On a Representation of Vector Continued Fractions

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Abstract

Vector Padé approximants to power series with vector coefficients may be calculated using the three-term recurrence relations of vector continued fractions if formulated in the framework of Clifford algebras. We show that the numerator and denominator polynomials of these fractions take particularly simple forms which require just a few degrees of freedom in their representation. The new description also allows the calculation of "hybrid" approximants.

1 Introduction

Given a power series with vector coefficients in $\mathbb{R}^d$ we may construct rational approximations to it using vector Padé approximants [5,15]. The theory of these approximants [6,13,16,19] parallels that of the scalar case [*e.g.* 2] if vectors are treated as objects in an appropriate algebra — Clifford algebras allow multiplication as well as addition of vectors. In particular, such approximants may be realised using vector continued fractions, thus enabling advantage to be taken of three-term recurrence relations in their computation. However, the polynomials obeying these relations can be complicated elements in the Clifford algebra, with perhaps as many as 2^{d-1} components required to describe them. In this paper we use the fact that these polynomials belong to a particular group — the Lipschitz group — and are then able to represent them fully using far fewer degrees of freedom. For example, the denominator requires a scalar polynomial together with an antisymmetric matrix of order d with polynomial entries — *i.e.* $1 + d(d-1)/2$ Clifford coefficients. This is important in numerical applications in particular where large dimensions may be encountered [3].

In the next section we introduce Clifford algebras and the Lipschitz group, before defining in the following section vector continued fractions which form approximants to a given power series with vector coefficients. We also prove that the numerator and denominator polynomials of *corresponding* vector continued fractions (with appropriate normalisation) belong to the Lipschitz group. In section 4 we derive a new representation for these polynomials using a result discovered by Lipschitz(1886). Finally, we present some examples, including numerical, illustrating the use of the representation to derive "hybrid" approximants [4,7].

2 Clifford algebras

The real Clifford algebra of $\mathbb{R}^d$, $C\ell_d$,is the associative algebra over $\mathbb{R}$ generated by the orthonormal basis of $\mathbb{R}^d$, $\{\mathbf{e}_1, \mathbf{e}_2 \cdots \mathbf{e}_d\}$, which satisfies the anti-commutation relations

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = 2\delta_{i,j} \qquad\qquad i, j = 1, 2 \cdots, d \tag{2.1}$$

where the algebra identity is 1 [11,12]. The universality property, $\mathbf{e}_1\mathbf{e}_2 \cdots \mathbf{e}_d \neq \pm 1$, ensures that $C\ell_d$ is a linear space of dimension 2^d spanned by the basis elements

$$\mathbf{e}_I = \mathbf{e}_{i_1 i_2 \cdots i_k} = \mathbf{e}_{i_1}\mathbf{e}_{i_2} \cdots \mathbf{e}_{i_k} \tag{2.2}$$

where $I = \{i_1, i_2, \cdots, i_k\}$ and $1 \leq i_1 < i_2 < \cdots i_k \leq d$ for $k = 1, 2 \cdots, d$. The identity element corresponds to the empty set *i.e.* $k = 0$. A general element of $C\ell_d$ is given by

$$a = \sum_I a_I \mathbf{e}_I \qquad a_I \in \mathbb{R} \tag{2.3}$$

where the summation is over the 2^d different ordered multi-indices I.

For given k, the $\mathbf{e}_I$ form the basis of a subspace, $\bigwedge^k \mathbb{R}^d$, whose elements are called $k-$vectors. $C\ell_d$ is the direct sum of the spaces $\bigwedge^k \mathbb{R}^d$ for $k = 0, 1, \cdots, d$. The k-vector part of the Clifford element a is denoted by $< a >_k$. The coefficient $a_0 :=< a >_0$ is called the real or scalar part of a, and is also denoted by $Re(a)$.

The spinor norm or absolute value of an element is given by the Euclidean norm on $C\ell_d$

$$|a| = \sqrt{\sum_I |a_I|^2}. \tag{2.4}$$

Following [1,10], we may define an inner and outer product in $C\ell_d$. We first of all consider two multivectors $a \in \bigwedge^i \mathbb{R}^d$ and $b \in \bigwedge^j \mathbb{R}^d$ and note that their product is in the direct sum

$$\bigwedge^{i+j} \mathbb{R}^d + \bigwedge^{i+j-2} \mathbb{R}^d + \cdots + \bigwedge^{|i-j|} \mathbb{R}^d.$$

The component in $\bigwedge^{i+j} \mathbb{R}^d$ is called the outer product $a \wedge b$ while the component in $\bigwedge^{|i-j|} \mathbb{R}^d$ is the inner product $a \cdot b$. The two products may be extended to all of $C\ell_d$ by linearity. We note that the outer product is associative [1,10]*i.e.* $(a \wedge b) \wedge c = a \wedge (b \wedge c)$. We shall only use the inner product in situations where at least one of a, b belongs to $\mathbb{R}^d$.

Each vector $(v_1, v_2, \cdots, v_d) \in \mathbb{R}^d$ is identified with an element, $\sum_{i=1}^d v_i \mathbf{e}_i$, of $C\ell_d$, using the common label $\mathbf{v}$. We use the Euclidean norm in $\mathbb{R}^d$ which is consistent with the spinor norm applied to vectors.

The anti-commutation relations (2.1) imply

$$\mathbf{uv} + \mathbf{vu} = 2(\mathbf{u} \cdot \mathbf{v}) \tag{2.5}$$

where $\mathbf{u} \cdot \mathbf{v}$ coincides with the usual scalar product, $\sum_{i=1}^d u_i v_i$, and

$$\mathbf{uvu} = 2(\mathbf{u} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{v} \tag{2.6}$$

i.e. $\mathbf{uvu} \in \mathbb{R}^d$. We have the identity

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} \tag{2.7}$$

where $\mathbf{u} \wedge \mathbf{v}$ denotes the *bivector* (2-vector)

$$\frac{1}{2}[\mathbf{uv} - \mathbf{vu}] = \sum_{i<j} (u_i v_j - v_i u_j)\mathbf{e}_{ij} \quad \in \bigwedge^2 \mathbb{R}^d. \tag{2.8}$$

There are two involutions on $C\ell_d$ which we shall need. The first, called the *main involution*, is the isomorphism : $a \mapsto \hat{a}$ in which each $\mathbf{e}_i$ is replaced by $-\mathbf{e}_i$; hence $\widehat{ab} = \hat{a}\hat{b}$. The other one is the anti-isomorphism : $a \mapsto \tilde{a}$ obtained by reversing the order of factors in $\mathbf{e}_I$, and is called *reversion*; hence $\widetilde{ab} = \tilde{b}\tilde{a}$.

The set of products of invertible vectors forms a group under multiplication — the *Lipschitz group*, Γ_d [1,10,11]. If $a \in \Gamma_d$ then $a\tilde{a} = \tilde{a}a = |a|^2$. Hence,

$$a^{-1} = \frac{\tilde{a}}{|a|^2}. \tag{2.9}$$

If $a = \mathbf{v} \in \mathbb{R}^d$ then we obtain the Moore-Penrose generalised inverse of a real vector

$$\mathbf{v}^{-1} = \frac{\mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \tag{2.10}$$

For elements in the Lipschitz group we have [1]

$$|ab| = |a||b| \qquad\qquad \forall\, a, b \in \Gamma_d. \tag{2.11}$$

3 Vector Continued Fractions

We are interested in rational approximations, in the form of continued fractions, to vector-valued functions $\mathbf{f}(x)$ whose Maclaurin series expansions are known

$$\mathbf{f}(x) = \mathbf{c}_0 + x\mathbf{c}_1 + x^2\mathbf{c}_2 + \ldots, \qquad x \in \mathbb{R}, \quad \mathbf{c}_i \in \mathbb{R}^d, \quad i = 0,\, 1,\, \ldots \tag{3.1}$$

and are valid in some neighbourhood of the origin. We consider continued fractions

$$b_0 + xa_1[b_1 + xa_2[b_2 + \cdots]^{-1}]^{-1} \tag{3.2}$$

with elements in $C\ell_d$ which *correspond* to $\mathbf{f}(x)$. That is

$$\mathbf{f}(x) - \mathbf{C}_n(x) = O(x^{n+1}) \qquad\qquad n = 0, 1, 2\cdots \tag{3.3}$$

where $\mathbf{C}_n(x)$ is the n^{th} convergent of (3.2). In particular we discuss two types:
(i) one in which the partial numerators are simply x — *i.e.* $a_i = 1, i = 1, 2, \cdots$,
(ii) and the other with unit partial denominators — *i.e.* $b_i = 1, i = 1, 2, \cdots$.

For (i) we obtain

$$\mathbf{C}_n(x) = \boldsymbol{\pi}_0 + x[\boldsymbol{\pi}_1 + x[\boldsymbol{\pi}_2 + \cdots + x[\boldsymbol{\pi}_n]^{-1}\cdots]^{-1}]^{-1} = p_n(x)[q_n(x)]^{-1} \tag{3.4}$$

with $\mathbf{C}_0(x) = \boldsymbol{\pi}_0$, where $p_n(x), q_n(x)$ are polynomials in $C\ell_d[x]$, of degrees $[n + 1/2]$ and $[n/2]$, respectively, for $n \geq 1$ — the square brackets [ ] denoting the integer part. These polynomials satisfy the three-term recurrence relations

$$\left.\begin{array}{lll} p_n(x) = p_{n-1}(x)\boldsymbol{\pi}_n + xp_{n-2}(x) & p_{-1}(x) := 1 & p_0(x) := \boldsymbol{\pi}_0 = \mathbf{c}_0 \\ q_n(x) = q_{n-1}(x)\boldsymbol{\pi}_n + xq_{n-2}(x) & q_{-1}(x) := 0 & q_0(x) := 1 \end{array}\right\} \tag{3.5}$$

for $n = 1, 2\cdots$. Hence, $q_n(0) = \boldsymbol{\pi}_1\boldsymbol{\pi}_2\cdots\boldsymbol{\pi}_n \in \Gamma_d$. In fact, $\mathbf{C}_n(x)$ is the $[l/m]$ vector Padé approximant where $l = [n + 1/2]$ and $m = [n/2]$ [6,13].

We shall restrict ourselves to the non-degenerate case, in which $\boldsymbol{\pi}_i \neq \mathbf{0}$, $i = 1, 2\cdots$. The more general case is considered in [6], where it is shown that the elements of the continued fraction may be evaluated using vector versions of the Viskovatov and modified Euclidean algorithms. However, the methods developed here remain valid in the more general case.

It may be shown, using the methods of [13], that

$$Q_n(x) := q_n(x)\widetilde{q_n(x)} \in \mathbb{R}[x] \tag{3.6}$$

and

$$\mathbf{P}_n(x) := p_n(x)\widetilde{q_n(x)} \in \mathbb{R}^d[x] \tag{3.7}$$

Hence, the denominator is invertible

$$[q_n(x)]^{-1} = \frac{\widetilde{q_n(x)}}{Q_n(x)} \tag{3.8}$$

allowing the n^{th} convergent to be written as the vector-valued rational function

$$\mathbf{C}_n(x) = \frac{\mathbf{P}_n(x)}{Q_n(x)} \tag{3.9}$$

which is in the form of a generalised inverse Padé approximant, first defined and studied by Graves-Morris *e.g.* [5]. $Q_n(x)$ is of degree $2[n/2]$ and each component of $\mathbf{P}_n(x)$ is of maximum degree n.

In [14] it is shown that

$$p_n(x), q_n(x) \in \Gamma_d \quad \text{for each } x \in \mathbb{R}. \tag{3.10}$$

Here we present a proof using the definition of the Lipschitz group given above. From (3.5) we obtain

$$\mathbf{v}_n(x) = \boldsymbol{\pi}_n + x[\mathbf{v}_{n-1}(x)]^{-1} \qquad n = 1, 2, \cdots \tag{3.11}$$

where

$$\mathbf{v}_n(x) := [q_{n-1}(x)]^{-1} q_n(x) \qquad n = 1, 2, \cdots. \tag{3.12}$$

Since $\mathbf{v}_1(x) := \boldsymbol{\pi}_1 \in \mathbb{R}^d$, it follows that each $\mathbf{v}_n(x)$ is a vector in $\mathbb{R}^d$. Furthermore, since $\mathbf{v}_n(0) = \boldsymbol{\pi}_n$, our assumption of non-degeneracy ensures that none of these vectors is identically zero. Hence,

$$q_n(x) = \mathbf{v}_1(x)\mathbf{v}_2(x)\cdots\mathbf{v}_n(x) \quad \in \quad \Gamma_d. \tag{3.13}$$

In a similar manner we may demonstrate that $\mathbf{u}_n(x) := [p_{n-1}(x)]^{-1} p_n(x)$ is also a vector in $\mathbb{R}^d$ for $n = 0, 1, \cdots$. Therefore, assuming that $\boldsymbol{\pi}_0 \neq \mathbf{0}$,

$$p_n(x) = \mathbf{u}_0(x)\mathbf{u}_1(x)\cdots\mathbf{u}_n(x) \quad \in \quad \Gamma_d. \tag{3.14}$$

The second type of continued fraction is of the form

$$b_0' + xa_1'[1 + xa_2'[1 + \cdots xa_{n-1}'[1 + xa_n']^{-1} \cdots]^{-1}]^{-1} \tag{3.15}$$

which is equivalent to (3.4). This may be seen by employing the *equivalence transformation* :

$$\left.\begin{array}{c} b_0 = b_0' \;\;,\;\; a_1 = a_1'\alpha_1 \;\;,\;\; b_1 = b_1'\alpha_1 \;\;,\;\; a_2 = a_2'\alpha_2 \\ b_i = (\alpha_{i-1})^{-1} b_i'\alpha_i \quad i \geq 2 \quad \text{and} \quad a_i = (\alpha_{i-2})^{-1} a_i'\alpha_i \quad i \geq 3 \end{array}\right\} \tag{3.16}$$

Here, each α_i $i = 1, 2, \cdots$ is the invertible element

$$\alpha_i = \boldsymbol{\pi}_1 \boldsymbol{\pi}_2 \cdots \boldsymbol{\pi}_i \in \Gamma_d \qquad i = 1, 2, \cdots. \tag{3.17}$$

and

$$b_0 := \mathbf{c}_0 \;\;,\;\; a_i := 1 \;\;,\;\; b_i := \boldsymbol{\pi}_i \quad i = 1, 2, \cdots \tag{3.18}$$

with $b'_i = 1$ for $i = 1, 2, \cdots$.

The n^{th} numerator $A_n(x)$, and denominator $B_n(x)$ of (3.15) satisfy the recurrence relations

$$\left.\begin{array}{l} A_n(x) := A_{n-1}(x) + xA_{n-2}(x)a'_n \\ B_n(x) := B_{n-1}(x) + xB_{n-2}(x)a'_n \end{array}\right\} \tag{3.19}$$

for $n = 1, 2 \cdots$, with the initial conditions

$$\left.\begin{array}{ll} A_{-1}(x) := 1, & A_0(x) := \mathbf{c}_0 \\ B_{-1}(x) := 0, & B_0(x) := 1 \end{array}\right\} \tag{3.20}$$

It then follows that

$$A_n(x) = p_n(x)[\alpha_n]^{-1} \quad \text{and} \quad B_n(x) = q_n(x)[\alpha_n]^{-1} \tag{3.21}$$

Therefore, from (3.13) and (3.14) we obtain

$$B_n(x) = \mathbf{v}_1(x)\mathbf{v}_2(x)\cdots\mathbf{v}_n(x)[\boldsymbol{\pi}_n]^{-1}\cdots[\boldsymbol{\pi}_2]^{-1}[\boldsymbol{\pi}_1]^{-1} \quad \in \quad \Gamma_d^+ \tag{3.22}$$

and

$$A_n(x) = \mathbf{u}_0(x)\mathbf{u}_1(x)\cdots\mathbf{u}_n(x)[\boldsymbol{\pi}_n]^{-1}\cdots[\boldsymbol{\pi}_2]^{-1}[\boldsymbol{\pi}_1]^{-1} \quad \in \quad \Gamma_d^- \tag{3.23}$$

where $\Gamma_d^+(\Gamma_d^-)$ denotes the set of products of even (odd) numbers of vectors. For later use we note that $B_n(0) = 1$ and $A_n(0) = \boldsymbol{\pi}_0 = \mathbf{c}_0$.

It is a simple matter [18] to use the recurrence relations to show that the continued fraction elements a'_n are ratios of vectors in $\mathbb{R}^d$

$$a'_n = -[\mathbf{s}_{n-1}]^{-1}\mathbf{s}_n \tag{3.24}$$

which are linked to the leading coefficient in the Padé error

$$\mathbf{f}(x) - A_n(x)[B_n(x)]^{-1} = \mathbf{s}_{n+1}x^{n+1} + O(x^{n+2}) \quad n = 0, 1, 2\cdots \tag{3.25}$$

In [18] it is shown how a vector version of the quotient-difference algorithm allows the computation of the $\mathbf{s}_n$ using *only* vector operations of scalar multiplication, scalar product as well as of addition. Thus, the components of the elements a'_n may be calculated without using Clifford numbers.

Example 1: The [1/1] vector Padé approximant is given by the third convergent $\mathbf{C}_3$, which may be derived as follows :

$$\mathbf{f}(x) = \mathbf{c}_0 + x\mathbf{c}_1 + x^2\mathbf{c}_2 + \ldots,$$

$$= \mathbf{c}_0 + x\mathbf{c}_1[1 - x{\mathbf{c}_1}^{-1}\mathbf{c}_2]^{-1} + O(x^3)$$

Hence,

$$B_3(x) = 1 - x{\mathbf{c}_1}^{-1}\mathbf{c}_2 \tag{3.26}$$

and

$$A_3(x) = \mathbf{c}_0[1 - x{\mathbf{c}_1}^{-1}\mathbf{c}_2] + x\mathbf{c}_1 \tag{3.27}$$

Or, using (2.7)

$$B_3(x) = [1 - x{\mathbf{c}_1}^{-1} \cdot \mathbf{c}_2] - x{\mathbf{c}_1}^{-1} \wedge \mathbf{c}_2 \tag{3.28}$$

i.e. a scalar together with a bivector; and

$$A_3(x) = \{\mathbf{c}_0 + x[\mathbf{c}_1 - (\mathbf{c}_0 \cdot {\mathbf{c}_1}^{-1})\mathbf{c}_2 + (\mathbf{c}_0 \cdot \mathbf{c}_2){\mathbf{c}_1}^{-1} - (\mathbf{c}_2 \cdot {\mathbf{c}_1}^{-1})\mathbf{c}_0]\} - x\mathbf{c}_0 \wedge {\mathbf{c}_1}^{-1} \wedge \mathbf{c}_2 \tag{3.29}$$

i.e. a vector and a 3-vector. In the general case the polynomials might require 2^{d-1} Clifford terms in the expansion (2.3). We shall employ the Lipschitz group to develop representations of these polynomials which require far fewer degrees of freedom.

4 The Lipschitz Group

From (2.6) and (2.10) we may deduce that the reflection of $\mathbf{v}$ in the hyperplane orthogonal to $\mathbf{u}$ is given by $\mathbf{u}\mathbf{v}\hat{\mathbf{u}}^{-1}$. Since an isometry of $\mathbb{R}^d$ may be accomplished by a sequence of reflections *c.f.* [11,12] a rotation of a vector $\mathbf{v}$ may be represented by

$$a\mathbf{v}\hat{a}^{-1} \qquad\qquad \text{for some} \quad a \in \Gamma_d. \tag{4.1}$$

Alternatively, it is clear from repeated use of (2.6) and (2.10) that the Clifford element of (4.1) is a vector $\mathbf{u}$ whose square $\mathbf{u}\tilde{\mathbf{u}}$ is equal to that of $\mathbf{v}$, thus showing that the transformation $\mathbf{v} \to \mathbf{u}$ is a Euclidean isometry. This rotation corresponds to an orthogonal transformation of $\mathbf{v}$ denoted by $U\mathbf{v}$, where $U \in O(d)$. It is shown in [1] that if $a \in \Gamma_d^+$ then it is determined up to a real factor by the orthogonal map it induces. This association is made precise in the theorem below. In order to present this result we introduce some definitions *c.f.* [1,9,10].
The Cayley Transform of an antisymmetric matrix M, of order d, is defined to be the orthogonal matrix

$$U := [I - M][I + M]^{-1} \quad \in \quad SO(d) \tag{4.2}$$

where I is the unit matrix of order d.

We may construct a bivector $\mu \in C\ell_d$ corresponding to M by

$$\mu := \sum_{i<j} M_{i,j}\mathbf{e}_i\mathbf{e}_j \tag{4.3}$$

so that the d-tuple $M\mathbf{v}$ corresponds to the Clifford element $\mu \cdot \mathbf{v}$ *i.e.*

$$\mu \cdot \mathbf{v} = \sum_{i,j=1}^{d} M_{i,j} v_j \mathbf{e}_i. \tag{4.4}$$

We introduce the notation

$$\bigwedge^{k} a := a \wedge a \wedge \cdots \wedge a \qquad (k - \text{factors}) \tag{4.5}$$

which vanishes if k is greater than the number of distinct $\mathbf{e}_i$ in the expansion of a in (2.3). The outer exponential of μ, $\exp[\mu]$, is the exponential series with the outer product as multiplication

$$\exp[\mu] := \sum_{k=0}^{\infty} \frac{\bigwedge^k \mu}{k!}. \tag{4.6}$$

Unless it is zero the k^{th} term is a $2k$-vector. Hence, (4.6) is a finite sum of at most $1 + [d/2]$ terms. The reader is referred to [1,9,10,11] for a discussion of links with Pfaffians.

We are now in a position to state a theorem first given by Lipschitz (1886) [9,10]. We adapt the statement of the theorem given in section 3 of [1] for our purposes.
Theorem 4.1 *Given a bivector μ in $C\ell_d$, there is one and only one element $a \in \Gamma_d^+$ with $< a >_0 = 1$ and $< a >_2 = \mu$: this element is the outer exponential of μ i.e. $a = exp[\mu]$.*

For a modern proof the reader is referred to [1,10]. Furthermore [9,10], the spinor norm of $a \in \Gamma_d^+$ with real part 1 such that

$$U\mathbf{v} = a\mathbf{v}\hat{a}^{-1} \tag{4.7}$$

is the square root of $a\tilde{a}$ which is given by

$$\det(I + M) = \det\Bigl(\frac{I+U}{2}\Bigr)^{-1} \tag{4.8}$$

We now apply the Lipschitz theorem to the denominator polynomial $B_n(x)$ of a vector Padé approximant satisfying the condition $B_n(0) = 1$. In the following, where appropriate, vectors are represented by column matrices and the superscript T denotes the matrix transpose; for simplicity we relabel the vectors involved in the continued fraction element a'_n by

$$\mathbf{u}_n := [\mathbf{s}_{n-1}]^{-1} \qquad , \qquad \mathbf{v}_n := \mathbf{s}_n \tag{4.9}$$

so that $a'_n = -\mathbf{u}_n\mathbf{v}_n$. We introduce the scalar polynomial of degree $m := [n/2]$

$$\sigma_n(x) := < B_n(x) >_0 \quad \in \quad \mathbb{R}[x] \ , \tag{4.10}$$

and the bivector polynomial

$$x\delta_n(x) \quad = \quad \sum_{i<j} \{x\Delta_n(x)\}_{i,j}\mathbf{e}_i\mathbf{e}_j \quad := \quad < B_n(x) >_2 \tag{4.11}$$

in which $\Delta_n(x)$ is an antisymmetric matrix of order d with polynomial entries each of maximum degree $m-1$. We have used the fact that $B_n(0) = 1$,thus implying that the bivector in (4.11) vanishes at the origin. Note also that $\sigma_n(0) = 1$. We further define the bivector

$$\beta_n(x) := \frac{\delta_n(x)}{\sigma_n(x)} \tag{4.12}$$

and the square matrix of order d

$$D_n(x) := \sigma_n(x)I + x\Delta_n(x). \tag{4.13}$$

Corresponding to the bivector

$$\mathbf{u}\wedge\mathbf{v} = \sum_{i<j}[u_iv_j - u_jv_i]\mathbf{e}_i\mathbf{e}_j$$

we introduce the wedge product of two column matrices $\mathbf{u},\mathbf{v}$ as the antisymmetric matrix

$$[\mathbf{u}\wedge\mathbf{v}]_{i,j} := [u_iv_j - u_jv_i]. \tag{4.14}$$

Theorem 4.2 *The denominator polynomials $B_n(x)$ may be represented by*

$$B_n(x) = \sigma_n(x)\exp[x\beta_n(x)] = \sigma_n(x) + x\delta_n(x) + \frac{x^2}{2}\delta_n(x)\wedge\beta_n(x) + \cdots \tag{4.15}$$

where $\sigma_n(x)$ and $\delta_n(x)$ are defined by (4.10) *and* (4.11)*.*

The scalar polynomial $\sigma_n(x)$ and the antisymmetric matrix $\Delta_n(x)$, corresponding to $\delta_n(x)$, satisfy the recurrence relations

$$\sigma_n(x) - \sigma_{n-1}(x) = -x\mathbf{v}_n^T D_{n-2}\mathbf{u}_n \tag{4.16}$$

$$\Delta_n(x) - \Delta_{n-1}(x) = \frac{1}{\sigma_{n-2}(x)}\{[\sigma_n(x) - \sigma_{n-1}(x)]\Delta_{n-2}(x) + [D_{n-2}(x)\mathbf{v}_n]\wedge[D_{n-2}(x)\mathbf{u}_n]\} \tag{4.17}$$

for $n = 2, 3, \cdots$.

The initialisations are

$$\sigma_0(x) = 1 \quad , \quad \sigma_1(x) = 1 \tag{4.18}$$

$$\Delta_0(x) = O \quad , \quad \Delta_1(x) = O \tag{4.19}$$

where O is the null matrix of order d.

Proof It has been shown that $B_n(x) \in \Gamma_d^+$. Hence, the Clifford element

$$a := \frac{B_n(x)}{\sigma_n(x)} \quad \in \quad \Gamma_d^+$$

has real part 1, thus satisfying the conditions of Theorem 4.1. Therefore, since $< a >_2 = x\beta_n(x)$, we have

$$\frac{B_n(x)}{\sigma_n(x)} = \exp[x\beta_n(x)] \tag{4.20}$$

which yields (4.15) on expansion of the outer exponential. That is, each denominator polynomial $B_n(x)$ is characterised by a scalar polynomial $\sigma_n(x)$ of maximum degree $[n/2]$ and an antisymmetric matrix of polynomials $\Delta_n(x)$ each of maximum degree one less. Recurrence relations for $\sigma_n(x)$ and $\Delta_n(x)$ are derived by taking the scalar and bivector parts of the recurrence relation for $B_n(x)$. From (3.19), (3.24) and (4.9)

$$B_n(x) = B_{n-1}(x) - xB_{n-2}(x)\mathbf{u}_n\mathbf{v}_n \tag{4.21}$$

The second term on the right-hand side is

$$-x[\sigma_{n-2}(x) + x\delta_{n-2}(x) + \frac{x^2}{2}\delta_{n-2}(x) \wedge \beta_{n-2}(x) + \cdots]\mathbf{u}_n\mathbf{v}_n \tag{4.22}$$

The scalar part of this expression is

$$-x\{\sigma_{n-2}(x)\mathbf{v}_n \cdot \mathbf{u}_n + x[\delta_{n-2}(x) \cdot \mathbf{u}_n] \cdot \mathbf{v}_n\} \tag{4.23}$$

which, in matrix form, equals

$$-x[\sigma_{n-2}(x)\mathbf{v}_n^T\mathbf{u}_n + x\mathbf{v}_n^T\Delta_{n-2}(x)\mathbf{u}_n] \quad = \quad -x\mathbf{v}_n^T D_{n-2}(x)\mathbf{u}_n \tag{4.24}$$

thus establishing (4.16). After extracting a factor of $(-x)$ the bivector part of (4.22) is given by

$$\sigma_{n-2}(x)\mathbf{u}_n \wedge \mathbf{v}_n + x < \delta_{n-2}(x)\mathbf{u}_n\mathbf{v}_n >_2 + \frac{x^2}{2\sigma_{n-2}(x)} < \delta_{n-2}(x) \wedge \delta_{n-2}(x)\mathbf{u}_n\mathbf{v}_n >_2.$$

In order to compute this expression we use the following identities involving a bivector μ and vectors $\mathbf{u}, \mathbf{v}$:

$$< \mu\mathbf{u}\mathbf{v} >_2 = [\mu \cdot \mathbf{u}] \wedge \mathbf{v} + \mathbf{u} \wedge [\mu \cdot \mathbf{v}] + \mu[\mathbf{u} \cdot \mathbf{v}] \tag{4.25}$$

$$\frac{1}{2} < \mu \wedge \mu\mathbf{u}\mathbf{v} >_2 = [\mu \cdot \mathbf{u}] \wedge [\mu \cdot \mathbf{v}] - [\mathbf{u} \cdot (\mu \cdot \mathbf{v})]\mu \tag{4.26}$$

which may be proved using the anti-commutation relations (2.1). Denoting the matrix associated with the bivector μ by M, the corresponding antisymmetric matrices of (4.25) and (4.26) are given by

$$[M\mathbf{u}] \wedge \mathbf{v} + \mathbf{u} \wedge [M\mathbf{v}] + M[\mathbf{u}^T\mathbf{v}] \tag{4.27}$$

and

$$[M\mathbf{u}] \wedge [M\mathbf{v}] - [\mathbf{u}^T M\mathbf{v}]M \tag{4.28}$$

respectively. Then, after some manipulation, we may prove that $\Delta_n(x)$ satisfies the recurrence relation (4.17).

The initialisations follow from $B_0(x) = 1$, $B_1(x) = 1$. □

We note that since each term in the expansion (4.15) is a polynomial $2k$-vector we must have

$$\sigma_n(x)^k \quad | \quad \bigwedge^{k+1} \delta_n(x) \tag{4.29}$$

for $k = 1, 2, \cdots$.

Corollary 4.3 *The numerator polynomial $A_n(x)$ corresponding to the denominator $B_n(x)$ of Theorem 4.2 may be represented by*

$$A_n(x) = \mathbf{c}_0\sigma'_n(x)\exp[x\beta'_n(x)] \tag{4.30}$$

where

$$\beta'_n(x) := \frac{\delta'_n(x)}{\sigma'_n(x)}$$

The scalar polynomial $\sigma'_n(x)$, of maximum degree $[n+1/2]$, and the antisymmetric matrix of polynomials, each of max degree $[n-1/2]$, $\Delta'_n(x)$, corresponding to the bivector $\delta'_n(x)$, satisfy the same recurrence relations as the denominator polynomials (4.16,17), with the initialisations

$$\sigma'_0(x) = 1 \quad , \quad \sigma'_1(x) = 1 + x\mathbf{c}_0{}^{-1}\mathbf{c}_1 \tag{4.31}$$

$$\Delta'_0(x) = O \quad , \quad \Delta'_1(x) = \mathbf{c}_0{}^{-1} \wedge \mathbf{c}_1 \tag{4.32}$$

Proof We note that $A'_n(x) := \mathbf{c}_0{}^{-1}A_n(x)$ belongs to Γ_d^+ and that $A'_n(0) = 1$. Theorem 4.2 implies that there is a scalar polynomial $\sigma'_n(x)$ and an antisymmetric matrix of order d, $\Delta'_n(x)$, corresponding to the bivector $\delta'_n(x)$, such that

$$A'_n(x) = \sigma'_n(x)\exp[x\beta'_n(x)]$$

with

$$\beta'_n(x) := \frac{\delta'_n(x)}{\sigma'_n(x)}$$

The primed polynomials satisfy the same recurrence relations (4.16, 17) as the unprimed. The degrees stated follow from the fact that the maximum degree of $A_n(x)$ is $[n+1/2]$. The initialisations follow from $A'_0(x) = 1$, $A'_1(x) = 1 + x\mathbf{c}_0{}^{-1}\mathbf{c}_1$. □

5 Examples

We first of all demonstrate the representation involving the outer exponential using the illustrative example introduced earlier.
Example 2: From (3.28) we have

$$< B_3(x) >_0 = 1 - x\mathbf{c}_1{}^{-1} \cdot \mathbf{c}_2 = \sigma_3(x) \tag{5.1}$$

$$< B_3(x) >_2 = -x\mathbf{c}_1{}^{-1} \wedge \mathbf{c}_2 = x\delta_3(x) \tag{5.2}$$

We note that

$$\delta_3(x) \wedge \delta_3(x) = 0 \tag{5.3}$$

which follows from the associativity of the outer product and the fact that $\mathbf{u} \wedge \mathbf{u}$ vanishes for any vector $\mathbf{u} \in \mathbb{R}^d$. Hence,

$$\sigma_3(x)\exp[x\beta_3(x)] = \sigma_3(x)[1 + x\beta_3(x)] = \sigma_3(x) + x\delta_3(x) = B_3(x) \tag{5.4}$$

For the numerator (3.27) we find that

$$\sigma_3'(x) = 1 + x\{\mathbf{c}_0{}^{-1} \cdot \mathbf{c}_1 - \mathbf{c}_1{}^{-1} \cdot \mathbf{c}_2\} \tag{5.5}$$

and

$$\delta_3'(x) = \mathbf{c}_0{}^{-1} \wedge \mathbf{c}_1 - \mathbf{c}_1{}^{-1} \wedge \mathbf{c}_2 \tag{5.6}$$

A argument similar to that above shows that

$$\mathbf{c}_0\sigma_3'(x)\exp[x\beta_3'(x)] = \mathbf{c}_0[\sigma_3'(x) + x\delta_3'(x)] = A_3(x). \tag{5.7}$$

The other examples come from an implementation of the recurrence relations $(4.16, 17)$ for the denominator of approximants to vector-valued functions of the form

$$\mathbf{f}(z) = \frac{\mathbf{g}(z)}{R(z)} \qquad z \in \mathbb{C} \tag{5.8}$$

where each $g_i(z), \quad i = 1, \cdots, d$ is analytic in $D_\rho := \{z \in \mathbb{C} : |z| < \rho\}$ for some $\rho > 0$ and $R(z) \in \mathbb{R}[z]$ normalised by $R(0) = 1$, with zeroes $z_k \in D_\rho \quad k = 1, 2 \cdots, m$. This is the type of generating function encountered in the iterative solution to systems of linear equations [4,15].

We wish to investigate the behaviour of the Clifford denominators of the $[l/m]$ vector Padé approximants to $\mathbf{f}(x)$ as $l \to \infty$. These denominators are obtained by considering the $2m^{th}$ convergents to the series starting at $\mathbf{c}_{l-m+1}$ rather than $\mathbf{c}_0$ [2,13] — a superscript l denotes the corresponding quantities. From [17] these polynomials over $C\ell_d$ tend to the scalar $R(z)$. To be more precise,

$$lim_{l\to\infty} B_{2m}^l(z) = R(z) \tag{5.9}$$

the convergence being uniform in compact subsets of the complex plane E. That is, using the spinor norm (2.4) for $a_I \in \mathbb{C}$, there is an integer L such that, given any $\epsilon > 0$

$$|R(z) - B^l_{2m}(z)| < \epsilon \quad z \in E \text{ and } l > L \tag{5.10}$$

However, from (2.4,4.15) and the statement following (4.6) we may write

$$\begin{array}{rcl} |R(z) - B^l_{2m}(z)|^2 & = & |R(z) - \sigma^l_{2m}(z)|^2 + |z\delta^l_{2m}(z)|^2 + \frac{1}{2}|z\delta^l_{2m}(z) \wedge \delta^l_{2m}(z)|^2 + \cdots \\ & \geq & |R(z) - \sigma^l_{2m}(z)|^2 + |z\delta^l_{2m}(z)|^2 \end{array} \tag{5.11}$$

From this it follows that

$$lim_{l\to\infty}\sigma^l_{2m}(z) = R(z) \tag{5.12}$$

$$lim_{l\to\infty}\Delta^l_{2m}(z) = O \tag{5.13}$$

each convergence being uniform for $z \in E$. The denominator of the generalised inverse version of the approximant (3.9) may be calculated using (4.8)

$$Q^l_{2m}(z) = B^l_{2m}(x)\widetilde{B^l_{2m}(x)} = [\sigma^l_{2m}(z)]^{2-d}\det[D^l_{2m}(z)] \tag{5.14}$$

using an obvious notation. From [17] equation (3.33) we may write

$$\lim_{l\to\infty} Q^l_{2m}(z) = R(z)^2 \tag{5.15}$$

the convergence being uniform for $z \in E$. Note that the determinant is a polynomial which must be divisible by the $(d-2)^{th}$ power of $\sigma^l_n(z)$.

The particular functions we consider are of the form

$$\mathbf{f}(x) = \sum_{i=1}^{4} \frac{\mathbf{r}_i}{(1-\lambda_i x)} = \frac{\mathbf{g}(x)}{R(x)} \tag{5.16}$$

where

$$[z_i]^{-1} := \lambda_i = 6 - i, \quad \text{for } i = 1, \cdots 4, \tag{5.17}$$

$$R(x) := (1 - x\lambda_1)(1 - x\lambda_2)(1 - x\lambda_3) \tag{5.18}$$

and $\mathbf{r}_i, i = 1,2,3,4$ are various vectors in $\mathbb{R}^6$ defined below. The functions $g_k(z), k = 1, \cdots 6$ are analytic for $|z| < 0.5$.

Using Theorem 4.2 we calculate the polynomial $\sigma^l_6(x)$ and the matrix of polynomials $\Delta^l_6(x)$. The zeroes of the cubic polynomial $\sigma^l_6(x)$ are denoted by z^l_i, $i = 1,2,3$. To check (5.13) the Frobenius norm of $\Delta^l_6(x)$, denoted by $||\cdot||$, is computed at $x = 0.1, 0.3, 1.0$. We note that $\sqrt{2}|\delta^l_6| = ||\Delta^l_6||$.

The poles of the vector Padé approximant are calculated from (5.14), which is of the 6^{th} degree, thus yielding twice as many poles as required. They are denoted by $\alpha^l_i, \alpha^{l*}_i$ $i = 1,2,3$, since they occur, for these examples, in complex conjugate pairs.

To enable comparison these were also evaluated; the error for each pair is quoted in the tables.

Two sets of vectors $\mathbf{r}_i, i = 1,2,3,4$ are considered. The first group contains vectors satisfying $\mathbf{r}_i \cdot \mathbf{r}_j \neq 0 \quad \forall i,j$ while the second consists of mutually orthogonal vectors.

The calculations were performed using MAPLE and the some of the results presented in tables 1 and 2.

Example 3 :

$$\left.\begin{array}{ll} \mathbf{r}_1 := [1,1,1,0,0,0]^T & \mathbf{r}_2 := [1,1,0,1,0,0]^T \\ \mathbf{r}_3 := [1,1,0,1,0,2]^T & \mathbf{r}_4 := [1,0,0,1,1,0]^T \end{array}\right\} \tag{5.19}$$

$$\mathbf{r}_i \cdot \mathbf{r}_j \neq 0 \quad \forall i,j.$$

Example 4 :

$$\left.\begin{array}{ll} \mathbf{r}_1 := [1,1,0,1,1,0]^T & \mathbf{r}_2 := [1,-1,1,0,0,0]^T \\ \mathbf{r}_3 := [1,0,-1,-1,0,1]^T & \mathbf{r}_4 := [1,0,-1,0,-1,-2]^T \end{array}\right\} \tag{5.20}$$

$$\mathbf{r}_i \cdot \mathbf{r}_j = 0 \quad \text{for} \quad i \neq j.$$

It may be seen that the data demonstrate the validity of the statements $(5.12, 13, 15)$. The differing accuracies of the two examples is an illustration of a generalisation of a result first observed experimentally by Graves-Morris [4] and explained by Roberts [15] for the [1/1] case. This result concerns the behaviour of the denominators of the "hybrid" approximant, which are given by the scalar part of the Clifford denominator — the cubic polynomial $\sigma_6^l(x)$ in our notation. In example 4, where the vector residues are mutually orthogonal, the zeroes of this polynomial offer much more accurate approximations to the poles of $\mathbf{f}(z)$ than do the poles of the vector Padé approximant itself — given by the zeroes of $Q_6^l(z)$. In fact, we discover that for example 4 $Re\{\alpha_i^l\}$ and z_i^l have, asymptotically, twice as many significant figures of accuracy as $Im\{\alpha_i^l\}$. Whereas, in example 3 $Re\{\alpha_i^l\}, Im\{\alpha_i^l\}$ and z_i^l all have the same order of accuracy. A more detailed study of the convergence rates will be given in a later paper.

l	$\lvert z_1 - z_1^l \rvert$ ($\lvert z_1 - \alpha_1^l \rvert$)	$\lvert z_2 - z_2^l \rvert$ ($\lvert z_2 - \alpha_2^l \rvert$)	$\lvert z_3 - z_3^l \rvert$ ($\lvert z_3 - \alpha_3^l \rvert$)	$\lvert\lvert \Delta_6^l(0.1) \rvert\rvert$	$\lvert\lvert \Delta_6^l(0.3) \rvert\rvert$	$\lvert\lvert \Delta_6^l(1.0) \rvert\rvert$
14	$6.8 \cdot 10^{-7}$	$7.6 \cdot 10^{-5}$	$5.3 \cdot 10^{-3}$	$2.0 \cdot 10^{-2}$	$6.3 \cdot 10^{-3}$	$7.8 \cdot 10^{-1}$
	$(1.8 \cdot 10^{-6})$	$(1.6 \cdot 10^{-4})$	$(7.5 \cdot 10^{-3})$			
18	$1.7 \cdot 10^{-8}$	$4.9 \cdot 10^{-6}$	$1.0 \cdot 10^{-3}$	$4.1 \cdot 10^{-3}$	$1.3 \cdot 10^{-3}$	$1.6 \cdot 10^{-1}$
	$(5.1 \cdot 10^{-8})$	$(1.1 \cdot 10^{-5})$	$(1.5 \cdot 10^{-3})$			
22	$4.4 \cdot 10^{-10}$	$3.1 \cdot 10^{-7}$	$2.0 \cdot 10^{-4}$	$8.1 \cdot 10^{-4}$	$2.7 \cdot 10^{-4}$	$3.3 \cdot 10^{-2}$
	$(1.3 \cdot 10^{-9})$	$(7.0 \cdot 10^{-7})$	$(3.0 \cdot 10^{-4})$			

Table 1: Errors of the estimated zeroes for Example 3 and Frobenius norm of the matrix Δ_6^l

l	$\lvert z_1 - z_1^l \rvert$ ($\lvert z_1 - \alpha_1^l \rvert$)	$\lvert z_2 - z_2^l \rvert$ ($\lvert z_2 - \alpha_2^l \rvert$)	$\lvert z_3 - z_3^l \rvert$ ($\lvert z_3 - \alpha_3^l \rvert$)	$\lvert\lvert \Delta_6^l(0.1) \rvert\rvert$	$\lvert\lvert \Delta_6^l(0.3) \rvert\rvert$	$\lvert\lvert \Delta_6^l(1.0) \rvert\rvert$
14	$1.9 \cdot 10^{-11}$	$6.0 \cdot 10^{-7}$	$7.1 \cdot 10^{-4}$	$3.6 \cdot 10^{-2}$	$1.2 \cdot 10^{-2}$	$1.4 \cdot 10^{0}$
	$(1.5 \cdot 10^{-6})$	$(2.7 \cdot 10^{-4})$	$(8.9 \cdot 10^{-3})$			
18	$1.3 \cdot 10^{-14}$	$2.4 \cdot 10^{-9}$	$2.8 \cdot 10^{-5}$	$6.9 \cdot 10^{-3}$	$2.3 \cdot 10^{-3}$	$2.8 \cdot 10^{-1}$
	$(3.9 \cdot 10^{-8})$	$(1.7 \cdot 10^{-5})$	$(1.8 \cdot 10^{-3})$			
22	$8.3 \cdot 10^{-18}$	$9.2 \cdot 10^{-12}$	$1.1 \cdot 10^{-6}$	$1.4 \cdot 10^{-3}$	$4.5 \cdot 10^{-4}$	$5.4 \cdot 10^{-2}$
	$(1.0 \cdot 10^{-9})$	$(1.1 \cdot 10^{-6})$	$(3.4 \cdot 10^{-4})$			

Table 2: Errors of the estimated zeroes for Example 4 and Frobenius norm of the matrix Δ_6^l

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