \mathbb{R}_+^q \quad \text{with} \quad \|z\| \leq k,$$

and $A(w) + B(w)\Delta E(w)$ is irreducible.

Then the conclusions of Theorem 2.1 hold.

In words, hypothesis **(M3)'** allows for the function $\Theta(w, \cdot)$ to be linearly bounded from below only on bounded sets, provided that a boundedness/growth condition holds. Possibly the simplest situation wherein the first two properties of hypothesis **(M3)'** are satisfied is when $\rho(A(0)) < 1$ and the function $\Theta(0, \cdot)$ is bounded (so that Γ_1 may be chosen above to equal zero).

The assumption **(M3)'** can be further relaxed by replacing 0 which appears as an argument of $A(0)$, $B(0)$ etc. by some $\sigma_1 > 0$, provided that, roughly, for all $w \in [0, \sigma_1]$ a lower bound of the form

$$A(w)z + B(w)\Theta(w, E(w)z) \geq \Lambda(w)z \quad \forall z \in \mathbb{R}_+^n,$$

holds, for some irreducible $\Lambda(w) \in \mathbb{R}_+^{n \times n}$. For the sake of brevity, we do not give the details.

In a discrete-time control system the update law for the state determines the state at the next time-step in terms of the current state and the control, as well as any external variables. The formulation of the right hand side depends on *when*, throughout the time-step, these processes take place, particularly if the processes do not commute (which is likely to be the situation in the matrix-valued case). We now discuss how the model (2.1) may capture the timings of control actions.

Specifically, we assume that the quantity x is governed by (1.1) when uncontrolled, and that control action reduces the state proportionally via multiplication with $\Gamma(u)$, modelled as

$$x^+ = \Gamma(u)(A_0 x + B_0 \Theta_0(E_0 x)) + B_e v, \quad x(0) = x^0, \quad (2.11)$$

for some fixed $A_0 \in \mathbb{R}_+^{n \times n}$, $B_0 \in \mathbb{R}_+^{n \times m}$, $E_0 \in \mathbb{R}_+^{q \times n}$ with $m, n, q \in \mathbb{N}$. The function Γ is not expected to be known exactly, but is expected to have certain qualitative properties, captured as

(D) $\Gamma : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ is zero on the off-diagonal and its diagonal components are either unity, or non-increasing continuous functions $\mathbb{R}_+ \rightarrow (0, 1]$ which are unity at zero.

Consequently, Γ as in **(D)** satisfies $\Gamma(0) = I$, and the difference equation (2.11) coincides with the original model (1.1) when $u = 0$. In a biological context, the motivation for the model (2.11), and the qualitative properties of Γ , is that the control action of multiplication by $\Gamma(u)$, corresponding to proportional removal or reduction, occurs *after* the other biological processes which take place over each time-step. The external signal $B_e v$ is assumed independent of the control action.

The difference equation (2.11) is a special case of the dynamics for x in (2.1), with functions A , B , and E given by

$$A(w) := \Gamma(w)A_0, \quad B(w) := \Gamma(w)B_0 \quad \text{and} \quad E(w) := E_0.$$

Here E and $\Theta := \Theta_0 : \mathbb{R}_+^q \rightarrow \mathbb{R}_+^m$ are independent of w .

If, in fact, the control action of proportional removal is assumed to occur *before* the other biological processes over a time-step, then a more appropriate model is to replace the update law for x in (2.11) by

$$x^+ = A_0 \Gamma(u)x + B_0 \Theta_0(E_0 \Gamma(u)x) + B_e v, \quad x(0) = x^0, \quad (2.12)$$

where the interpretation of the terms in (2.12) is otherwise the same as that in (2.11).

In a given situation, it may not be clear whether (2.11) or (2.12) is the most appropriate model. However, the hypotheses **(M2)**, **(M4)** and **(M5)** for (2.1) are independent of whether the x dynamics are specified by (2.11) or (2.12). Further, our next result states that the remaining key hypotheses of Theorem 2.1, namely, properties **(M1)**, **(M3)** and **(S)**, are the same for the models (2.11) and (2.12).

Proposition 2.7. *Consider the adaptive control system (2.1) in the special case of (2.11) or (2.12), where Γ satisfies **(D)**. The hypotheses **(M1)**, **(M3)** and **(S)** hold in the context of (2.11) if, and only if, they hold in the context of (2.12).*

We conclude this section by investigating the limiting control effort $u_\infty := \lim_{t \rightarrow \infty} u(t)$ (the so-called limiting gain) of (2.1), which is guaranteed to exist by statement (i) of Theorem 2.1. We shall consider the special case that $v = 0$ and $\lambda = 0$. In particular, Theorem 2.1 does not address any properties of u_∞ , but much attention in the adaptive control literature has been devoted to establishing whether the limiting gain generated by (2.1) is itself stabilising, that is, investigating the stability properties of the so-called “limit system”. Recall that in [18], which considers the linear positive case, the limiting gain is exponentially stabilising (under some structural assumptions), for all non-zero initial conditions x^0 . Without further assumptions on the nonlinear system (1.1) considered presently, however, the limiting gain need not be stabilising, and only local results should be expected.

Our next result broadly shows that the limiting gain fulfills a necessary condition for local stability of the limit system. As we shall discuss, a slight strengthening of this necessary condition for stability is in fact sufficient for local exponential stability.

Proposition 2.8. *Imposing the notation and assumptions of Theorem 2.1 in the case that $v = 0$ and $\lambda = 0$, and for given $(x^0, u^0) \in \mathbb{R}_+^n \times \mathbb{R}_+$ with $x^0 \neq 0$, let $u_\infty \in \mathbb{R}_+$ denote the limit of the control variable u in (2.1). In general, u_∞ depends on x^0 and u^0 . Suppose that there is some nonempty neighbourhood of zero in \mathbb{R}_+^q , denoted U , and $\Gamma \in \mathbb{R}_+^{m \times q}$ such that*

$$\Gamma z \leq \Theta(u_\infty, z) \quad \forall z \in U, \quad (2.13)$$

and $G := A(u_\infty) + B(u_\infty)\Gamma E(u_\infty)$ is irreducible. Then $\rho(G) < 1$.

The hypothesis that $x^0 \neq 0$ is required in the above proposition, as the unique solution of (2.1) is $(0, u^0)$ when $v = 0$ and $x^0 = 0$. Note that the condition (2.13) is independent of u_∞ if Θ in fact only depends on its second argument. In the case that $m = q = 1$ and $\Theta(u_\infty, \cdot)$ is differentiable at zero, the condition (2.13) is satisfied if $\Gamma \geq 0$ is such that

$$\Gamma < \frac{\partial \Theta}{\partial z}(u_\infty, z)|_{z=0}.$$

The inequality $\rho(G) \leq 1$ is necessary for zero to be a stable equilibrium of the limit system

$$x^+ = A(u_\infty)x + B(u_\infty)\Theta(u_\infty, E(u_\infty)x), \quad (2.14)$$

for Θ which satisfy (2.13). If, in fact, Θ and $\Gamma \in \mathbb{R}_+^{m \times q}$ are such that there is some $\Lambda \in \mathbb{R}_+^{m \times q}$ with the property that

$$\Gamma z \leq \Theta(u_\infty, z) \leq (\Gamma + \Lambda)z \quad \forall z \in U,$$

and $\rho(G + B\Lambda E) < 1$, then zero is a locally exponentially stable equilibrium of the limit system (2.14).

Therefore, the value of the above result is that, under the above conditions, a single experiment can determine (in other words, learn) a stabilising control level u_∞ .

3 Examples

We illustrate our results through two worked examples. All following simulations were performed in Mathworks MATLAB 2020a, and in the sequel “randomly generated numbers” are actually pseudo-randomly generated.

3.1 Scalar difference equations

We consider (2.1) in the simple case that the underlying model for x is scalar, often called a (discrete) map in the difference equations literature. As is well-known, difference equations have been proposed as suitable models for species with non-overlapping generations; see, for instance [22, 35]. A motivating biological example is the economically important pest, the Colorado potato beetle (*Leptinotarsa decemlineata*), which is described in detail in [1] and, to quote, “can completely destroy potato crops”.

3.1.1 Introducing the model

Forced (nonlinear) difference equations take the form

$$x^+ = g(x) + v_1, \quad (3.1)$$

where $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is some function and v_1 is an external signal. Including a control term updates the model (3.1) to

$$x^+ = f(u, x) + v_1, \quad (3.2)$$

which falls within the scope of (1.1) with $n = 1$, $A = 0$, $B = E = 1$, $B_e = (1 \ 0)$, and $\Theta = f$. In the absence of control ($u = 0$), the models (3.1) and (3.2) coincide, captured by the equality $g(z) = f(0, z)$ for all $z \geq 0$. In this scalar case, we assume that the output y is equal to the state, up to measurement error, leading to $y = x + v_2$ so that $C = 1$ and $D_e = (0 \ 1)$.

3.1.2 Verifying hypotheses

We discuss the key model hypotheses **(M)** and **(S)** in the context of the scalar model (3.2). Assumptions **(M1)** and **(M4)** are satisfied, and assumption **(M5)** relates to the difference equation for u , and so is independent of (3.2). Assumptions **(M2)** and **(M3)**/**(M3)'** relate to the nonlinear term f . The former is satisfied if $f(\cdot, 0) = 0$, f is continuous in its second variable for each fixed first variable, and $f(w_1, \cdot) \leq f(w_2, \cdot)$ for all $w_1 \geq w_2 \geq 0$.

In these applied settings, assumption **(M3)'** is often more relevant than **(M3)**, and the former is satisfied here if $f(0, z) \leq \gamma_1 z + \gamma_2$ for some $\gamma_1 \in [0, 1)$ and any $\gamma_2 \in \mathbb{R}_+$ and, for each $w \geq 0$ and $\gamma > 0$, there exists $\delta > 0$ such that

$$f(w, z) \geq \delta z \quad \forall z \in [0, \gamma].$$

In the typical setting that $f(w, \cdot)$ is continuously differentiable, the above condition is satisfied if $f(w, \cdot)$ is positive definite and, for each $w \geq 0$,

$$\frac{\partial f}{\partial z}(w, 0) > 0.$$

In the context of (3.2), assumption **(S)** requires $\sigma \geq 0$ and $\kappa \in (0, 1)$ such that

$$f(\sigma, z) \leq \kappa z \quad \forall z \geq 0. \quad (3.3)$$

To simplify the condition (3.3) further requires bespoke assumptions on f .

3.1.3 A numerical example

As a concrete example, we consider (3.1) where g is the so-called Hassell map $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$g(z) = \frac{\ell z}{(1 + az)^\beta} \quad \forall z \geq 0. \quad (3.4)$$

Here $\ell = g'(0)$ (derivative from the right) is the “finite net rate of increase”² and “ a and β are constants defining the density dependent feedback term” (quotes from [22, p.472]). In our numerical simulations, we take

$$\ell = 75, \quad \beta = 30, \quad a = 7 \times 10^{-5},$$

which are based on the values from [22, Fig. 1, Table 2] for the Colorado potato beetle (*Leptinotarsa decemlineata*).

To arrive at the controlled model (3.2) requires some assumptions on the effect of the control effort u . We suppose that the application of control effort u leads to the removal of the proportion $p(u)$ from the population, and gives rise to the model (3.2) with

$$f(w, z) = (1 - p(w))g(z) \quad \forall w, z \geq 0. \quad (3.5)$$

²The symbol λ is used in [22] instead of ℓ . We are using ℓ as λ denotes the output threshold in (2.1).

Natural assumptions on the function $p : \mathbb{R}_+ \rightarrow [0, 1]$ are that it is continuous, non-decreasing with $p(0) = 0$. In light of our choice of g in (3.4), condition (3.3) is satisfied if there exists $\sigma > 0$ such that

$$(1 - p(\sigma))\ell < 1. \quad (3.6)$$

In the following numerical simulation, we consider the adaptive feedback control system (2.1) applied to the model (3.2), specified by (3.4) and (3.5), with initial data and $\Psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$x^0 = 1000, \quad u^0 = 0, \quad \text{and} \quad \Psi(w) = 400 \ln(1 + w) \quad \forall w \geq 0. \quad (3.7)$$

The function Ψ is based on that used in [18, Section 4]. We assume that the control efficacy function is given by

$$p(w) = 1 - e^{-\alpha w} \quad w \geq 0 \quad \text{where} \quad \alpha := 8 \times 10^{-5}.$$

We set v_1 as a uniformly randomly generated number between 0 and 50, modelling a low-level of per time-step immigration. The measurement error v_2 is assumed to be proportional to the true output x , meaning that

$$y = x + v_2 = x + \varepsilon x = (1 + \varepsilon)x,$$

where $\varepsilon(t) \in [-0.05, 0.05]$ is determined randomly for every $t \in \mathbb{Z}_+$, corresponding to an at-most 5% per time step measurement error. Note that with these assumptions on v_2 , it follows that $y(t)$ is always nonnegative, and so $\phi(y(t)) = y(t)$ in (2.1).

The results of our first numerical simulations are contained in Figure 3.1. Three simulations are performed corresponding to the thresholds

$$\lambda_1 = 6000, \quad \lambda_2 = 2000 \quad \text{and} \quad \lambda_3 = 200. \quad (3.8)$$

The data in (3.7) and (3.8) have been chosen somewhat arbitrarily to illustrate the qualitative conclusions of Theorem 2.1. Figure 3.1a plots the observed population abundance $y(t)$ against time t — using a logarithmic axis for the vertical axis for ease of inspection. In each case, it is observed that

$$\max \{y(t) - \lambda, 0\} \rightarrow 0, \quad (3.9)$$

as time increases. In words, the observed pest population is asymptotically controlled to the desired target level. This is in contrast to the uncontrolled population, given by (3.1), which is plotted in a black dotted line in Figure 3.1a, and is seen to fluctuate. The control efficacy $p(u(t))$ is plotted against time t in Figure 3.1b, and is seen to increase from zero initially and to converge over time. Remember that in this illustrative example u does not contain units, and so arguably $p(u)$ is more informative than u .

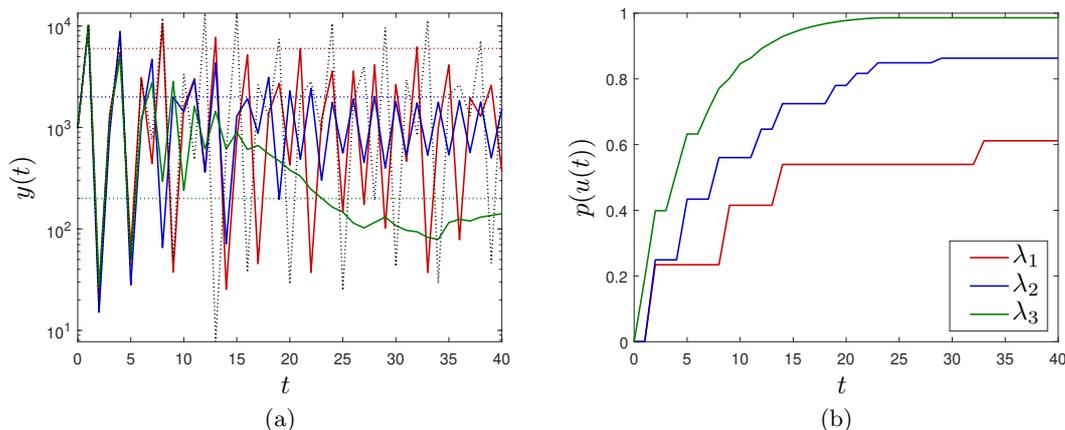


Figure 3.1: Numerical simulations from the example in Section 3.1 for varying λ . Model data are detailed in Section 3.1.3. In panel (a) the dotted horizontal lines are the λ_i levels, and the black dotted line corresponds to the uncontrolled model (3.1).

With the above assumptions, all the hypotheses of Proposition 2.6 are satisfied, apart from knowledge of whether the forcing tolerance threshold conditions (2.3) or (2.8) hold. The following heuristic argument

suggests that (2.8) may well be satisfied. Since v_1 is “small” relative to x and y , the proof of Theorem 2.1 shows that $\alpha(\lambda) \approx \lambda$, and so the condition (2.8) is likely to be satisfied if

$$\limsup_{t \rightarrow \infty} |v_2(t)| = 0.05 \limsup_{t \rightarrow \infty} |x(t)| < \lambda.$$

This above inequality appears to be satisfied in each simulation.

We illustrate next the asymptotic stabilisation of the scheme (2.1) in the case that $\lambda = 0$ and $v = 0$, guaranteed by statement (v) of Theorem 2.1. For which purpose, we again simulate (2.1) specified by (3.2), (3.4) and (3.5), only now with $\lambda = 0$ and $v = 0$. Three simulations are performed corresponding to the functions $\Psi = \Psi_j \in \mathcal{K}$ given by

$$\Psi_1(w) = 400 \ln(1 + w), \quad \Psi_2(w) = 2w \quad \text{and} \quad \Psi_3(w) = \frac{1200w}{1 + w} \quad \forall w \geq 0.$$

These functions are again based on those used in [18, Section 4]. Each is Lipschitz, and so the growth condition (2.5) is satisfied. All other model parameters are as above. The results of our second numerical simulations are contained in Figure 3.2. By Proposition 2.6, statement (v) of Theorem 2.1 applies, and we see that $x(t) = y(t) \rightarrow 0$ as $t \rightarrow \infty$.

Noting that $g'(0) = \ell$, the hypotheses of Proposition 2.8 are satisfied with $\Gamma := \ell - \varepsilon$, for every $\varepsilon \in (0, \ell)$, so the limiting control effort u_j^∞ should be stabilising, here meaning that

$$(1 - p(u_j^\infty))\ell \leq 1 \quad j \in \{1, 2, 3\},$$

which is the condition $\rho(G) < 1$ from Proposition 2.8 in the current context. Taking the control values $u_j(60)$ as a proxy for u_j^∞ for $j \in \{1, 2, 3\}$, each of which are of the order 10^4 , it is readily computed that

$$(1 - p(u_1^\infty))\ell = 0.389, \quad (1 - p(u_2^\infty))\ell = 0.955 \quad \text{and} \quad (1 - p(u_3^\infty))\ell = 0.387,$$

which are all less than one. In other words, in each case a stabilising control effort has been found. However, noting that each control effort results in $p(u_j^\infty) \approx 0.98$, in order to asymptotically eradicate the population requires removing approximately 98% of the population per time step, which is likely not practicable.

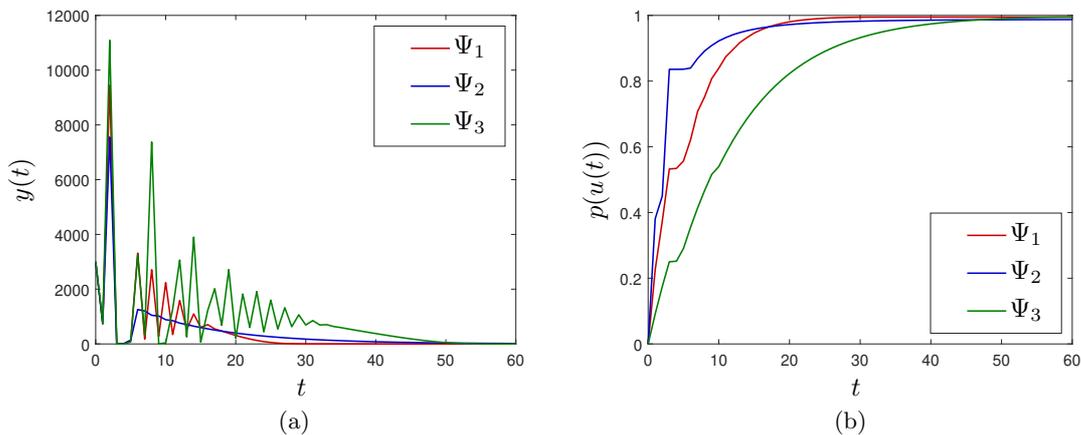


Figure 3.2: Numerical simulations from the example in Section 3.1 for varying Ψ . Model data are detailed in Section 3.1.3, now with $\lambda = 0$ and $v = 0$.

We comment that determining analytically the time taken for $\max\{y(t) - \lambda, 0\}$ to converge to zero seems intractable to us, even in this simple scalar example. This is a consequence of the presence of forcing terms, and the nonlinear interplay between the measured variable y , the tolerance λ , the update law for u , and the consequent effect on x .

To explore one of these relationships numerically, Figure 3.3 plots the (minimal) time taken for $\max\{y(t) - \lambda, 0\}$ to converge against varying λ , for all other variables fixed, and in the absence of forcing terms. All

simulations were run over 100 time steps, and the time for convergence recorded is the first after which $\max\{y(t) - \lambda, 0\}$ is always zero. Interestingly, no clear pattern emerges, and time taken for convergence is constant (or nearly so) for some ranges of λ , and highly oscillatory for other ranges of λ . Intuitively (and very roughly), on the one hand, we expect smaller λ to require a larger control effort u to achieve convergence, which would take longer all other variables being equal. On the other hand, smaller λ results in u increasing faster, as $\max\{y(t) - \lambda, 0\}$ is larger, suggesting faster convergence. It seems that these factors broadly balance out.

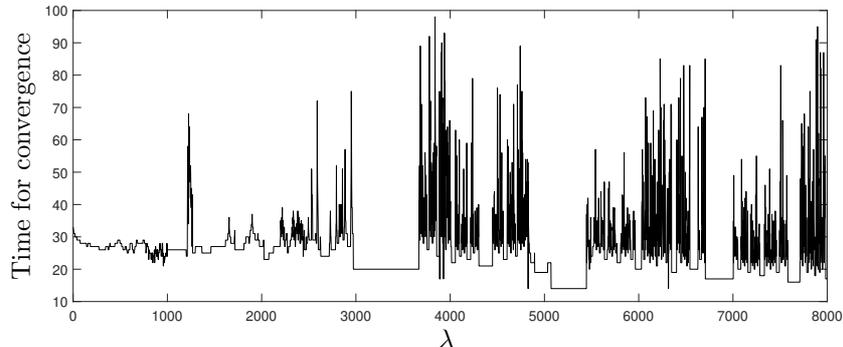


Figure 3.3: Numerical simulation results from the example in Section 3.1. Time for convergence is plotted against varying λ . Model data are detailed in Section 3.1.3, with $v = 0$.

We conclude the example by commenting that in this scalar setting, the condition (3.6) is sufficient for the zero solution of the uncontrolled ($u = 0$) and unforced ($v_1 = 0$) difference equation (3.1) specified by (3.4) and (3.5) to be exponentially stable. Evidently, if the control efficacy function p and net rate of increase ℓ are known with certainty, then a constant (and minimal) level of control $u = w$ can be chosen to satisfy (3.6). What is arguably less clear from the condition (3.6) is how to choose a constant control effort $u = w$ which leads to (3.9), for a prescribed level λ . Indeed, revisiting the first numerical simulation, and taking $u(40; \lambda_j)$ as a proxy for the limiting control effort under λ_j , we compute that

$$(1 - p(u(40; \lambda_1)))\ell = 29.6423, \quad (1 - p(u(40; \lambda_2)))\ell = 9.8207, \quad (1 - p(u(40; \lambda_3)))\ell = 1.0462.$$

In other words, none of the simulations in Figure 3.1 have led to a limiting control effort which satisfies (3.6), yet (3.9) holds.

3.2 Structured population models

As a second illustrative example, we consider (2.1) for a stage-structured population model, corresponding to vector-valued $x(t)$. We continue the study of potential pest control strategies of the insect pest, the Diaprepes root weevil (DRW; *Diaprepes abbreviatus*). The present example builds on [18, Section 4], where the simple adaptive feedback control scheme proposed in [18] is applied to a linear DRW model. The present example also complements the work [12], where optimal control approaches are presented for the management of DRW. The model we currently use is a nonlinear version of that used in [12], which itself is based on the model proposed in [36]. We refer the reader to [12, 18, 36], and the references therein, for more background and context on DRW and potential control strategies.

3.2.1 Introducing the model

We first discuss the uncontrolled model, which is of the form (1.1). The model has four stage-classes, namely eggs, larvae, pupae, and adults, leading to $n = 4$, and the time-steps t denote time in weeks. Letting $x(t)$ denote the structured population abundance at time-step t , we propose the following nonlinear

matrix population projection model, given by

$$\begin{pmatrix} x_1^+ \\ x_2^+ \\ x_3^+ \\ x_4^+ \end{pmatrix} = \begin{pmatrix} s_1 & 0 & 0 & \theta(x_4) \\ g_1 & s_2 & 0 & 0 \\ 0 & g_2 & s_3 & 0 \\ 0 & 0 & g_3 & s_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ v_1 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ x_4(0) \end{pmatrix} = \begin{pmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \\ x_4^0 \end{pmatrix}. \quad (3.10)$$

Here the s_i terms are per time-step stasis probabilities of stage class i . The g_i terms are per time-step growth probabilities from stage i to $i + 1$. Combined, $s_i + g_i$ is the survival probability of an individual who starts a time step in stage i . The s_i and g_i are all assumed positive and satisfy $0 < s_i + g_i < 1$ for every i . The θ term is the per adult number of eggs recruited into the population each time step, and is assumed to be a function of the number of adults, owing to crowding effects at higher abundances. Therefore, at each time step $\theta(x_4)x_4$ new eggs are recruited into the population. The term v_1 models an adult immigration term, reflecting the most motile stage class.

We comment that the theory developed in this paper extends to models with more density dependent terms, although verifying the hypotheses becomes more involved.

The uncontrolled model (3.10) may be written in the form (1.1) with

$$A := \begin{pmatrix} s_1 & 0 & 0 & 0 \\ g_1 & s_2 & 0 & 0 \\ 0 & g_2 & s_3 & 0 \\ 0 & 0 & g_3 & s_4 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad E := (0 \ 0 \ 0 \ 1), \quad B_e = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (3.11)$$

so that $n = 4$, $m = q = 1$ and $s = 2$. (The second zero column of B_e shall reflect that v has two components, the second component corresponding to measurement noise.) The nonlinear function $\Theta_0 : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is given by

$$\Theta_0(z) = \theta(z)z \quad \forall z \in \mathbb{R}_+.$$

The scalar function $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is unlikely to be known in practice, but is assumed to be non-increasing, continuous, and positive for positive arguments.

As is the case in [18], we assume that the number of adults is (noisily) measured at each time step, leading to

$$y = x_4 + v_2 = Cx + D_e v \quad \text{with} \quad C = (0 \ 0 \ 0 \ 1) \quad \text{and} \quad D_e = (0 \ 1), \quad (3.12)$$

where v_2 is a measurement error term. We assume that control in this example corresponds to the release of entomopathogenic nematodes, which attack DRW larvae, see [12]. The control terms $u(t)$ denotes the quantity of nematodes to release. The control is modelled as acting before the biological processes in each time-step, leading to the model (2.12) with

$$\Gamma(w) := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta(w) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \forall w \geq 0, \quad (3.13)$$

where $\delta : \mathbb{R}_+ \rightarrow [0, 1]$ is assumed to be non-increasing, continuous, positive for positive arguments and satisfy $\delta(0) = 1$. This final condition captures the property that no control effort leads to no control effect on the pest.

With a slight abuse of notation, the matrix-valued functions A , B and E in (2.1) are thus given by

$$A(w) = A\Gamma(w), \quad B(w) = B \quad \text{and} \quad E(w) = E\Gamma(w) = E \quad \forall w \geq 0,$$

where the matrices on the right hand sides are as in (3.11) and (3.13). The nonlinear function Θ in (2.1) is given by $\Theta = \Theta_0$ (so is in fact independent of the first variable w).

3.2.2 Verifying hypotheses

We proceed to verify that the assumptions **(M)** and **(S)** are satisfied in this example. Since δ is assumed continuous and non-increasing, it follows that Γ is continuous and non-increasing, and hence so are A ,

B and E , so that **(M1)** holds. The function $\Theta = \Theta_0$ is independent of w (its first variable) and our hypotheses on θ ensures that **(M2)** holds. Assumption **(M4)** is evidently satisfied, and assumption **(M5)** relates to the difference equation in (2.1) for u , and so is independent of the model described so far.

The growth hypotheses **(M3)** and **(M3)'** depend on the properties of the nonlinear term θ . The assumptions on θ guarantee that it has a nonnegative limit at infinity, denoted θ_∞ , which is the per adult number of eggs recruited per time step at (infinitely) large population sizes. In fact, Θ satisfies the bounds

$$\theta_\infty z \leq \Theta(z) = \theta(z)z \leq \theta(0)z \quad \forall z \in \mathbb{R}_+. \quad (3.14)$$

If $\theta_\infty > 0$ (and so necessarily Θ is unbounded), then we claim that property **(M3)** holds, which follows from the lower bound for Θ in (3.14) holds with constant $\Delta := \theta_\infty$ and as the matrix

$$A(w) + B(w)\Delta E(w) = A_0\Gamma(w) + B_0\theta_\infty E_0 = \begin{pmatrix} s_1 & 0 & 0 & \theta_\infty \\ g_1 & s_2\delta(w) & 0 & 0 \\ 0 & g_2\delta(w) & s_3 & 0 \\ 0 & 0 & g_3 & s_4 \end{pmatrix}, \quad (3.15)$$

is irreducible for all $w \geq 0$. If $\theta_\infty = 0$, then we claim that **(M3)'** holds. For which purpose, since $A(0)$ is lower triangular, with diagonal entries less than unity, it is clear that $\rho(A(0)) < 1$. Let $\varepsilon > 0$ be sufficiently small so that $\rho(A(0) + \varepsilon BE) < 1$. Since $\theta_\infty = 0$, it is routine to see that there exists $\gamma > 0$ such that

$$\Theta(z) = z\theta(z) \leq \varepsilon z + \gamma \quad \forall z \in \mathbb{R}_+.$$

Further, for all $w, k \geq 0$, we have

$$\Theta(z) = \theta(z)z \geq \theta(k)z \quad \forall z \in [0, k],$$

and, akin to (3.15), the matrix $A(w) + B(w)\theta(k)E(w)$ is irreducible. Therefore, assumption **(M3)'** holds.

To verify the stabilisability assumption **(S)**, we use Lemma 2.2 as in this example $m = q = 1$. That $\rho(A(0)) < 1$ has already been discussed. A consequence of the upper bound for Θ in (3.14) is that the linear bound for Θ in (2.6) holds with $\Sigma := \theta(0)$. The key quantity for Lemma 2.2 is the function $\mathbf{H} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$\mathbf{H}(w) = E(w)(I - A(w))^{-1}B(w) = \frac{g_1 g_2 g_3 \delta(w)}{(1 - s_1)(1 - \delta(w)s_2)(1 - s_3)(1 - s_4)} \quad \forall w \in \mathbb{R}_+. \quad (3.16)$$

The values of \mathbf{H} have the following biological interpretation: they are the total number of adults, predicted by the model (2.12) with $v = 0$ and constant control effort $u = w$, that a single egg produces over all time. Note that a single adult can hatch from each egg, that adult will be counted in each time step it survives, and so $\mathbf{H}(w) > 1$ is possible. Since the survival of eggs to adults is assumed to be density independent, this quantity scales linearly. If $\mathbf{H}(0)\theta_\infty < 1$, so that at large population sizes each egg is surviving and then recruiting fewer than one egg, then an application of [15, Theorem 4.4 (a)] yields that the solutions of the uncontrolled model (3.10) are bounded when v_1 is bounded.

Noting that \mathbf{H} is non-increasing as δ is assumed non-increasing, if there exists $\sigma \geq 0$ such that

$$\mathbf{H}(\sigma)\theta(0) < 1, \quad (3.17)$$

then, in light of (3.14), the second inequality in (2.6) holds, and hypothesis **(S)** follows from an application of Lemma 2.2. We comment that the condition (3.17) holds independently of $\theta(0)$ if $\delta(w) \rightarrow 0$, and hence $\mathbf{H}(w) \rightarrow 0$, as $w \rightarrow \infty$.

3.2.3 A numerical example

For a numerical simulation we take parameter values s_i and g_i as in [12, Section 2.5.1], which are reproduced in (D.01) in Appendix D. The nonlinear term θ does not appear in [12], as that paper considers a linear (density independent) DRW model. The function we choose is given in (D.02), and is such that **(M3)'** holds and the linearisation around zero has the same slope as the corresponding parameters in [12]. For simplicity, we further assume that $v_1 = 0$.

For the control terms, first, we assume that the function δ is given by

$$\delta(w) = e^{-\alpha w} \quad \forall w \geq 0 \quad \text{with} \quad \alpha := 1.655 \times 10^{-8},$$

the motivation for which is discussed in [12, Section 2.5.2]³. This choice of δ ensures that \mathbf{H} satisfies assumption (3.17), the upshot being that the stabilisability assumption **(S)** holds. The adult stage class is measured per time step, as in (3.12), and we assume a random per time-step proportional measurement error of at most 5%, meaning that $v_2 = \varepsilon Cx$ with uniformly randomly chosen $\varepsilon(t) \in [-0.05, 0.05]$ for all $t \in \mathbb{Z}_+$.

Since, with these parameters $\mathbf{H}(0)\theta(0) > 1$, the uncontrolled model (3.10) (meaning $v_1 = 0$) has a unique non-zero steady state

$$x_* := (I - A(0))^{-1}B\Theta(z_*),$$

where z_* is the unique positive solution of the scalar equation $\mathbf{H}(0)\Theta(z) = z$. We used x_* as the initial DRW population x^0 for the controlled model (2.1), the rationale being that the pest population is assumed to be endemic before control is applied. The norm of x_* has order of magnitude 10^5 and is given in (D.03). Moreover, with our choice of θ in (D.02) it follows from an application of [15, Theorem 5.2] that the equilibrium x_* of (3.10) with $v_1 = 0$ attracts all non-zero solutions and is semi-globally exponentially stable in the following sense. Namely, for every nonempty, compact set $X \subseteq \mathbb{R}_+^n$ with $0 \notin X$, there exist $M \geq 1$ and $\mu \in (0, 1)$ such that, for all $x^0 \in X$, the solution x of (3.10) with $v_1 = 0$ satisfies

$$\|x(t) - x_*\| \leq M\mu^t \|x^0 - x_*\| \quad \forall t \in \mathbb{Z}_+.$$

We performed three simulations by varying u^0 , the initial control effort. In this example $u(t)$ denotes the nematodes applied per hectare per time step, with typical values of order of magnitude 10^8 , see [12]. Since our control effort u is non-decreasing, we take

$$u_1^0 = 2.5 \times 10^6, \quad u_2^0 = 5 \times 10^6, \quad u_3^0 = 7.5 \times 10^6,$$

as under estimates of a limiting control effort which is stabilising. Finally, across the simulations we fixed

$$\lambda = 5 \times 10^4 \quad \text{and} \quad \Psi(z) = z.$$

With these choices, all of the hypotheses of Proposition 2.6 are satisfied, apart from possibly the threshold conditions (2.3) or (2.8). We comment on this shortly. Our numerical results are plotted in Figure 3.4. We see in Figure 3.4a that $\max\{y(t) - \lambda, 0\} \rightarrow 0$, as time increases, certainly for the initial control efforts u_2^0 and u_3^0 whilst the simulation length is too short for the initial control effort u_1^0 . The lower and higher black dotted lines in Figure 3.4a are the tolerance level λ and the true output of the uncontrolled model (3.10), respectively, the latter of which remains at its initial equilibrium value. This line is included for comparison purposes. The control effort $u(t)$ is plotted against time t in Figure 3.4b, with line colours corresponding to those in Figure 3.4a.

Here we see that whilst u_3^0 achieves the control objective reasonably quickly, the initial control efforts u_1^0 and u_2^0 are arguably too small and performance is sluggish. Indeed, the control efforts $u(t; u_1^0)$ and $u(t; u_2^0)$ have not converged over the course of the simulation but, understandably, are at lower levels than u_3^∞ . In applications, the tradeoff between the cost of control and the cost of pest population would need to be considered. We comment that although the theory presented is qualitatively independent of, in this case, the functional form of θ , the numerical simulations are somewhat sensitive to θ . Practically, this highlights the importance of identifying parameters as much as possible, and extensive numerical testing over a range of possible scenarios. Interestingly, in this example, performance is somewhat sluggish as the adult population, modelled by x_4 , is determined via

$$x_4(t+1) = s_4 x_4(t) + g_3 x_3(t) = 0.98 x_4(t) + 0.202 x_3(t) \quad \forall t \in \mathbb{Z}_+.$$

In particular, as $0.98 \approx 1$, once an adult population is established, as it is in this example via the assumption that $x(0) = x_*$, it decays comparatively slowly, and independently of the control effort $u(t)$.

³The symbol α is used in [12] in this context, and we use it here as well — it should not be confused with the forcing threshold in Theorem 2.1.

Regarding the threshold conditions (2.3) and (2.8), in this case, as $v_1 = 0$, it follows that $\theta(\lambda) = \lambda$. Again, arguing heuristically, in light of our simulations, since

$$\limsup_{t \rightarrow \infty} |v_2(t)| = 0.05 \limsup_{t \rightarrow \infty} |Cx(t)| \approx 2 \times 10^3 < \lambda,$$

it appears that (2.8) is satisfied.

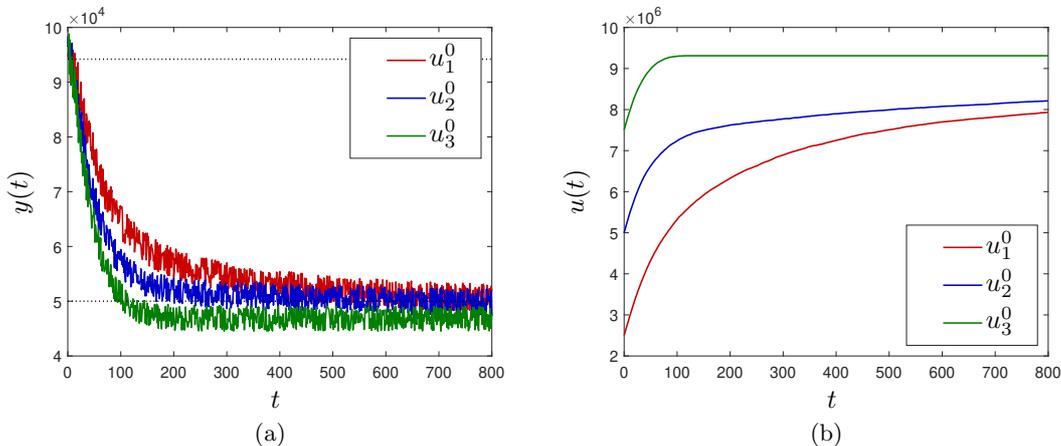


Figure 3.4: Numerical simulations from the DRW model from Section 3.2 with model data as described in Section 3.2.3. The higher and lower dotted lines in panel (a) denote the measured uncontrolled population, which is at equilibrium, and the threshold λ , respectively.

4 Discussion

A robust adaptive feedback controller has been proposed for the control of a class of systems of positive nonlinear forced difference equations. The difference equations are of so-called Lur’e type and contain both a linear component and a structured nonlinear component, as well as an additive external forcing term. Our motivating application has been to the dynamic and robust control of pests and weeds — a timely and societally relevant problem in the context of global food production — and our present work substantially improves our earlier paper [18] where the same problem was studied in the context of linear models only, and with a simpler controller which had no tolerance or robustness with respect to persistent exogenous forcing terms or measurement error. The adaptive feedback control system (2.1) considered presently admits a quite general description of control which, practically, we believe can model a number of different control actions or management strategies. Thus, on the one hand, the techniques described here are portable across a number of applied settings. On the other hand, the present paper pursues the specific problem of the management of the invasive economic pest species *Diaprepes root weevil* (DRW; *Diaprepes abbreviatus*), and so complements our existing works [12, 18, 36].

The controller considered here is based on a so-called simple adaptive controller and uses a measured variable, denoted y , to dynamically update the control effort u . The controller contains a forcing/noise tolerance term λ . The combined feedback system is (2.1) and our main result is Theorem 2.1 which presents stability and convergence properties of (2.1). Our key assumptions are structural properties of the model **(M)** and a stabilisability assumption **(S)**. This latter assumption essentially requires that there is a level of control effort which is stabilising and, when satisfied, the feedback system (2.1) informally “finds” such a level. Moreover, Proposition 2.8 shows that, under some simplifying assumptions, the control variable u associated with (2.1) converges and the limiting control effort is itself stabilising in a sense we describe. From this perspective, our work can be viewed as “learning control” for positive systems of Lur’e difference equations.

As with [18], our work is in the spirit of robust control. We study systems where the terms in the model, and the model structure itself, are likely to be uncertain, and the measured variable is likely to be subject to error or noise. Our hypotheses are structural, and do not require exact knowledge of the model

parameters or structure to be verified. We made a case in [18] for the value of considering robust control techniques in the context of pest management, not least to broaden the theoretical discussion and add another perspective. Consequently, given how little information this control uses from the system, our results are not expected to be optimal in any sense, but may have utility when models are so poor that optimal controls may not function or perform as intended. This comment also naturally raises a future research direction, which we hope to address, and is to combine elements of the robust control methods developed here with the optimal control approaches in [12], to design controllers with an agreeable and adjustable blend of robustness and optimality.

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Appendix

The Appendix is divided into four sections. The first gathers additional notation and estimates used in our proofs. The second and third contain proofs of our mathematical results. The fourth contains supporting information for the example presented Section 3.2.

A Preliminaries

Given strictly positive $\xi \in \mathbb{R}_+^n$, we let

$$|x|_\xi := \xi^T |x| = \sum_{j=1}^n \xi_j |x_j| \quad \forall x = (x_1 \ \dots \ x_n)^T \in \mathbb{R}^n,$$

where the i -th component of $|x| \in \mathbb{R}^n$ is defined to be $|x_i|$. We note that $|\cdot|_\xi$ is a norm on \mathbb{R}^n and, if $x \in \mathbb{R}_+^n$, then $|x|_\xi = \xi^T x$.

Throughout the proofs we shall use (often without explicit mention) the well-known monotonicity of the spectral radius for nonnegative matrices $M_1, M_2 \in \mathbb{R}^{n \times n}$, namely, if $0 \leq M_1 \leq M_2$, then $\rho(M_1) \leq \rho(M_2)$ (see, for instance [6, Corollary 1.5, p.27]).

We shall also use that if $z \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^n)$ is nonnegative and satisfies the difference inequality

$$z(t+1) \leq (\geq) A_0 z(t) + F(t, z(t)) + d(t) \quad \forall t \in \mathbb{Z}_+, t \geq s, \quad (\text{A.01})$$

for some $s \in \mathbb{Z}_+$, (constant) $A_0 \in \mathbb{R}_+^{n \times n}$, function $F : \mathbb{Z}_+ \times \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ and nonnegative sequence $d \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}_+^n)$, then z satisfies the variation-of-parameters inequality

$$z(t+k) \leq (\geq) A_0^t z(k) + \sum_{j=k}^{t+k-1} A_0^{t+k-1-j} (F(j, z(j)) + d(j)) \quad \forall t \in \mathbb{N}, \forall k \in \mathbb{Z}_+, k \geq s, \quad (\text{A.02})$$

which is easily established by induction.

B Proof of Theorem 2.1

The proof of Theorem 2.1 is facilitated by three technical lemmas. Using several smaller results shall help in proving later results where variations of an argument are used, without excessive repetition.

The first lemma contains a coupling condition between C , capturing the measured output, and bounds for the dynamics for x in (2.1).

Lemma B.1. *Let $C \in \mathbb{R}_+^{p \times n}$ be non-zero, and let $P \in \mathbb{R}_+^{n \times n}$ be irreducible. There exists $\mu > 0$ such that, for all $i \in \underline{n}$, there exists $k_i \in \underline{n}$ such that*

$$\|CP^{k_i}z\| \geq \mu\rho(P)^{k_i}z_i \quad \forall z := (z_1 \ \dots \ z_n)^T \in \mathbb{R}_+^n.$$

Proof. Fix $(\ell, s) \in \underline{p} \times \underline{n}$ such that $C_{\ell s} > 0$, which is possible as $C \neq 0$. Note that, as $P \in \mathbb{R}_+^{n \times n}$,

$$\|CP^kz\| \geq (CP^kz)_\ell = \sum_{j,r=1}^n C_{\ell r}(P^k)_{rj}z_j \geq C_{\ell s}(P^k)_{si}z_i \quad \forall z \in \mathbb{R}_+^n, \forall i \in \underline{n}, \forall k \in \mathbb{Z}_+. \quad (\text{B.01})$$

Since P is irreducible, it follows that $\rho(P) > 0$ and $P/\rho(P)$ is itself irreducible with spectral radius equal to one. Consequently,

$$k_i := \operatorname{argmin}_{k \in \underline{n}} \left\{ ((P/\rho(P))^k)_{si} : (P^k)_{si} > 0 \right\} \quad \forall i \in \underline{n} \quad \text{and} \quad \beta := \min_{i \in \underline{n}} ((P/\rho(P))^{k_i})_{si} > 0,$$

are well-defined, and have the property that

$$(P^{k_i})_{si}z_i \geq \beta\rho(P)^{k_i}z_i \quad \forall z \in \mathbb{R}_+^n, \forall i \in \underline{n}. \quad (\text{B.02})$$

The claimed inequality now follows with $\mu := C_{\ell s}\beta > 0$ from the conjunction of (B.01) and (B.02). \square

Broadly, the second lemma contains consequences for x and y in the adaptive feedback control system (2.1) when the control variable u is bounded.

Lemma B.2. *Consider (2.1) with $\lambda \geq 0$, $v \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^s)$ and $B_e v \geq 0$, and assume that (M1), (M2), (M4) and (M5) hold. Assume further that u is bounded. The following statements hold.*

- (a) $\max\{0, \|\phi(y(t))\| - \lambda\} \rightarrow 0$ as $t \rightarrow \infty$.
- (b) *If there exists an irreducible $P \in \mathbb{R}_+^{n \times n}$ such that*

$$x(t+1) \geq Px(t) \quad \forall t \in \mathbb{Z}_+, \quad (\text{B.03})$$

then x is bounded.

- (c) *If x is bounded, then so is y .*

Proof. Let $w \in \mathbb{R}_+$ denote an upper bound for u , which exists by hypothesis. From the definition of u in (2.1), it follows that

$$\sum_{j=0}^t \Psi(\max\{0, \|\phi(y(j))\| - \lambda\}) = u(t+1) - u^0 \leq w - u^0 \quad \forall t \in \mathbb{N},$$

and hence, the nonnegative sequence $(\Psi(\max\{0, \|\phi(y(k))\| - \lambda\}))_{k \in \mathbb{Z}_+}$ is summable. Therefore,

$$\Psi(\max\{0, \|\phi(y(t))\| - \lambda\}) \rightarrow 0 \quad \text{as } t \rightarrow \infty,$$

and as $\Psi \in \mathcal{K}$, it is invertible in a neighbourhood of zero, with a strictly increasing and continuous inverse. We conclude that statement (a) holds.

We now prove statement (b). Since $Cx(t) \geq 0$ for all $t \in \mathbb{Z}_+$, by arguing componentwise, it follows that

$$\|\phi(y(t))\|_1 = \|\phi(Cx(t) + D_e v(t))\|_1 \geq \|Cx(t)\|_1 - \|D_e v(t)\|_1 \quad \forall t \in \mathbb{Z}_+, \quad (\text{B.04})$$

(where $\|\cdot\|_1$ denotes the usual vector one-norm). Seeking a contradiction, assume that x is unbounded. Therefore, there exist sequences $(t_j)_{j \in \mathbb{N}} \subseteq \mathbb{N}$ and $(i_j)_{j \in \mathbb{N}} \subseteq \underline{n}$ such that

$$(x(t_j))_{i_j} \geq j \quad \forall j \in \mathbb{N}. \quad (\text{B.05})$$

The given lower bound (B.03) for the dynamics of x yields that

$$Cx(t_j + k) \geq CP^k x(t_j) \quad \forall j, k \in \mathbb{N}. \quad (\text{B.06})$$

An application of Lemma B.1 yields a bounded sequence $(k_j)_{j \in \mathbb{N}}$ and $\omega := \mu \min_{r \in \underline{n}} \rho(P)^r > 0$ such that, in light of (B.05),

$$\|CP^{k_j} x(t_j)\| \geq \omega x(t_j)_{i_j} \geq \omega j \quad \forall j \in \mathbb{N}. \quad (\text{B.07})$$

Taking norms in (B.06) and appealing to (B.07) yields that

$$\|Cx(t_j + k_j)\| \geq \omega j \quad \forall j \in \mathbb{N}. \quad (\text{B.08})$$

However, the conjunction of (B.04) and (B.08) contradicts statement (a). We conclude that x is bounded. Once x is bounded, it is clear from $y = Cx + D_e v$ and $v \in \ell^\infty(\mathbb{Z}_+, \mathbb{R}^s)$ that y is bounded as well. \square

The third and final lemma is an exponential input-to-state stability estimate for x in (2.1) in the situation that $u(t)$ is large enough and the stabilisability assumption (S) holds.

Lemma B.3. *Imposing the notation and assumptions of Theorem 2.1 apart from assumption (M3), if $\tau \in \mathbb{Z}_+$ is such that $u(\tau) \geq \sigma$, then there exist $M > 0$ and $\gamma \in (0, 1)$ such that*

$$\|x(t + \tau)\| \leq M(\gamma^t \|x(\tau)\| + \|B_e v\|_{\ell^\infty(\tau, t + \tau - 1)}) \quad \forall t \in \mathbb{N}. \quad (\text{B.09})$$

The constants M and γ in (B.09) depend on the model data in (2.1) (including on $u(\tau) \geq \sigma$), but are independent of x^0 , u^0 and v .

Proof. Note that the sequence u is non-decreasing, and so $u(t + s) \geq u(s)$ for all $s, t \in \mathbb{Z}_+$. A consequence of the monotonicity in assumptions (M1) and (M2) is that x admits the upper bound

$$x(t + 1) \leq A(u(s))x(t) + B(u(s))\Theta(u(s), E(u(t))x(t)) + B_e v(t) \quad \forall s, t \in \mathbb{Z}_+, t \geq s. \quad (\text{B.10})$$

Set $A_0 := A(u(\tau)) + B(u(\tau))\Sigma E(u(\tau))$, where Σ is as in (S). Since $u(\tau) \geq \sigma$ by hypothesis, the stabilisability assumption (S), when combined with the monotonicity of A , B and E in (M1), yields that $\rho(A_0) < 1$ and

$$\Theta(u(\tau), E(u(t))x(t)) \leq \Sigma E(u(t))x(t) \leq \Sigma E(u(\tau))x(t) \quad \forall t \in \mathbb{Z}_+, t \geq \tau.$$

Combining the above two estimates, the former with $s = \tau$, gives

$$x(t + 1) \leq A_0 x(t) + B_e v(t) \quad \forall t \in \mathbb{Z}_+, t \geq \tau.$$

The above inequality is a special case of (A.01) with $F := 0$ and $d := B_e v$, and so the variation-of-parameters inequality (A.02) holds, here with $k = s = \tau$. The estimate (B.09) now follows by taking norms in (A.02) and majorising, and critically uses that $\rho(A_0) < 1$. \square

We are now in position to prove Theorem 2.1.

Proof of Theorem 2.1. We record that the function ϕ which appears in (2.2) is Lipschitz with Lipschitz constant equal to one.

(i): Seeking a contradiction, we posit that u is unbounded. Then there exists $\tau_1 \in \mathbb{N}$ such that $u(\tau_1) \geq \sigma$, where σ is as in (S). An application of Lemma B.3 ensures the existence of $M > 0$ and $\gamma \in (0, 1)$ such that (B.09) holds. In preparation for the proof of statement (v), where $\lambda = 0$ and $v = 0$, we consider two exhaustive cases.

CASE 1: $\lambda > 0$. Define α by

$$\alpha(\lambda) := \frac{\lambda}{\|C\|M + 1}, \quad (\text{B.11})$$

and assume that (2.3) holds, meaning there exists $\varepsilon > 0$ such that

$$\varepsilon + (\|C\|M + 1) \max \{ \|B_e v\|_{\ell^\infty}, \|D_e v\|_{\ell^\infty} \} \leq \lambda. \quad (\text{B.12})$$

Since $\gamma < 1$, we can choose sufficiently large $\tau_2 \in \mathbb{N}$ so that

$$\|C\|M\gamma^{t+\tau_2}\|x(\tau_1)\| \leq \varepsilon \quad \forall t \in \mathbb{Z}_+.$$

We invoke (B.09) with t and τ there given by $t + \tau_2$ and τ_1 , respectively, to estimate that

$$\begin{aligned} \|\phi(y(t + \tau_2 + \tau_1))\| &\leq \|y(t + \tau_2 + \tau_1)\| = \|Cx(t + \tau_2 + \tau_1) + D_e v(t + \tau_2 + \tau_1)\| \\ &\leq \|C\|\|x(t + \tau_2 + \tau_1)\| + \|D_e v\|_{\ell^\infty} \\ &\leq \|C\|M\gamma^{t+\tau_2}\|x(\tau_1)\| + \|C\|M\|B_e v\|_{\ell^\infty} + \|D_e v\|_{\ell^\infty} \\ &\leq \varepsilon + (\|C\|M + 1) \max\{\|B_e v\|_{\ell^\infty}, \|D_e v\|_{\ell^\infty}\} \leq \lambda \quad \forall t \in \mathbb{Z}_+, \end{aligned}$$

where we have also majorised the $B_e v$ and $D_e v$ terms by their ℓ^∞ norms and invoked (B.12). Therefore, we see that

$$\|\phi(y(t + \tau_2 + \tau_1))\| - \lambda \leq 0 \quad \forall t \in \mathbb{Z}_+,$$

and, consequently, the update law for u in (2.1) yields that $u(t + \tau_2 + \tau_1) = u(\tau_2 + \tau_1)$ for all $t \in \mathbb{Z}_+$. In particular, u is constant after some finite time, which contradicts the assumed unboundedness of u .

CASE 2: $\lambda = 0$. In this case we also assume that $v = 0$ and that the growth condition (2.5) holds for Ψ . In particular, as $v = 0$, the inequality (B.09) with τ there equal to τ_1 yields

$$\|y(t + \tau_1)\| = \|Cx(t + \tau_1)\| \leq \|C\|M\gamma^t\|x(\tau_1)\| \quad \forall t \in \mathbb{Z}_+.$$

Since $\Psi \in \mathcal{K}$, we invoke the above inequality to estimate that

$$u(t + \tau_1) - u(\tau_1) = \sum_{j=0}^{t-1} \Psi(\|y(j + \tau_1)\|) \leq \sum_{j \in \mathbb{Z}_+} \Psi(\|C\|M\gamma^j\|x(\tau_1)\|) < \infty \quad \forall t \in \mathbb{N},$$

where we have also appealed to (2.5) to conclude that the infinite series above is finite. The above inequalities yield that u is bounded, which again contradicts the assumed unboundedness of u . Therefore, we have established statement (i), and that u is bounded under the hypotheses of statement (v).

For the proof of the remaining statements, we record that as u is bounded by part (i), and non-decreasing, u is convergent with limit $w \in \mathbb{R}_+$. In particular, $u(t) \leq w$ for all $t \in \mathbb{Z}_+$ which, when carefully combined with the monotonicity in **(M1)** and **(M2)**, and the lower bound in **(M3)**, leads to the estimate

$$\begin{aligned} x^+ &= A(u)x + B(u)\Theta(u, E(u)x) + B_e v \geq A(w)x + B(w)\Theta(w, E(u)x) + B_e v \\ &\geq (A(w) + B(w)\Delta E(u))x + B_e v \geq (A(w) + B(w)\Delta E(w))x + B_e v. \end{aligned} \quad (\text{B.13})$$

For notational convenience, set $P := A(w) + B(w)\Delta E(w)$, which is irreducible by hypothesis **(M3)**.

(ii) and (iii): In light of statement (i) and the lower bound (B.13) with irreducible P , the hypotheses of Lemma B.2 are satisfied and the claims follow from this result.

(iv): The hypotheses of Lemma B.3 are satisfied and, noting that the estimate (2.4) is the same as (B.09), the statement follows from Lemma B.3.

(v): Here we assume that $\lambda = 0$, $v = 0$ and that (2.5) holds. We have already proven that u is bounded under these assumptions, and the proof of statements (ii)–(iv) hold in the case that $\lambda = 0$. It remains to prove that $x(t) \rightarrow 0$ as $t \rightarrow \infty$. For which purpose, we claim that

$$\rho(A(w)) < 1. \quad (\text{B.14})$$

Seeking a contradiction, suppose that (B.14) fails. Then $\rho(P) \geq \rho(A(w)) \geq 1$. Consequently, letting ξ^T denote a strictly positive left eigenvector of the irreducible P corresponding to the spectral radius of P , it follows from (B.13) that

$$Cx(t+k) \geq CP^t x(k) \quad \text{and} \quad |x(t+k)|_\xi \geq (\rho(P))^t |x(k)|_\xi \geq |x(k)|_\xi \quad \forall t, k \in \mathbb{Z}_+.$$

Hence, repeated applications of Lemma B.1 to the first inequality above yields $\varepsilon > 0$ and a sequence $(t_k)_{k \in \mathbb{N}} \subseteq \mathbb{Z}_+$ with $t_k \nearrow \infty$ as $k \rightarrow \infty$ such that

$$\|\phi(y(t_k))\| = \|y(t_k)\| = \|Cx(t_k)\| \geq \varepsilon \quad \forall k \in \mathbb{N},$$

which contradicts the convergence in statement (ii), as here $\lambda = 0$ and $v = 0$. We conclude that (B.14) holds. Therefore, by continuity of A and the spectral radius, there exists $\tau \in \mathbb{Z}_+$ such that $\rho(A(u(\tau))) < 1$.

We next claim that

$$\limsup_{t \rightarrow \infty} \|E(u(t))x(t)\| = 0, \quad (\text{B.15})$$

which evidently yields that $E(u(t))x(t) \rightarrow 0$ as $t \rightarrow \infty$. The continuity of $\Theta(u(\tau), \cdot)$ then gives

$$\lim_{t \rightarrow \infty} \Theta(u(\tau), E(u(t))x(t)) = 0. \quad (\text{B.16})$$

Note that the upper bound (B.10) is of the form (A.01) with $s = \tau$, $A_0 := A(u(\tau))$ and

$$F(t, z) := B(u(\tau))\Theta(u(\tau), E(u(t))z) \quad \forall (t, z) \in \mathbb{Z}_+ \times \mathbb{R}_+^n,$$

and so the variation-of-parameters inequality (A.02) holds. The conjunction of $\rho(A(u(\tau))) < 1$ and (B.16), when combined with the inequality (A.02), gives $x(t) \rightarrow 0$ as $t \rightarrow \infty$. Again, seeking a contradiction, we suppose that (B.15) fails, meaning there exist $\varepsilon > 0$ and a sequence $(t_j)_{j \in \mathbb{N}} \subseteq \mathbb{Z}_+$ such that $t_j \nearrow \infty$ as $j \rightarrow \infty$ and

$$\|E(u(t_j))x(t_j)\| \geq \varepsilon \quad \forall j \in \mathbb{N}.$$

From the monotonicity of the norm and of E , it follows that $\|E(0)x(t_j)\| \geq \|E(u(t_j))x(t_j)\|$ for every $j \in \mathbb{N}$ and, consequently, there exists $\sigma > 0$ such that for every $j \in \mathbb{Z}_+$ there exists $i_j \in \underline{n}$ such that

$$x(t_j)_{i_j} \geq \sigma \quad \forall j \in \mathbb{N}. \quad (\text{B.17})$$

The inequality (B.13) guarantees that (B.06) holds. Repeated application of Lemma B.1 to this latter inequality yields the existence of a bounded sequence $(k_j)_{j \in \mathbb{Z}_+}$ and $\omega > 0$ such that

$$\|Cx(t_j + k_j)\| \geq \|CP^{k_j}x(t_j)\| \geq \omega x(t_j)_{i_j} \quad \forall j \in \mathbb{N}. \quad (\text{B.18})$$

Combining (B.17) and (B.18), it follows that

$$\|y(t_j + k_j)\| = \|Cx(t_j + k_j)\| \geq \omega\sigma > 0 \quad \forall j \in \mathbb{N},$$

which contradicts the convergence $\|y(t)\| = \|\phi(y(t))\| \rightarrow 0$ as $t \rightarrow \infty$ in statement (ii), as here $\lambda = 0$ and $v = 0$. The proof is complete. \square

C Proofs of remaining results

Proof of Lemma 2.2. The following argument is in the spirit of estimates of the spectral radius of a nonnegative matrix using a single point estimate; see, for example [30, Theorem 16.3]. It suffices to prove that $s := \rho(A(\sigma) + B(\sigma)\Sigma E(\sigma)) < 1$. For clarity, we suppress the fixed argument σ . Let non-zero $\xi \in \mathbb{R}_+^n$ be such that

$$(A + B\Sigma E)\xi = s\xi.$$

The eigenvector $\xi \in \mathbb{R}^n$ can be chosen to be nonnegative as $A + B\Sigma E \in \mathbb{R}_+^{n \times n}$. If $E\xi = 0$, then s is an eigenvalue of A , and so less than one by hypothesis. We therefore assume that $E\xi \neq 0$. Forming products and routine algebra gives

$$\begin{aligned} s\zeta^T E(I - A)^{-1}\xi &= \zeta^T E(I - A)^{-1}(A + B\Sigma E)\xi = \zeta^T E(I - A)^{-1}A\xi + \zeta^T \mathbf{H}\Sigma E\xi \\ &< \zeta^T E(I - A)^{-1}A\xi + \zeta^T E\xi = \zeta^T E(I - A)^{-1}\xi, \end{aligned} \quad (\text{C.01})$$

where we have used the second inequality in (2.6) and that $E\xi \neq 0$. Noting that, as $\rho(A) < 1$,

$$\zeta^T E(I - A)^{-1}\xi = \zeta^T E\left(\sum_{k \in \mathbb{Z}_+} A^k\right)\xi \geq \zeta^T E\xi > 0,$$

because ζ^T is strictly positive and $E\xi > 0$ by hypothesis, it follows from (C.01) that $s < 1$, as required. \square

Lemma C.1. *Let A, B, E satisfy **(M1)** and assume that $\sigma \geq 0$ is such that $\rho(A(\sigma)) < 1$. Then \mathbf{H} given by (2.7) is non-increasing on $[\sigma, \infty)$ in the sense of (1.3).*

Proof. A consequence of the hypothesis that $\rho(A(\sigma)) < 1$ is that $\mathbf{H}(w)$ may be expressed as a convergent Neumann series, namely

$$\mathbf{H}(w) = E(w)(I - A(w))^{-1}B(w) = E(w)\left(\sum_{j \in \mathbb{Z}_+} A(w)^j\right)B(w) \geq 0 \quad \forall w \in [\sigma, \infty).$$

Invoking the hypothesis **(M1)** that the functions A, B and E are non-increasing, we now see that the desired monotonicity holds: if $w_1, w_2 \geq \sigma$ with $w_1 \leq w_2$, then

$$\mathbf{H}(w_1) = E(w_1)\left(\sum_{j \in \mathbb{Z}_+} A(w_1)^j\right)B(w_1) \geq E(w_2)\left(\sum_{j \in \mathbb{Z}_+} A(w_2)^j\right)B(w_2) = \mathbf{H}(w_2). \quad \square$$

Proof of Proposition 2.6. First assume that $\lambda > 0$. The proofs of statements (i) and (iv) are the same as in Theorem 2.1, and the latter appeals to Lemma B.3. Neither uses assumption **(M3)**. Let $w := \lim_{t \rightarrow \infty} u(t)$. The proof of the convergence in statement (ii) follows from an application of statement (a) of Lemma B.2 (which, note, does not impose **(M3)**).

Set $A_0 := A(0) + B(0)\Gamma_1 E(0)$. Since $u(\tau) \geq 0$ for all $\tau \in \mathbb{Z}_+$, the hypotheses in **(M3)**' carefully combined with the upper bound (B.10) and the monotonicity in **(M1)** and **(M2)**, yield the following estimate for x , namely,

$$\begin{aligned} x^+ &\leq A(0)x + B(0)\Theta(0, E(u)x) + B_e v \leq A(0)x + B(0)(\Gamma_1 E(u)x + \Gamma_2) + B_e v \\ &\leq A_0 x + B(0)\Gamma_2 + B_e v \end{aligned}$$

The above upper bound and the resulting variation-of-parameters inequality (A.02) with $F := 0$ and $d := B(0)\Gamma_2 + B_e v$ now yield that x is bounded as $\rho(A_0) < 1$ and d is bounded. Thus, statement (iii) holds, and now y is bounded by statement (c) of Lemma B.2.

Once x is bounded, there exists $k = k(x^0, u^0, v) > 0$ such that

$$\|E(u(t))x(t)\| \leq \|E(0)x(t)\| \leq k \quad \forall t \in \mathbb{Z}_+,$$

and the hypothesis **(M3)**' guarantees the existence of $\Delta = \Delta(w, k)$ such that (B.13) holds.

The proof of statement (v) when $\lambda = 0$ is now the same as that in Theorem 2.1. \square

Proof of Proposition 2.7. Consider (2.1) in the special cases of (2.11) or (2.12). Associated with the former and latter are the functions

$$A_1(w) := \Gamma(w)A_0, \quad B_1(w) := \Gamma(w)B_0 \quad \text{and} \quad E_1(w) := E_0, \quad (\text{C.02})$$

and

$$A_2(w) := A_0\Gamma(w), \quad B_2(w) := B_0 \quad \text{and} \quad E_2(w) := E_0\Gamma(w), \quad (\text{C.03})$$

respectively. Both (2.11) and (2.12) have the same nonlinear term Θ_0 . We need to show that the assumptions **(M1)**, **(M3)** and **(S)** hold for (C.02) and Θ_0 if, and only if, they hold for (C.03) and Θ_0 . In this case, for brevity in the current proof, we informally say that the properties are equivalent.

Since Γ is assumed to satisfy **(D)**, it follows that Γ is continuous and non-increasing in the sense of (1.3). Moreover, $\Gamma(w)$ is diagonal with positive diagonal entries for all $w \in \mathbb{R}_+$. Consequently, that property **(M1)** is equivalent is clear. To investigate property **(M3)**, note that $\Gamma(w)$ is invertible with a diagonal inverse with positive diagonal entries for all $w \in \mathbb{R}_+$. Hence, for fixed $w \in \mathbb{R}_+$, which we suppress as an argument for clarity, we have

$$A_1 + B_1\Delta E_1 = \Gamma(A_0 + B_0\Delta E_0). \quad (\text{C.04})$$

is irreducible if, and only if, $A_0 + B_0\Delta E_0$ is irreducible, which occurs if, and only if,

$$A_2 + B_2\Delta E_2 = (A_0 + B_0\Delta E_0)\Gamma, \quad (\text{C.05})$$

is irreducible. Since $w \in \mathbb{R}_+$ was arbitrary, we conclude that property **(M3)** is equivalent.

Finally, note that the matrices on the right-hand sides of (C.04) and (C.05) (with Δ replaced by Σ) are themselves the product of the same two matrices in different orders. Therefore, $A_1 + B_1\Sigma E_1$ and $A_2 + B_2\Sigma E_2$ have the same non-zero eigenvalues and, hence, property **(S)** is equivalent as well. The proof is complete. \square

Proof of Proposition 2.8. Fix $(x^0, u^0) \in \mathbb{R}_+^n \times \mathbb{R}_+$ with $x^0 \neq 0$, and let (x, u) denote the solution of (2.1) with $v = 0$ and $\lambda = 0$. That the control variable u is convergent follows from Theorem 2.1. We denote its limit by u_∞ . Let $G := A(u_\infty) + B(u_\infty)\Gamma E(u_\infty)$, where $\Gamma \in \mathbb{R}_+^{m \times q}$ is as in (2.13). Seeking a contradiction, assume that the claim is false, meaning $\rho(G) \geq 1$.

The estimate (B.13) holds with $w = u_\infty$, and letting $\zeta \in \mathbb{R}_+^n$ denote a strictly positive eigenvector corresponding to the spectral radius of $P := A(u_\infty) + B(u_\infty)\Delta E(u_\infty)$, which is irreducible by assumption **(M3)**, it follows that

$$|x(t)|_\zeta \geq (\rho(P))^t |x^0|_\zeta > 0 \quad \forall t \in \mathbb{Z}_+.$$

In particular, $x(t) \neq 0$ for every $t \in \mathbb{Z}_+$.

An application of statement (v) of Theorem 2.1 gives that $x(t) \rightarrow 0$ as $t \rightarrow \infty$ and, hence, $Ex(t) \rightarrow 0$ as $t \rightarrow \infty$. In particular, there exists $\tau \in \mathbb{Z}_+$ such that $Ex(t + \tau) \in U$ for all $t \in \mathbb{Z}_+$. Therefore, invoking the estimate (2.13) for Θ and the monotonicity of u , as in the derivation of (B.13), gives the lower bound

$$x(t + 1) \geq (A(u_\infty) + B(u_\infty)\Gamma E(u_\infty))x(t) = Gx(t) \quad \forall t \geq \tau.$$

Since G is irreducible, letting $\xi \in \mathbb{R}_+^n$ denote a strictly positive eigenvector of corresponding to $\rho(G)$, it follows that

$$|x(t + \tau)|_\xi \geq (\rho(G))^t |x(\tau)|_\xi \geq |x(\tau)|_\xi > 0 \quad \forall t \in \mathbb{Z}_+.$$

The above inequality contradicts $x(t) \rightarrow 0$ as $t \rightarrow \infty$, completing the proof. \square

D Further information for Section 3.2

We present additional information for the numerical simulations in Section 3.2.3. The parameter values used in (3.10) are

$$s_1 = 0.305, \quad s_2 = 0.899, \quad s_3 = 0.778, \quad s_4 = 0.980, \quad g_1 = 0.530, \quad g_2 = 0.02, \quad g_3 = 0.202, \quad (\text{D.01})$$

which are the same as those in [12, Section 2.5.1]. For the nonlinear term θ , we take

$$\theta(z) = \frac{80.477}{1 + (6 \times 10^{-5})z^{\frac{7}{5}}} \quad \forall z \geq 0, \quad (\text{D.02})$$

which satisfies $\theta(0) = 80.477$ — equal to the corresponding term from the linear (density-independent) model in [12, Section 2.5.1]. We note that $z \mapsto \theta(z)z$ is bounded.

The initial DRW population is taken to be

$$x^0 = (0.1973 \quad 1.0353 \quad 0.0933 \quad 0.9420)^T \times 10^5. \quad (\text{D.03})$$

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