Incremental input-to-state stability for Lur'e systems and asymptotic behaviour in the presence of Stepanov almost periodic forcing

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Abstract. We prove (integral) input-to-state stability results for a class of forced Lur'e differential inclusions and use them to investigate incremental (integral) input-to-state stability properties of Lur'e differential equations. The latter provide a basis for the derivation of convergence results for trajectories of Lur'e equations generated by Stepanov almost periodic inputs.

Keywords. Absolute stability, almost periodic functions, circle criterion, differential inclusions, incremental (integral) input-to-state stability, (integral) input-to-state stability, Lur'e systems.

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1 Introduction

We consider stability and convergence properties of the feedback interconnection shown in Figure 1.1, which comprises a linear system Σ in the forward path and a static nonlinearity N in the feedback path. Such systems are often termed Lur'e systems.

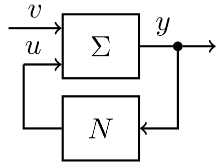

Figure 1.1: Forced Lur'e system

The term v is an exogenous signal which we call a forcing term and, depending on the context, may be control or disturbance signal. In the case wherein the nonlinearity N is a set-valued map (the scenario considered in Section 2), we will refer to the system as a Lur'e (differential) inclusion, whilst we will use the term Lur'e (differential) equation when N is a single-valued map (the situation focussed on in Sections 3 and 4). The stability and convergence properties of Lur'e systems is a much researched area. The study of the stability of such systems is called absolute stability theory, which seeks to conclude stability of the feedback system given in Figure 1.1, via the interplay of frequency-domain properties of the linear component and sector properties of the nonlinearity. Lyapunov approaches have been used to deduce global asymptotic stability of unforced (that is, $v = 0$) Lur'e systems (see, for example, [21, 23, 26, 39]), and input-output methods, pioneered by Sandberg and Zames in the

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1960s, have been used to infer L^2 and L^∞ stability (see, for example, [12, 39]). More recently, forced Lur'e systems have been analysed in the context of input-to-state stability (ISS) theory, with attention focussed on the extent to which results from classical absolute stability theory can be generalised to ensure certain ISS properties [4, 14, 16, 17, 19, 22, 23, 33]. Originating in the paper [34], ISS and its variants, including integral input-to-state stability (iISS), are properties of general controlled nonlinear systems and, roughly, ensure a natural boundedness property of the state, in terms of initial conditions and inputs, see also the survey papers [10, 35].

Incremental (integral) ISS is concerned with bounding the difference of two state trajectories in terms of the difference of initial conditions and the difference of inputs. For background information regarding incremental ISS for general nonlinear systems, we refer the reader to [2]. Recently, in the context of discrete-time Lur'e systems, it has been shown in [15] that a certain "nonlinear" incremental small-gain condition guarantees semi-global incremental ISS and this was exploited to show that the response to almost periodic inputs is asymptotically almost periodic. The concept of semi-global incremental ISS used in [15] means that, for arbitrary bounded sets of initial conditions and input functions there exist comparison functions such that an incremental ISS estimate holds. A similar concept will be used here in an integral ISS setting. For all practical purposes, semi-global stability notions seem to be at least as interesting as global stability concepts. We mention that, under a classical incremental L^2-type small-gain condition, a stronger incremental ISS property holds, namely (global) exponential incremental ISS, even in the infinite-dimensional case, see [14, 16, 19].

In this paper, we consider finite-dimensional forced continuous-time Lur'e differential equations and the stability property of semi-global incremental iISS which is considerably weaker than the notion of semi-global incremental ISS studied in [15]. The main result of Section 3, Theorem 3.3, asserts that the condition

$$g(\Sigma^K) \sup_{t\geq 0} \|N(t,y_1) - N(t,y_2) - K(y_1 - y_2)\| < \|y_1 - y_2\| \quad \text{for all } y_1 \neq y_2, \tag{1.1}$$

is sufficient for semi-global incremental iISS (with linear incremental iISS-gain), where, in (1.1), K is a stabilizing output feedback matrix for Σ, Σ^K denotes the corresponding linear feedback system and $g(\Sigma^K)$ denotes the L^2-gain of Σ^K (that is, the H^∞-norm of the transfer function of Σ^K). We emphasize that this hypothesis is considerably less restrictive than those given in [14, 15, 16, 19]; in particular, the quotient of the LHS of (1.1) and $\|y - y_2\|$, $y_1 \neq y_2$, is not required to be bounded away from 1, and therefore, (1.1) could be described as a "weak small-gain" condition.

Furthermore, we show that under the additional assumption that there exists $y^\dagger$ such that

$$\big(\|y\| - g(\Sigma^K) \sup_{t\geq 0} \|N(t, y + y^\dagger) - N(t, y^\dagger) - Ky\|\big) \to \infty \quad \text{as } \|y\| \to \infty,$$

the Lur'e differential equation depicted in Figure 1.1 is semi-globally incrementally ISS. The proofs of the results on semi-global incremental iISS and ISS in Section 3 are based on the iISS and ISS theory for forced Lur'e inclusions developed in Section 2. Using a differential inclusions setting is a convenient framework for addressing certain uniformity issues which arise in Section 3 in the context of establishing incremental iISS and ISS estimates which apply to whole families of Lur'e differential equations. Results from [37] play a crucial role in the development our ISS theory for Lur'e inclusions.

The key result on semi-global incremental iISS, Theorem 3.3, is used in Section 4 to obtain certain convergence properties of Lur'e differential equations when the forcing is almost periodic in the sense of Stepanov [1, 9] (which is a weaker notion of almost periodicity than that of Bohr [1, 7, 9]). In particular, Theorem 4.3 provides a criterion which guarantees that, for every Stepanov almost periodic input w, there exists a unique Bohr almost periodic state trajectory $z^{\rm ap}$ such that any state trajectory x generated by an input v with $v - w \in L^1(\mathbb{R}_+)$ satisfies $\lim_{t\to\infty}(x(t) - z^{\rm ap}(t)) = 0$. As such, our work provides a contribution to the analysis of almost periodic differential equations from the perspective of mathematical systems and control theory.

The layout of the paper is as follows. In Section 2, we prove basic iISS and ISS results for a class of forced Lur'e differential inclusions. Underpinned by the stability properties of forced Lur'e differential inclusions established in Section 2, a theory of semi-global incremental iISS and ISS is developed in Section 3. As an application of this theory, we analyze the asymptotic behaviour of Lur'e differential equations under Stepanov almost periodic forcing in Section 4. An example is presented in Section 5. To avoid disruptions to the flow of the presentation, the proofs of several technical lemmas are presented in the Appendix which forms Section 6.

Notation. We denote the set of positive integers by $\mathbb{N}$ and the fields of real and complex numbers by $\mathbb{R}$ and $\mathbb{C}$, respectively, and define $\mathbb{C}_0 := \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ and $\mathbb{R}_+ := [0, \infty)$. A square matrix $M \in \mathbb{C}^{n\times n}$ is said to be Hurwitz if the eigenvalues of M have negative real parts, that is, all eigenvalues are in $-\mathbb{C}_0$. The conjugate transpose of M is denoted by M^*.

For $K \in \mathbb{F}^{m\times p}$, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, and $r > 0$, we define

$$\mathbb{B}_{\mathbb{F}}(K, r) := \{L \in \mathbb{F}^{m\times p} : \|K - L\| < r\},$$

the open ball centered at K of radius r. The closure of $\mathbb{B}_{\mathbb{F}}(K, r)$ (that is, the closed ball centered at K of radius r) is denoted by $\operatorname{cl}\mathbb{B}_{\mathbb{F}}(K, r)$. For a non-empty compact set $S \subset \mathbb{R}^m$ we define

$$|S|_{\mathrm{m}} := \max\{\|z\| : z \in S\}.$$

The Hardy space $H^\infty_{p\times m}$ is the set of all holomorphic functions $\mathbf{H} : \mathbb{C}_0 \to \mathbb{C}^{p\times m}$ with

$$\|\mathbf{H}\|_{H^\infty} := \sup_{s\in\mathbb{C}_0} \|\mathbf{H}(s)\| < \infty.$$

We further define $\mathcal{P} := \{\alpha \in C(\mathbb{R}_+, \mathbb{R}_+) : \alpha(0) = 0 \text{ and } \alpha(s) > 0 \ \forall\, s > 0\}$,

$$\mathcal{K} := \{\alpha \in \mathcal{P} : \alpha \text{ strictly increasing}\} \quad \text{and} \quad \mathcal{K}_\infty := \{\alpha \in \mathcal{K} : \lim_{s\to\infty} \alpha(s) = \infty\}.$$

Obviously, $\mathcal{K}_\infty \subset \mathcal{K} \subset \mathcal{P}$. A function $\alpha : \mathbb{R}_+ \to (0, \infty)$ is said be of class $\mathcal{L}$ if α is continuous and non-increasing and $\alpha(s) \to 0$ as $s \to \infty$. The symbol $\mathcal{KL}$ stands for the set of functions $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for each fixed $t \in \mathbb{R}_+$, $\psi(\cdot, t) \in \mathcal{K}$ and for each fixed $s > 0$, $\psi(s, \cdot) \in \mathcal{L}$.

For X a Banach space, $p \in [1, \infty]$, and $J \subset \mathbb{R}$ an interval, $L^p(J, X)$ stands for the usual Lebesgue space and the norm of $x \in L^p_{\mathrm{loc}}(J, X)$ is denoted by $\|x\|_{L^p(J)}$. For $R = \mathbb{R}_+$ or $\mathbb{R}$, we simply write $\|x\|_{L^p(R)} := \|x\|_{L^p}$. The local version of $L^p(R, X)$ is denoted by $L^p_{\mathrm{loc}}(R, X)$. Further, for $\tau \in R$, we define the shift operator $\mathbf{S}_\tau : L^1_{\mathrm{loc}}(R, X) \to L^1_{\mathrm{loc}}(R, X)$ by $(\mathbf{S}_\tau v)(t) := v(t + \tau)$ for all $t \in R$ and $v \in L^1_{\mathrm{loc}}(R, X)$. Finally, let $W^{1,1}_{\mathrm{loc}}(R, \mathbb{R}^n)$ be the local version of the Sobolev space $W^{1,1}(R, \mathbb{R}^n)$. Strictly speaking, elements in $W^{1,1}_{\mathrm{loc}}(R, \mathbb{R}^n)$ are equivalence classes of functions which coincide almost everywhere in R, but it is well-known that $W^{1,1}_{\mathrm{loc}}(R, \mathbb{R}^n)$ can be identified with the space of absolutely continuous functions on R (see, for example, [28, Corollary 7.20]), and thus, $x \in W^{1,1}_{\mathrm{loc}}(R, \mathbb{R}^n)$ if, and only if, there exists $y \in L^1_{\mathrm{loc}}(R, \mathbb{R}^n)$ such that $x(t) = x(0) + \int_0^t y(s)\mathrm{d}s$ for all $t \in R$.

2 ISS results for Lur'e differential inclusions

The main results of this paper can be found in Sections 3 and 4 in which Lur'e differential equations are considered. However, in order to guarantee certain uniformity properties (the derivation of incremental ISS estimates which apply to a whole family of Lur'e differential equations), it is convenient to first develop an ISS theory for Lur'e differential inclusions.

Let m, n and p be positive integers and set $\mathbb{L} := \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n}$. Consider the forced Lur'e differential inclusion

$$\dot{x}(t) - Ax(t) - v(t) \in BF(Cx(t)), \quad t \geq 0, \tag{2.1}$$

where $(A, B, C) \in \mathbb{L}$, $v \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$, and F is an upper semi-continuous set-valued map defined on $\mathbb{R}^p$ and its values are non-empty, compact and convex subsets of $\mathbb{R}^m$. We say that (2.1) is unforced if $v = 0$. Occasionally, we will refer to (2.1) as Lur'e inclusion (A, B, C, F).

Let $v \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ be given and let $0 < \tau \leq \infty$. A function $x \in W^{1,1}_{\text{loc}}([0, \tau), \mathbb{R}^n)$ is said to be a solution of (2.1) on $[0, \tau)$ if (2.1) holds for almost every $t \in [0, \tau)$. A solution of (2.1) on $[0, \infty) = \mathbb{R}_+$ is called a global solution.

We denote the transfer function of (A, B, C) by $\mathbf{G}$, *i.e.*, $\mathbf{G}(s) = C(sI-A)^{-1}B$. For $L \in \mathbb{C}^{m\times p}$, we define $A^L := A+BLC$, denote the transfer function of (A^L, B, C) by $\mathbf{G}^L$ and note that $\mathbf{G}^L = \mathbf{G}(I - L\mathbf{G})^{-1}$. The set of all stabilizing output feedback matrices for (A, B, C) is denoted by $\mathbb{S}_{\mathbb{C}}(A, B, C)$, that is,

$$\mathbb{S}_{\mathbb{F}}(A, B, C) := \{L \in \mathbb{F}^{m\times p} : A^L = A + BLC \text{ is Hurwitz}\}, \quad \text{where } \mathbb{F} = \mathbb{R}, \mathbb{C}.$$

Furthermore, for $K \in \mathbb{C}^{m\times p}$ and $r > 0$, we obtain from [33, Lemma 2.1] that $\mathbb{B}(K, r) \subseteq \mathbb{S}_{\mathbb{C}}(A, B, C)$ if, and only if, $\|\mathbf{G}^K\|_{H^\infty} \leq 1/r$ and (A, B, C) is stabilizable and detectable. In particular, if $\mathbf{G}^K(s) \not\equiv 0$ and (A, B, C) is stabilizable and detectable, then the largest $r > 0$ such that $\mathbb{B}_{\mathbb{C}}(K, r) \subset \mathbb{S}_{\mathbb{C}}(A, B, C)$ is given by $r := 1/\|\mathbf{G}^K\|_{H^\infty}$. In this context, we note that $\mathbf{G}^K(s) \not\equiv 0$ if, and only if, $\mathbf{G}(s) \not\equiv 0$. Furthermore, if (A, B, C) is controllable and $C \neq 0$, then $\mathbf{G}(s) \not\equiv 0$. By duality, observability of (A, B, C) together with $B \neq 0$ will also ensure that $\mathbf{G}(s) \not\equiv 0$.

We define the behaviour of (2.1) by

$$\mathcal{B}_{\text{inc}} := \{(v, x) \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n) \times W^{1,1}_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n) \ : \ (v, x) \text{ satisfies (2.1) a.e. on } \mathbb{R}_+\}\,.$$

An element $(v, x) \in \mathcal{B}_{\text{inc}}$ is also called a trajectory of (2.1). We note that $\mathcal{B}_{\text{inc}}$ is left-shift invariant, that is,

$$(v, x) \in \mathcal{B}_{\text{inc}} \implies (\mathbf{S}_\tau v, \mathbf{S}_\tau x) \in \mathcal{B}_{\text{inc}} \quad \forall\, \tau \in \mathbb{R}_+. \tag{2.2}$$

The following proposition addresses the existence and extendability of solutions of (2.1).

Proposition 2.1. *Let $v \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$.*

(1) *For every $x^0 \in \mathbb{R}^n$, there exist $0 < \tau \leq \infty$ and a solution x of (2.1) on $[0, \tau)$ such that $x(0) = x^0$. If $\tau < \infty$ and x is bounded, then the solution x can be extended beyond τ, that is, there exist $\tau < \tilde{\tau} \leq \infty$ and a solution $\tilde{x}$ of (2.1) on $[0, \tilde{\tau})$ such that $\tilde{x}(t) = x(t)$ for all $t \in [0, \tau)$.*

(2) *If F is affinely linearly bounded, that is, there exist $a, b \geq 0$ such that*

$$|F(y)|_{\text{m}} \leq a + b\|y\| \quad \forall\, y \in \mathbb{R}^p,$$

then, every solution of (2.1) can be extended to a global solution.

Statement (1) is an immediate consequence of [11, Corollary 5.2] and statement (2) can be be obtained by a routine argument involving part (a), Filippov's selection theorem (see, for example, [40, p. 72]) and Gronwall's lemma.

The next result will be a useful tool in the following. A proof can be found in the Appendix.

Lemma 2.2. *Assume that $F(0) = \{0\}$ and*

$$|F(y) - Ky|_{\text{m}} < r\|y\| \quad \forall\, y \in \mathbb{R}^p,\ y \neq 0\,, \tag{2.3}$$

where r is a positive constant. Then there exists continuously differentiable $\gamma \in \mathcal{P}$ such that

$$|F(y) - Ky|_{\text{m}} \leq r\|y\| - \gamma(\|y\|) \quad \forall\, y \in \mathbb{R}^p\,. \tag{2.4}$$

Under the additional assumption that

$$r\|y\| - |F(y) - Ky|_{\text{m}} \to \infty \quad \text{as } \|y\| \to \infty\,, \tag{2.5}$$

the inequality (2.4) holds with some continuously differentiable $\gamma \in \mathcal{K}_\infty$.

The following theorem provides an ISS result for the Lur'e inclusion (2.1).

Theorem 2.3. *Let $K \in \mathbb{R}^{m\times p}$ and $r > 0$ be such that $\mathbb{B}_{\mathbb{C}}(K,r) \subset \mathbb{S}_{\mathbb{C}}(A,B,C)$. If $F(0) = \{0\}$ and (2.3) and (2.5) are satisfied, then there exist $\zeta \in \mathcal{KL}$ and $\theta \in \mathcal{K}$ such that*

$$\|x(t)\| \le \zeta(\|x(0)\|,t) + \theta(\|v\|_{L^\infty(0,t)}) \quad \forall\, t \ge 0,\ \forall\,(v,x) \in \mathcal{B}_{\text{inc}} \ \textit{with}\ v \in L^\infty_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)\,. \tag{2.6}$$

Proof. Lemma 2.2 guarantees the existence of $\gamma \in \mathcal{K}_\infty$ such that

$$|F(y) - Ky|_{\text{m}} \le r\|y\| - \gamma(\|y\|) \quad \forall\, y \in \mathbb{R}^p\,.$$

An inspection of the proof of [33, Theorem 3.2] (which establishes (2.6) for forced Lur'e differential equations) shows that it carries over to our Lur'e inclusions setting. In particular, there exist $\alpha, \beta \in \mathcal{K}_\infty$ and a continuously differentiable radially unbounded function $U : \mathbb{R}^n \to \mathbb{R}_+$ with $U(0) = 0$ and $U(z) > 0$ for all $z \neq 0$ and such that

$$\langle(\nabla U)(z), w\rangle \le -\alpha(\|z\|) + \beta(\|u\|) \quad \forall\, w \in Az + BF(Cz) + u,\ \forall\, z \in \mathbb{R}^n. \tag{2.7}$$

The existence of $\zeta \in \mathcal{KL}$ and $\theta \in \mathcal{K}$ such that (2.6) holds now follows as in the case of differential equations (see, for example, the proof of [29, Theorem 5.41]). □

The next theorem guarantees the existence of certain stability properties and Lyapunov functions for the unforced Lur'e inclusion (2.1).

Theorem 2.4. *Let $K \in \mathbb{R}^{m\times p}$ and $r > 0$ be such that $\mathbb{B}_{\mathbb{C}}(K,r) \subset \mathbb{S}_{\mathbb{C}}(A,B,C)$. If (A,B,C) is controllable or observable, $v = 0$, $F(0) = \{0\}$ and (2.3) is satisfied, then the following statements hold.*

(1) *The exists a positive definite matrix $P = P^* \in \mathbb{R}^{n\times n}$ such that*

$$\big(A^K\big)^* P + PA^K + C^*C + r^2 PBB^*P = 0\,, \tag{2.8}$$

and the associated quadratic form $V(z) := \langle Pz, z\rangle$ satisfies

$$\langle(\nabla V)(z), w\rangle \le 0 \quad \forall\, w \in Az + BF(Cz),\ \forall\, z \in \mathbb{R}^n\,. \tag{2.9}$$

Furthermore, for each compact set $\Gamma \subset \mathbb{R}^n$ such that $\min_{z\in\Gamma}\|Cz\| > 0$, there exists $\nu > 0$ such that

$$\langle(\nabla V)(z), w\rangle \le -\nu \quad \forall\, w \in Az + BF(Cz),\ \forall\, z \in \Gamma\,. \tag{2.10}$$

(2) *There exists $\kappa > 0$ such that $\|x(t)\| \le \kappa\|x(0)\|$ for all $t \ge 0$ and all $x \in W^{1,1}_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$ such that $(0,x) \in \mathcal{B}_{\text{inc}}$, that is, 0 is uniformly stable in the large.*

(3) *For every $\varepsilon > 0$ and every $\rho > 0$, there exists $\tau \ge 0$ such that, for each $x \in W^{1,1}_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$ with $(0,x) \in \mathcal{B}_{\text{inc}}$, the following implication holds*

$$\|x(0)\| \le \rho \quad \Rightarrow \quad \|x(t)\| \le \varepsilon\ \ \forall\, t \ge \tau\,,$$

that is, 0 is uniformly globally attractive.

(4) *There exists a radially unbounded smooth function $U : \mathbb{R}^n \to \mathbb{R}_+$ such that $U(z) > 0$ for all $z \neq 0$ and*

$$\langle(\nabla U)(z), w\rangle \le -U(z) \quad \forall\, w \in Az + BF(Cz),\ \forall\, z \in \mathbb{R}^n\,. \tag{2.11}$$

Note that in statement (4), as $F(0) = \{0\}$, we may conclude that $U(0) = 0$, and the hypotheses on U guarantee that there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(\|z\|) \le U(z) \le \alpha_2(\|z\|)$ for all $z \in \mathbb{R}^n$. It follows from (2.11) that

$$\langle (\nabla U)(z), w\rangle \le -\alpha_1(\|z\|) \quad \forall\, w \in Az + BF(Cz),\ \forall\, z \in \mathbb{R}^n\,.$$

Proof of Theorem 2.4. (1) It is well-known that there exists $P = P^* \in \mathbb{R}^{n\times n}$ satisfying (2.8), see [21, Theorem 5.3.25 and Remark 5.3.27]. Furthermore, P satisfies

$$P = \int_0^\infty e^{(A^K)^* t}\big(C^*C + r^2 PBB^*P\big)e^{A^K t}\mathrm{d}t \ge \int_0^\infty e^{(A^K)^* t} C^*C e^{A^K t}\mathrm{d}t\,. \tag{2.12}$$

Assume that (C, A) is observable. It then follows from (2.12) that P is positive definite. If (C, A) is not observable, but (A, B) is controllable, then it is natural to consider the dual system (A^*, C^*, B^*). The transfer function of this system is $\mathbf{H}(s) = \mathbf{G}^*(\bar{s}) = B^*(sI - A^*)^{-1}C^*$, (A^*, C^*, B^*) is stabilizable and detectable, (B^*, A^*) is observable, $\|\mathbf{H}\|_{H^\infty} = \|\mathbf{G}\|_{H^\infty}$ and $K^* \in \mathbb{S}(\mathbf{H})$. By an argument identical to that used above, we conclude that there exists a positive definite matrix $Q = Q^* \in \mathbb{R}^{n\times n}$ such that

$$A^K Q + Q\big(A^K\big)^* + B^*B + r^2 QC^*CQ = 0.$$

The matrix $P := r^{-2}Q^{-1}$ is positive definite and solves (2.8).

Now let $P = P^* \in \mathbb{R}^{n\times n}$ be a positive definite solution of (2.8). A routine calculation invoking (2.3), (2.8) and the identity $Az + BF(Cz) = A^K z + B(F(Cz) - KCz)$ shows that the quadratic form

$$V(z) := \langle Pz, z\rangle \quad \forall\, z \in \mathbb{R}^n\,,$$

satisfies

$$\langle (\nabla V)(z), Az + Bw\rangle \le -\big(r^2\|Cz\|^2 - \|w - KCz\|^2\big) \quad \forall\, w \in F(Cz),\ \forall\, z \in \mathbb{R}^n. \tag{2.13}$$

The inequality (2.9) now follows from (2.3).

To establish (2.10), let $\Gamma \subset \mathbb{R}^n$ be compact and such that $\min_{z\in\Gamma} \|Cz\| > 0$. Seeking a contradiction, suppose the claim is not true, in which case (2.13) implies the existence of a sequence $\big((z_j, w_j)\big)_j$ with $z_j \in \Gamma$ and $w_j \in F(Cz_j)$ such that

$$\lim_{j\to\infty} \big(r^2\|Cz_j\|^2 - \|w_j - KCz_j\|^2\big) = 0. \tag{2.14}$$

As the sequence $\big((z_j, w_j)\big)_j$ is bounded, it has a convergent subsequence, the limit of which we denote by (z_∞, w_∞). Using the compactness of Γ and $F(y)$ for all $y \in \mathbb{R}^p$ and the upper semicontinuity of F, we conclude that $z_\infty \in \Gamma$ and $w_\infty \in F(Cz_\infty)$. Consequently, $Cz_\infty \ne 0$, and, by (2.3), $r\|Cz_\infty\| > \|w_\infty - KCz_\infty\|$, yielding a contradiction to (2.14).

(2) For $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ such that $(0, x) \in \mathcal{B}_{\mathrm{inc}}$ it follows from (2.9) that $(\mathrm{d}(V \circ x)/\mathrm{d}t)(t) \le 0$ for almost every $t \ge 0$, and so

$$V(x(t)) \le V(x(0)) \quad \forall\, t \ge 0.$$

The claim now follows as the map $z \mapsto \sqrt{V(z)}$ is a norm on $\mathbb{R}^n$.

(3) Let $\varepsilon, \rho > 0$ be given. By statement (2), there exists $\kappa > 0$ such that,

$$\|x(t)\| \le \kappa\|x(0)\| \quad \forall\, t \ge 0,\ \forall\, x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n) \text{ with } (0, x) \in \mathcal{B}_{\mathrm{inc}},$$

and so,

$$\|Cx(t)\| \le \kappa\rho\|C\| =: b \quad \forall\, t \ge 0,\ \forall\, x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n) \text{ with } (0, x) \in \mathcal{B}_{\mathrm{inc}} \text{ and } \|x(0)\| \le \rho.$$

By Lemma 2.2, there exists continuously differentiable $\gamma_1 \in \mathcal{P}$ such that

$$|F(y) - Ky|_{\mathrm{m}} \le r\|y\| - \gamma_1(\|y\|) \quad \forall\, y \in \mathbb{R}^p\,.$$

An application of [25, Lemma 18] guarantees the existence of comparison functions $\gamma_2 \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{L}$ such that

$$\gamma_1(s) \ge \gamma_2(s)\sigma(s) \quad \forall\, s \ge 0.$$

The function $\gamma_3 := \sigma(b)\gamma_2$ is in $\mathcal{K}_\infty$ and

$$\gamma_1(s) \ge \gamma_3(s) \quad \forall\, s \in [0, b].$$

Therefore,

$$|F(y) - Ky|_{\mathrm{m}} \le r\|y\| - \gamma_3(\|y\|) \quad \forall\, y \in \mathbb{R}^p \text{ with } \|y\| \le b. \tag{2.15}$$

Setting

$$\gamma(s) := \begin{cases} \gamma_3(s), & s \in [0, b], \\ rs - (rb - \gamma_3(b)), & s > b, \end{cases}$$

it is evident that $\gamma \in \mathcal{K}_\infty$ and

$$rb - \gamma(b) = r\|y\| - \gamma(\|y\|) \quad \forall\, y \in \mathbb{R}^p \text{ with } \|y\| \ge b. \tag{2.16}$$

The set-valued map $F^\sharp$ defined by

$$F^\sharp(y) := \begin{cases} F(y), & \|y\| \le b, \\ F\Big(\dfrac{by}{\|y\|}\Big) + (\|y\| - b)K\dfrac{y}{\|y\|}, & \|y\| > b, \end{cases} \tag{2.17}$$

is upper semi-continuous and, for each $y \in \mathbb{R}^p$, the subset $F^\sharp(y) \subset \mathbb{R}^m$ is non-empty, compact and convex. Furthermore,

$$|F^\sharp(y) - Ky|_{\mathrm{m}} = \left|F\Big(\frac{by}{\|y\|}\Big) - K\frac{by}{\|y\|}\right|_{\mathrm{m}} \le rb - \gamma(b) = r\|y\| - \gamma(\|y\|) \quad \forall\, y \in \mathbb{R}^p \text{ with } \|y\| > b, \tag{2.18}$$

by (2.16) and (2.17).

The conjunction of (2.15)–(2.18) gives

$$|F^\sharp(y) - Ky|_{\mathrm{m}} \le r\|y\| - \gamma(\|y\|) \quad \forall\, y \in \mathbb{R}^p.$$

Denoting the behaviour of the Lur'e inclusion

$$\dot{x}(t) - Ax(t) - v(t) \in BF^\sharp(Cx(t)), \quad t \ge 0 \tag{2.19}$$

by $\mathcal{B}^\sharp_{\mathrm{inc}}$, an application of Theorem 2.3 to (2.19) shows that there exists $\zeta \in \mathcal{KL}$ such that

$$\|x(t)\| \le \zeta(\|x(0)\|, t) \quad \forall\, t \ge 0,\ \forall\, x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n) \text{ with } (0, x) \in \mathcal{B}^\sharp_{\mathrm{inc}}\,. \tag{2.20}$$

Noting that, by construction, each function $x \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ such that $(0, x) \in \mathcal{B}_{\mathrm{inc}}$ and $\|x(0)\| \le \rho$ satisfies $(0, x) \in \mathcal{B}^\sharp_{\mathrm{inc}}$, it follows from (2.20) that $\|x(t)\| \le \varepsilon$ for all $t \ge \tau$ with $\tau \ge 0$ being any number satisfying $\zeta(\rho, \tau) \le \varepsilon$.

(4) By Lemma 2.2 there exists a continuously differentiable function $\gamma \in \mathcal{P}$ such that

$$|F(y) - Ky|_{\mathrm{m}} \le r\|y\| - \gamma(\|y\|) \quad \forall\, y \in \mathbb{R}^p\,.$$

Setting

$$\tilde{F}(y) := \operatorname{cl}\mathbb{B}_{\mathbb{R}}\big(Ky, r\|y\| - \gamma(\|y\|)\big) = Ky + \operatorname{cl}\mathbb{B}_{\mathbb{R}}\big(0, r\|y\| - \gamma(\|y\|)\big) \quad \forall\, y \in \mathbb{R}^p\,,$$

it is clear that $\tilde{F}$ is upper semi-continuous, $\tilde{F}(0) = \{0\}$ and, for all $y \in \mathbb{R}^p$, the set $\tilde{F}(y)$ is non-empty, convex and compact, $|\tilde{F}(y) - Ky|_{\mathrm{m}} < r\|y\|$ and $F(y) \subset \tilde{F}(y)$. Consequently, the conclusions of statements (1)–(3) and Proposition 2.1 apply to the differential inclusion

$$\dot{x}(t) - Ax(t) - v(t) \in B\tilde{F}(Cx(t)), \quad t \geq 0\,, \tag{2.21}$$

and so, by [37, Proposition 1], (2.21) is $\mathcal{KL}$-stable [37, Definition 6].

Obviously, it is sufficient to prove that there exists a radially unbounded smooth function $U : \mathbb{R}^n \to \mathbb{R}_+$ such that $U(z) > 0$ for all $z \neq 0$ and

$$\langle (\nabla U)(z), w\rangle \leq -U(z) \quad \forall\, w \in Az + B\tilde{F}(Cz),\ \forall\, z \in \mathbb{R}^n\,,$$

that is, (2.11) with F replaced by $\tilde{F}$. According to [37, Theorem 1], the existence of such a function U is equivalent to the differential inclusion (2.21) being robustly $\mathcal{KL}$-stable in the sense of [37, Definition 8]. Since (2.21) is $\mathcal{KL}$-stable, we can use [37, Theorem 8] to establish robust $\mathcal{KL}$-stability by showing that $\tilde{F}$ is locally Lipschitz in the sense of [37], that is, we have to verify that, for every bounded subset $Y \subset \mathbb{R}^p$, there exists a number $L > 0$ such that $\tilde{F}(y_1) \subset \tilde{F}(y_2) + L\|y_1 - y_2\|\Delta$ for all $y_1, y_2 \in Y$, where Δ is the open unit ball in $\mathbb{R}^m$. Recalling that the set $\mathcal{C}$ of all compact nonempty subsets of $\mathbb{R}^m$ endowed with the Hausdorff metric d_{H} is a metric space, and using elementary properties of d_{H} (see, for example, [8, Subsection 7.3.1]), it is a routine exercise to show that the local Lipschitz concept in the sense of [37] is equivalent to $\tilde{F}$ being locally Lipschitz as a map $\mathbb{R}^p \to (\mathcal{C}, d_{\mathrm{H}})$. As $\tilde{F}(y) = Ky + \operatorname{cl}\mathbb{B}_{\mathbb{R}}(0, r\|y\| - \gamma(\|y\|))$ for all $y \in \mathbb{R}^p$, and the functions

$$\mathbb{R}^p \to \mathcal{C},\ y \mapsto \{Ky\}, \quad \mathbb{R}_+ \to \mathcal{C},\ \rho \mapsto \operatorname{cl}\mathbb{B}_{\mathbb{R}}(0,\rho) \quad \text{and} \quad \mathbb{R}^p \to \mathbb{R},\ y \mapsto r\|y\| - \gamma(\|y\|),$$

are locally Lipschitz (where we have used that γ, as a continuously differentiable function, is locally Lipschitz), we conclude that $\tilde{F}$ is locally Lipschitz. □

The next result establishes an iISS property for the Lur'e inclusion (2.1).

Theorem 2.5. *Let $K \in \mathbb{R}^{m\times p}$ and $r > 0$ be such that $\mathbb{B}_{\mathbb{C}}(K, r) \subset \mathbb{S}_{\mathbb{C}}(A, B, C)$. Assume that (A, B, C) is controllable or observable, $F(0) = \{0\}$ and* (2.3) *holds. Then there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that*

$$\|x(t)\| \leq \psi(\|x(0)\|, t) + \varphi\Big(\int_0^t \|v(s)\|\mathrm{d}s\Big) \quad \forall\, t \geq 0,\ \forall\, (v, x) \in \mathcal{B}_{\mathrm{inc}}\,. \tag{2.22}$$

The estimate (2.22) says that 0 is iISS with linear iISS-gain (which has been absorbed into φ). Theorem 2.5, says, roughly speaking, that linear stability (namely, $\mathbb{B}_{\mathbb{C}}(K, r) \subset \mathbb{S}_{\mathbb{C}}(A, B, C)$) implies iISS (with linear iISS-gain) for all (set-valued) nonlinearities F satisfying (2.3). In this sense, Theorem 2.5 is reminiscent of the complex Aizerman conjecture [20, 21] which addresses global asymptotic stability of unforced Lur'e differential equations. Following the argumentation in the proof of [33, Corollary 3.7], it can shown that that if B and C are non-negative and $A + BKC$ is a Metzler matrix, then Theorem 2.5 remains true when the complex condition $\mathbb{B}_{\mathbb{C}}(K, r) \subset \mathbb{S}_{\mathbb{C}}(A, B, C)$ is replaced by its real counterpart $\mathbb{B}_{\mathbb{R}}(K, r) \subset \mathbb{S}_{\mathbb{R}}(A, B, C)$.

Proof of Theorem 2.5. As in the proof of [3, Theorem 1], it can be shown that the existence of comparison functions $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that (2.22) holds is guaranteed provided it can be shown that there exist a radially unbounded continuously differentiable function $W : \mathbb{R}^n \to \mathbb{R}_+$, $\mu \in \mathcal{P}$ and $\beta > 0$ such that $W(0) = 0$, $W(z) > 0$ for all $z \neq 0$ and

$$\langle (\nabla W)(z), w\rangle \leq -\mu(\|z\|) + \beta\|u\| \quad \forall\, w \in Az + BF(Cz) + u,\ \forall\, (z, u) \in \mathbb{R}^n \times \mathbb{R}^n\,. \tag{2.23}$$

To this end, let $U : \mathbb{R}^n \to \mathbb{R}_+$ be a radially unbounded smooth function such that $U(z) > 0$ for all $z \neq 0$ and (2.11) holds. Let $\alpha \in \mathcal{K}_\infty$ be such that

$$\langle (\nabla U)(z), w \rangle \leq -\alpha(\|z\|) \quad \forall\, w \in Az + BF(Cz),\ \forall\, z \in \mathbb{R}^n\,. \tag{2.24}$$

Suitably modifying an argument given in the proof of [24, Lemma 3.4], we set

$$\gamma(s) := \max\{\|(\nabla U)(z)\| : z \in \mathbb{R}^n,\, U(z) \leq s\} \quad \forall\, s \geq 0\,,$$

and note that γ is continuous, $\gamma(s) > 0$ for $s > 0$ and

$$\|(\nabla U)(z)\| \leq \gamma(U(z)) \quad \forall\, z \in \mathbb{R}^n\,. \tag{2.25}$$

Furthermore, defining $\delta \in \mathcal{P}$ by

$$\delta(0) := 0, \quad \delta(s) := \min\{s, 1/\gamma(s)\}\ \forall\, s > 0\,,$$

it follows from (2.25) that

$$\delta(U(z))\|(\nabla U)(z)\| \leq 1 \quad \forall\, z \in \mathbb{R}^n\,. \tag{2.26}$$

The function $U_1 : \mathbb{R}^n \to \mathbb{R}_+$ given by

$$U_1(z) := \int_0^{U(z)} \delta(s)\mathrm{d}s \quad \forall\, z \in \mathbb{R}^n\,,$$

is continuously differentiable, $U_1(0) = 0$ and $U_1(z) > 0$ for all $z \neq 0$. Let $(z, u) \in \mathbb{R}^n \times \mathbb{R}^n$ and let $w \in Az + BF(Cz) + u$. Then, obviously, $w - u \in Az + BF(Cz)$, and,

$$\langle (\nabla U_1)(z), w \rangle = \delta(U(z))\langle (\nabla U)(z), w \rangle = \delta(U(z))\big(\langle (\nabla U)(z), w - u \rangle + \langle (\nabla U)(z), u \rangle\big).$$

Hence, invoking (2.24) and (2.26),

$$\langle (\nabla U_1)(z), w \rangle \leq -\delta(U(z))\alpha(\|z\|) + \|u\|.$$

Setting

$$\mu_1(s) := \alpha(s) \min\big\{\delta(U(z)) : \|z\| = s,\ z \in \mathbb{R}^n\big\} \quad \forall\, s \geq 0,$$

we have that $\mu_1 \in \mathcal{P}$ and

$$\langle (\nabla U_1)(z), w \rangle \leq -\mu_1(\|z\|) + \|u\| \quad \forall\, w \in Az + BF(Cz) + u,\ \forall\, (z, u) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{2.27}$$

Whilst (2.23) holds with $W = U_1$, $\mu = \mu_1$ and $\beta = 1$, this does not quite establish the claim because there is no guarantee that U_1 is radially unbounded.

By adopting an argument from the proof of [17, Proposition 3.10], we will construct a suitable function $U_2 : \mathbb{R}^n \to \mathbb{R}_+$ such that $W = U_1 + U_2$ has the desired properties. Let $P = P^*$ be a positive definite solution of (2.8) and let $V(z) := \langle Pz, z \rangle$ be the associated by quadratic form. Define $U_2 := h \circ V$, where

$$h(s) := \int_0^s \min\{t, (1 + t)^{-1}\}\, \mathrm{d}t \quad \forall\, s \geq 0.$$

Evidently, U_2 is continuously differentiable, $U_2(0) = 0$, $U_2(z) > 0$ for all $z \neq 0$ and U_2 is radially unbounded. Let $(z, u) \in \mathbb{R}^n \times \mathbb{R}^n$ and $w \in Az + BF(Cz) + u$. Then, trivially, $w - u \in Az + BF(Cz)$, and, invoking (2.9), we obtain.

$$\langle (\nabla U_2)(z), w \rangle = h'(V(z))\big(\langle (\nabla V)(z), w - u \rangle + \langle (\nabla V)(z), u \rangle\big) \leq h'(V(z))\langle (\nabla V)(z), u \rangle.$$

As

$$b := \sup_{z \in \mathbb{R}^n} h'(V(z))\|z\| < \infty\,,$$

and $(\nabla V)(z) = 2Pz$, we conclude that

$$\langle (\nabla U_2)(z), w\rangle \le 2b\|P\|\|u\| \quad \forall\, w \in Az + BF(Cz) + u,\ \forall\,(z,u) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Obviously, the function $W := U_1 + U_2$ is radially unbounded, $W(0) = 0$, $W(z) > 0$ for all $z \neq 0$ and, appealing to (2.27), we obtain

$$\langle (\nabla W)(z), w\rangle \le -\mu_1(\|z\|) + (2b\|P\| + 1)\|u\| \quad \forall\, w \in Az + BF(Cz) + u,\ \forall\,(z,u) \in \mathbb{R}^n \times \mathbb{R}^n,$$

showing that (2.23) holds with $\mu = \mu_1$ and $\beta = 2b\|P\| + 1$. □

The corollary below provides a "small-gain" interpretation of Theorem 2.5.

Corollary 2.6. *Let $K \in \mathbb{R}^{m\times p}$ be such that $K \in \mathbb{S}_{\mathbb{C}}(A,B,C)$ and set*

$$\nu(y) := \frac{|F(y) - Ky|_{\mathrm{m}}}{\|y\|} \quad \forall\, y \in \mathbb{R}^p,\ y \neq 0.$$

Assume that (A,B,C) is controllable or observable, $F(0) = \{0\}$ and that $\sup_{y\neq 0}\nu(y) < \infty$. If

$$\nu(y)\|\mathbf{G}^K\|_{H^\infty} < 1 \quad \forall\, y \in \mathbb{R}^p,\ y \neq 0\,,$$

then there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that (2.22) holds.

Proof. We distinguish two case: $\mathbf{G}(s) \equiv 0$ and $\mathbf{G}(s) \not\equiv 0$. If $\mathbf{G}(s) \equiv 0$, then, evidently, $\mathbb{B}_{\mathbb{C}}(K,r) \subset \mathbb{S}_{\mathbb{C}}(A,B,C)$, where $\sup_{y\neq 0}\nu(y) < r < \infty$, and the claim follows from Theorem 2.5.

Let us now assume that $\mathbf{G}(s) \not\equiv 0$. Then $\mathbf{G}^K(s) \not\equiv 0$ and $r^K := 1/\|\mathbf{G}^K\|_{H^\infty} \in (0,\infty)$. Noting that (A,B,C) is stabilizable and detectable (because $K \in \mathbb{S}_{\mathbb{C}}(A,B,C)$), we conclude that $\mathbb{B}_{\mathbb{C}}(K,r^K) \subset \mathbb{S}_{\mathbb{C}}(A,B,C)$. Since, by hypothesis, $\nu(y)\|\mathbf{G}^K\|_{H^\infty} < 1$ for all $y \neq 0$, it follows that $|F(y) - Ky|_{\mathrm{m}} < r^K\|y\|$ for all $y \in \mathbb{R}^p$, $y \neq 0$. An application of Theorem 2.5 yields the claim. □

Next we derive a corollary of Theorem 2.5 which shows that, under conditions very similar to those of the well-known circle criterion [23, 26, 39], the forced Lur'e inclusion (2.22) is iISS. To this end, we recall that a square rational matrix $\mathbf{H}(s)$ is said to be positive real if the matrix $\mathbf{H}(s) + (\mathbf{H}(s))^*$ is positive semi-definite for every $s \in \mathbb{C}_0$ which is not a pole of $\mathbf{H}$. We note that if $\mathbf{H}$ is positive real, then $\mathbf{H}$ is holomorphic in $\mathbb{C}_0$, see, for example, [18].

Corollary 2.7. *Let $K_1, K_2 \in \mathbb{R}^{m\times p}$ and assume that (A,B,C) is controllable and detectable or, alternatively, stabilizable and observable. If $(I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ is positive real, $F(0) = \{0\}$ and*

$$\langle w - K_1 y, w - K_2 y\rangle < 0 \quad \forall\, w \in F(y),\ \forall\, y \in \mathbb{R}^p,\ y \neq 0\,,$$

then there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that (2.22) holds.

Proof. Define matrices $L, M \in \mathbb{R}^{m\times p}$ by

$$L := \frac{1}{2}(K_1 - K_2), \quad M := \frac{1}{2}(K_1 + K_2)$$

and note that

$$\begin{aligned}\langle w - K_1 y, w - K_2 y\rangle &= \langle w - (L+M)y, w + (L-M)y\rangle \\ &= -\|Ly\|^2 + \|w - My\|^2 \quad \forall\, w \in F(y),\ \forall\, y \in \mathbb{R}^p\,. \end{aligned} \tag{2.28}$$

As $\langle w - K_1 y, w - K_2 y\rangle < 0$ for all $w \in F(y)$ and all non-zero $y \in \mathbb{R}^p$, we see that $\ker L = \{0\}$. Consequently, L is left invertible, with $L^\sharp := (L^*L)^{-1}L^* \in \mathbb{R}^{p\times m}$ being a left inverse of L. Furthermore, it follows from (2.28) that

$$|F(y) - My|_{\mathrm{m}}^2 = \|Ly\|^2 + \sup_{w\in F(y)} \langle w - K_1 y, w - K_2 y\rangle \quad \forall\, y \in \mathbb{R}^p\,,$$

and hence, using the compactness of the set $F(y)$, we obtain that

$$|F(y) - My|_{\mathrm{m}} < \|Ly\| \quad \forall\, y \in \mathbb{R}^p,\ y \neq 0\,.$$

Defining the set-valued map $\tilde F$ by $\tilde F(z) := F(L^\sharp z)$ for all $z \in \mathbb{R}^m$, we conclude that

$$|\tilde F(z) - ML^\sharp z|_{\mathrm{m}} < \|LL^\sharp z\| \quad \forall\, z \in (\ker L^\sharp)^\perp = \operatorname{im} L,\ y \neq 0\,.$$

Let $z \in \mathbb{R}^m$ and write $z = z_1 + z_2$ with $z_1 \in \operatorname{im} L$ and $z_2 \in (\operatorname{im} L)^\perp = \ker L^\sharp$. If $z_1 = 0$, then $|\tilde F(z) - ML^\sharp z|_{\mathrm{m}} = 0$, and if $z_1 \neq 0$, then we arrive at

$$|\tilde F(z) - ML^\sharp z|_{\mathrm{m}} < \|LL^\sharp z_1\| = \|LL^\sharp z\| \leq \|z\|\,,$$

where we have used that $\|LL^\sharp\| \leq 1$, which follows from the fact that $LL^\sharp$ is the orthogonal projection onto $\operatorname{im} L$ along $(\operatorname{im} L)^\perp$. Consequently,

$$|\tilde F(z) - ML^\sharp z|_{\mathrm{m}} < \|z\| \quad \forall\, z \in \mathbb{R}^m,\ z \neq 0\,. \tag{2.29}$$

By hypothesis $\mathbf{H} := (I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ is positive real, and therefore, by a well-known result (see, for example, [18, Corollary 3.6]), $\|(\mathbf{H} - I)(\mathbf{H} + I)^{-1}\|_{H^\infty} \leq 1$. As

$$(\mathbf{H} - I)(\mathbf{H} + I)^{-1} = L\mathbf{G}(I - M\mathbf{G})^{-1} = L\mathbf{G}(I - ML^\sharp L\mathbf{G})^{-1} = (L\mathbf{G})^{ML^\sharp}\,,$$

we see that

$$\|(L\mathbf{G})^{ML^\sharp}\|_{H^\infty} \leq 1\,. \tag{2.30}$$

Now $L\mathbf{G}$ is the transfer function of the state-space system (A, B, LC) and the hypothesis on (A, B, C) combined with the left invertiblity of L shows that (A, B, LC) controllable and detectable, or stabilizable and observable. Therefore, (2.30) implies that

$$\mathbb{B}_{\mathbb{C}}(ML^\sharp, 1) \subset \mathbb{S}_{\mathbb{C}}(A, B, LC)\,. \tag{2.31}$$

Equations (2.29) and (2.31) show that Theorem 2.5 (with $K = ML^\sharp$ and $r = 1$) applies to the Lur'e inclusion

$$\dot x(t) - Ax(t) - v(t) \in B\tilde F(LCx(t)), \quad t \geq 0\,. \tag{2.32}$$

The claim now follows since $(v, x) \in \mathcal{B}_{\mathrm{inc}}$ if, and only if, (v, x) is a trajectory of (2.32). □

In the proposition below, we strengthen the assumption on F to avoid the controllability/observability hypothesis for the linear system (A, B, C) imposed in Theorem 2.5. The stability property obtained is stronger than that guaranteed by Theorem 2.5 as the integral ISS estimate (2.22) continues to hold and, additionally, the Lur'e inclusion (2.1) is ISS.

Proposition 2.8. *Let $K \in \mathbb{R}^{m\times p}$ and $r > 0$ be such that $\mathbb{B}_{\mathbb{C}}(K, r) \subset \mathbb{S}_{\mathbb{C}}(A, B, C)$. If $F(0) = \{0\}$ and* (2.3) *and* (2.5) *are satisfied, then there exist $\zeta, \psi \in \mathcal{KL}$ and $\theta, \varphi \in \mathcal{K}$ such that* (2.6) *and* (2.22) *hold.*

Proof. The existence of $\zeta \in \mathcal{KL}$ and $\theta \in \mathcal{K}$ such that (2.6) holds follows immediately from Theorem 2.3. By the proof of Theorem 2.3, there there exist $\alpha, \beta \in \mathcal{K}_\infty$ and a continuously differentiable radially unbounded function $U : \mathbb{R}^n \to \mathbb{R}_+$ with $U(0) = 0$ and $U(z) > 0$ for all $z \neq 0$ and such that (2.7) is satisfied. Obviously, (2.7) implies that

$$\langle (\nabla U)(z), w \rangle \leq -\alpha(\|z\|) \quad \forall\, w \in Az + BF(Cz),\ \forall\, z \in \mathbb{R}^n$$

which is identical to (2.24). We can now proceed as in the proof of Theorem 2.5 to establish the existence of $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that (2.22) holds. □

Proposition 2.8 has corollaries which are similar to Corollaries 2.6 and 2.7. For the sake of brevity, we only state the result which corresponds to Corollary 2.6.

Corollary 2.9. *Let $K \in \mathbb{R}^{m\times p}$ be such that $K \in \mathbb{S}_\mathbb{C}(A, B, C)$ and let ν be as in* Corollary 2.6. *Assume that $F(0) = \{0\}$ and $\sup_{y\neq 0} \nu(y) < \infty$. If there exists $\gamma \in \mathcal{K}_\infty$ such that $\nu(y)\|\mathbf{G}^K\|_{H^\infty} < 1 - \gamma(\|y\|)/\|y\|$ for all $y \neq 0$, then there exist $\zeta, \psi \in \mathcal{KL}$ and $\theta, \varphi \in \mathcal{K}$ such that* (2.6) *and* (2.22) *hold.*

3 Incremental integral ISS results for Lur'e differential equations

In the sequel, we will consider the forced Lur'e differential equation

$$\dot{x}(t) = Ax(t) + Bf(t, Cx(t)) + v(t) \quad t \geq 0\,, \tag{3.1}$$

where $(A, B, C) \in \mathbb{L}$ and the time-varying nonlinearity $f : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ is assumed to have the following properties: for every compact set $\Gamma \subset \mathbb{R}^p$, there exists $\lambda \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}_+)$ such that $\|f(t, y_1) - f(t, y_2)\| \leq \lambda(t)\|y_1 - y_2\|$ for all $y_1, y_2 \in \Gamma$ and all $t \geq 0$, and, for each $y \in \mathbb{R}^p$, the function $t \mapsto f(t, y)$ is in $L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^m)$. We will sometimes refer to (3.1) as the Lur'e equation (A, B, C, f).

Let $v \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ be given and let $0 < \tau \leq \infty$. A function $x \in W^{1,1}_{\mathrm{loc}}([0, \tau), \mathbb{R}^n)$ is said to be a solution of (3.1) on $[0, \tau)$ if (3.1) holds for almost every $t \in [0, \tau)$. A solution of (3.1) on $[0, \infty) = \mathbb{R}_+$ is called a global solution. If f is uniformly affinely linearly bounded in the sense that there exist $a, b \geq 0$ such that

$$\|f(t, y)\| \leq a + b\|y\| \quad \forall\, (t, y) \in \mathbb{R}_+ \times \mathbb{R}^p\,,$$

then, for each $v \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ and each $x^0 \in \mathbb{R}^n$, there exists a unique global solution x of (3.1) satisfying $x(0) = x^0$, see, for example, [29, Proposition 4.12]. We define the behaviour of (3.1) by

$$\mathcal{B} := \left\{(v, x) \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n) \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^n) \ :\ (v, x) \text{ satisfies (3.1) a.e. on } \mathbb{R}_+\right\}.$$

In the case wherein f does not depend on time, we will (in Section 4) also make use of the bilateral behaviour $\mathcal{BB}$ of (3.1) defined by

$$\mathcal{BB} := \left\{(v, x) \in L^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \times W^{1,1}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n) \ :\ (v, x) \text{ satisfies (3.1) a.e. on } \mathbb{R}\right\}.$$

As, in the bilateral context, f is assumed to be time-independent, $\mathcal{BB}$ is shift-invariant, that is,

$$(v, x) \in \mathcal{BB} \implies (\mathbf{S}_\tau v, \mathbf{S}_\tau x) \in \mathcal{BB} \quad \forall\, \tau \in \mathbb{R}\,. \tag{3.2}$$

As in Section 2, we set $\mathbf{G}(s) = C(sI - A)^{-1}B$ and $\mathbf{G}^L(s) := \mathbf{G}(I - L\mathbf{G})^{-1}$ for $L \in \mathbb{C}^{m\times p}$. We will assume throughout that $\mathbf{G}(s) \not\equiv 0$, and so $\mathbf{G}^L(s) \not\equiv 0$ for every $L \in \mathbb{C}^{m\times p}$.

Let $K \in \mathbb{S}_\mathbb{R}(A, B, C)$ and set

$$r^K := 1/\|\mathbf{G}^K\|_{H^\infty} < \infty\,.$$

We introduce three assumptions which will be used throughout (but not all of them simultaneously in any of the following results).

(A1) $\sup_{t\geq 0}\|f(t,y+\xi)-f(t,\xi)-Ky\| < r^K\|y\|$ for all $y,\xi\in\mathbb{R}^p$, $y\neq 0$.

(A2) There exists $\xi_0\in\mathbb{R}^p$ such that

$$\big(r^K\|y\|-\sup_{t\geq 0}\|f(t,y+\xi_0)-f(t,\xi_0)-Ky\|\big)\to\infty \quad \text{as } \|y\|\to\infty.$$

(A3) (A,B,C) is controllable or observable.

Before we concern ourselves with the main results of this section, it is useful to state two technical lemmas, the proofs of which can be found in the Appendix.

Lemma 3.1. *If* (A1) *and* (A2) *hold, then, for all* $\xi\in\mathbb{R}^p$,

$$\big(r^K\|y\|-\sup_{t\geq 0}\|f(t,y+\xi)-f(t,\xi)-Ky\|\big)\to\infty \quad \textit{as } \|y\|\to\infty.$$

Lemma 3.2. *Assume that* (A1) *holds and let* $\Gamma\subset\mathbb{R}^p$ *be a compact set. Define* $\beta_\Gamma:\mathbb{R}_+\to\mathbb{R}_+$ *by*

$$\beta_\Gamma(s):=\sup_{t\geq 0,\,\xi\in\Gamma,\,\|y\|\leq s}\|f(t,y+\xi)-f(t,\xi)-Ky\|. \tag{3.3}$$

(1) *The function* β_Γ *is globally Lipschitz with Lipschitz constant* r^K*, the function* α *defined by* $\alpha(s):=r^Ks-\beta_\Gamma(s)$ *for all* $s\geq 0$ *is in* $\mathcal{P}$ *and*

$$\sup_{t\geq 0}\|f(t,y+\xi)-f(t,\xi)-Ky\|\leq r^K\|y\|-\alpha(\|y\|)\quad \forall\, y\in\mathbb{R}^p,\ \forall\,\xi\in\Gamma. \tag{3.4}$$

If additionally (A2) *is satisfied, then there exists* $\alpha_0\in\mathcal{K}_\infty$ *such that* $\alpha_0(s)\leq\alpha(s)$ *for all* $s\geq 0$ *and* (3.4) *holds with* α *replaced by* α_0.

(2) *If there exists* $\xi_0\in\mathbb{R}^p$ *such that*

$$\liminf_{\|y\|\to\infty}\big(r^K\|y\|-\sup_{t\geq 0}\|f(t,y+\xi_0)-f(t,\xi_0)-Ky\|\big)>0,$$

then there exists $\alpha_0\in\mathcal{K}$ *such that* $\alpha_0(s)\leq r^Ks-\beta_{\{\xi_0\}}(s)$ *for all* $s\geq 0$ *and*

$$\sup_{t\geq 0}\|f(t,y+\xi_0)-f(t,\xi_0)-Ky\|\leq r^K\|y\|-\alpha_0(\|y\|)\quad \forall\, y\in\mathbb{R}^p.$$

A pair $(x^\dagger,v^\dagger)\in\mathbb{R}^n\times L^1_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$ is said to be an equilibrium pair of (3.1) if $Ax^\dagger+Bf(t,Cx^\dagger)+v^\dagger(t)=0$ for almost every $t\geq 0$. Trivially, if $(x^\dagger,v^\dagger)$ is an equilibrium pair of (3.1), then the constant function $t\mapsto x^\dagger$ is a solution of (3.1) with forcing $v=v^\dagger$. For given $x^\dagger\in\mathbb{R}^n$, we define $e_{x^\dagger}\in L^1_{\text{loc}}(\mathbb{R}_+,\mathbb{R}^n)$ by $e_{x^\dagger}(t):=-(Ax^\dagger+Bf(t,Cx^\dagger))$ for all $t\geq 0$. Obviously, for each $x^\dagger\in\mathbb{R}^n$, the pair $(x^\dagger,e_{x^\dagger})$ is an equilibrium pair of (3.1).

The next theorem provides a sufficient condition for a certain semi-global incremental iISS property to hold.

Theorem 3.3. *Assume that* (A1) *and* (A3) *hold.*

(1) *For each* $\rho>0$ *there exist* $\psi\in\mathcal{KL}$ *and* $\varphi\in\mathcal{K}$ *such that, for every* $(w,z)\in\mathcal{B}$ *with* $\|z\|_{L^\infty}\leq\rho$ *and every* $t_0\geq 0$,

$$\|x(t)-z(t)\|\leq\psi(\|x(t_0)-z(t_0)\|,t-t_0)+\varphi\Big(\int_{t_0}^t\|v(s)-w(s)\|\mathrm{d}s\Big)\quad\forall\,(v,x)\in\mathcal{B},\ \forall\, t\geq t_0. \tag{3.5}$$

(2) *For each $x^\dagger \in \mathbb{R}^n$ and every $b > 0$ there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for every $(w, z) \in \mathcal{B}$ with $\|z(0) - x^\dagger\| + \|w - e_{x^\dagger}\|_{L^1} \le b$ and every $t_0 \ge 0$, (3.5) holds.*

Statement (2) of Theorem 3.3 says that (3.1) is semi-globally incrementally iISS with every semi-global incremental iISS-gain being linear (absorbed into φ on the RHS of the estimate (3.5)). The linearity of the semi-global incremental iISS-gains will play an important role in Section 4, see the proof of Theorem 4.3.

Proof of Theorem 3.3. (1) Let $\rho > 0$ and set

$$Y_\rho := \{C\eta : \|\eta\| \le \rho\}. \tag{3.6}$$

Invoking statement (1) of Lemma 3.2 (with $\Gamma = Y_\rho$) shows that there exists $\alpha \in \mathcal{P}$ such that

$$\sup_{t\ge 0} \|f(t, y + \xi) - f(t, \xi) - Ky\| \le r^K \|y\| - \alpha(\|y\|) \quad \forall\, y \in \mathbb{R}^p,\ \forall\, \xi \in Y_\rho. \tag{3.7}$$

Consequently, for any $(w, z) \in \mathcal{B}$ with $\|z\|_{L^\infty} \le \rho$, the function $f_z : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ defined by

$$f_z(t, y) := f(t, y + Cz(t)) - f(t, Cz(t)) \quad \forall\, (t, y) \in \mathbb{R}_+ \times \mathbb{R}^p\,, \tag{3.8}$$

satisfies

$$\sup_{t\in\mathbb{R}_+} \|f_z(t, y) - Ky\| \le r^K \|y\| - \alpha(\|y\|) \quad \forall\, y \in \mathbb{R}^p\,. \tag{3.9}$$

Let $(w, z) \in \mathcal{B}$ with $\|z\|_{L^\infty} \le \rho$ and let $(v, x) \in \mathcal{B}$ be arbitrary. Then

$$\dot{x}(t) - \dot{z}(t) = A(x(t) - z(t)) + Bf_z\big(t, C(x(t) - z(t))\big) + v(t) - w(t)\,, \quad \text{for a.e. } t \ge 0,$$

that is, $(v-w, x-z)$ is a trajectory of the Lur'e equation (A, B, C, f_z) and hence of the Lur'e inclusion (A, B, C, F) with $F(y) := \mathrm{cl}\,\mathbb{B}_{\mathbb{R}}\big(Ky, r^K\|y\| - \alpha(\|y\|)\big)$ for $y \in \mathbb{R}^p$. An application of Theorem 2.5 (or, alternatively, of Corollary 2.6) shows that there exist $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ (depending only on (A, B, C), K and ρ) such that

$$\|x(t) - z(t)\| \le \psi(\|x(0) - z(0)\|, t) + \varphi\Big(\int_0^t \|v(s) - w(s)\|\mathrm{d}s\Big) \quad \forall\, (v, x) \in \mathcal{B},\ \forall\, t \ge 0\,. \tag{3.10}$$

Finally, let $t_0 \ge 0$. Obviously, $\|\mathbf{S}_{t_0} z\|_{L^\infty} \le \rho$, and combining (3.10) with the left-shift invariance of the behaviour $\mathcal{B}_{\mathrm{inc}}$ of the Lur'e inclusion (A, B, C, F), cf. (2.2), we conclude that

$$\|x(t+t_0) - z(t+t_0)\| \le \psi(\|x(t_0) - z(t_0)\|, t) + \varphi\Big(\int_0^t \|v(s+t_0) - w(s+t_0)\|\mathrm{d}s\Big) \quad \forall\, (v, x) \in \mathcal{B},\ \forall\, t \ge 0\,.$$

Consequently,

$$\|x(t) - z(t)\| \le \psi(\|x(t_0) - z(t_0)\|, t - t_0) + \varphi\Big(\int_{t_0}^t \|v(s) - w(s)\|\mathrm{d}s\Big) \quad \forall\, (v, x) \in \mathcal{B},\ \forall\, t \ge t_0\,,$$

completing the proof of statement (1).

(2) Let $x^\dagger \in \mathbb{R}^n$ and $b > 0$. As $(e_{x^\dagger}, x^\dagger) \in \mathcal{B}$, it follows from statement (1) that there exist $\beta \in \mathcal{KL}$ and $\alpha \in \mathcal{K}$ such that, for every $t_0 \ge 0$,

$$\|x(t) - x^\dagger\| \le \beta(\|x(t_0) - x^\dagger\|, t - t_0) + \alpha\Big(\int_{t_0}^t \|v(s) - e_{x^\dagger}(s)\|\mathrm{d}s\Big) \quad \forall\, (v, x) \in \mathcal{B}\,,\ \forall\, t \ge t_0\,.$$

Hence, for all $(w, z) \in \mathcal{B}$ with $\|z(0) - x^\dagger\| + \|w - e_{x^\dagger}\|_{L^1} \le b$,

$$\|z(t) - x^\dagger\| \le \beta(\|z(0) - x^\dagger\|, 0) + \alpha(\|w - e_{x^\dagger}\|_{L^1}) \le \beta(b, 0) + \alpha(b) \quad \forall\, t \ge 0\,,$$

showing that, for all $(w,z) \in \mathcal{B}$ with $\|z(0)-x^\dagger\| + \|w - e_{x^\dagger}\|_{L^1} \le b$, we have that $\|z\|_{L^\infty} \le \rho$, where $\rho := \beta(b,0) + \alpha(b) + \|x^\dagger\|$. Finally, statement (1) guarantees the existence of $\psi \in \mathcal{KL}$ and $\varphi \in \mathcal{K}$ such that, for every $(w,z) \in \mathcal{B}$ with $\|z(0)-x^\dagger\| + \|w - e_{x^\dagger}\|_{L^1} \le b$ and every $t_0 \ge 0$, (3.5) holds. □

The next result is an immediate consequence of statement (1) of Theorem 3.3.

Corollary 3.4. *Assume that* (A1) *and* (A3) *hold and let $(w,z) \in \mathcal{B}$ with z bounded. Then, for every $(v,x) \in \mathcal{B}$ such that $v - w \in L^1(\mathbb{R}_+, \mathbb{R}^n)$, the difference $x(t) - z(t)$ converges to 0 as $t \to \infty$.*

The following corollary provides a circle-criterion interpretation of Theorem 3.3. The proof is similar to that of Corollary 2.7 (with Theorem 3.3 now playing the role of Theorem 2.5) and we therefore leave it to the reader.

Corollary 3.5. *Let $K_1, K_2 \in \mathbb{R}^{m\times p}$. Assume that (A,B,C) is controllable and detectable or, alternatively, stabilizable and observable. If $(I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ is positive real and*

$$\sup_{t\ge 0}\langle f(t,y+\xi) - f(t,\xi) - K_1 y, f(t,y+\xi) - f(t,\xi) - K_2 y\rangle < 0 \quad \forall\, y,\xi \in \mathbb{R}^p,\ y \ne 0\,,$$

then statements (1) *and* (2) *of* Theorem 3.3 *hold.*

We return briefly to Theorem 3.3. To this end, let $(w,z) \in \mathcal{B}$ and assume that there does not exist $x^\dagger \in \mathbb{R}^n$ such that $w - e_{x^\dagger} \in L^1(\mathbb{R}_+, \mathbb{R}^n)$ (so that statement (2) of Theorem 3.3 cannot be applied). In this case, boundedness of z is essential for (3.5) to hold, and it is natural to ask under what conditions is z bounded. The following two results provide such conditions.

Lemma 3.6. *Assume that* (A1) *holds and that there exists $y^\dagger \in \operatorname{im} C$ such that*

$$\liminf_{\|y\|\to\infty}\big(r^K\|y\| - \sup_{t\ge 0}\|f(t,y+y^\dagger) - f(t,y^\dagger) - Ky\|\big) > 0. \tag{3.11}$$

Let $x^\dagger \in \mathbb{R}^n$ be such that $y^\dagger = Cx^\dagger$. Then there exist $b > 0$ and $\alpha, \beta \in \mathcal{K}$ such that, for every $(w,z) \in \mathcal{B}$ with $\|w - e_{x^\dagger}\|_{L^\infty} \le b$, the state function z satisfies

$$\|z(t) - x^\dagger\| \le \alpha(\|z(0) - x^\dagger\|) + \beta(\|w - e_{x^\dagger}\|_{L^\infty}) \quad \forall\, t \ge 0\,.$$

Proof. Setting

$$\tilde f(t,y) := f(t,y+y^\dagger) - f(t,y^\dagger) \quad \forall\,(t,y) \in \mathbb{R}_+ \times \mathbb{R}^p\,,$$

and invoking (3.11) and statement (2) of Lemma 3.2 (with $\xi_0 = y^\dagger$), we conclude that there exists $\gamma \in \mathcal{K}$ such that

$$\sup_{t\ge 0}\|\tilde f(t,y) - Ky\| \le r^K\|y\| - \gamma(\|y\|) \quad \forall\, y \in \mathbb{R}^p\,.$$

Therefore, appealing to [17, Theorem 3.1 and Remark 3.12], the Lur'e equation $(A,B,C,\tilde f)$ is strongly iISS. In particular, $(A,B,C,\tilde f)$ is small-signal ISS, and consequently, there exist $b > 0$ and $\alpha,\beta \in \mathcal{K}$ such that if (u,x) is a trajectory of the Lur'e equation $(A,B,C,\tilde f)$ with $\|u\|_{L^\infty} \le b$, then

$$\|x(t)\| \le \alpha(\|x(0)\|) + \beta(\|u\|_{L^\infty}) \quad \forall\, t \ge 0\,. \tag{3.12}$$

Now let $w \in L^1_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ be such that $\|w - e_{x^\dagger}\|_{L^\infty} \le b$ and let $z \in W^{1,1}_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ be such that $(w,z) \in \mathcal{B}$. Then $(w - e_{x^\dagger}, z - x^\dagger)$ is a trajectory of the Lur'e equation $(A,B,C,\tilde f)$ and the claim now follows from (3.12). □

The next theorem shows that if (A1) and (A2) are assumed, then (3.1) is semi-globally incrementally stable, in both the ISS and iISS sense.

Theorem 3.7. *Assume that* (A1) *and* (A2) *hold and that there exists* $y^* \in \mathbb{R}^p$ *such that the function* $t \mapsto f(t, y^*)$ *is essentially bounded. Then, for every* $b > 0$*, there exist* $\psi, \zeta \in \mathcal{KL}$ *and* $\varphi, \theta \in \mathcal{K}$ *such that, for every* $(w, z) \in \mathcal{B}$ *with* $\|w\|_{L^\infty} + \|z(0)\| \le b$ *and every* $t_0 \ge 0$*, the estimate* (3.5) *holds, and, furthermore,*

$$\|x(t) - z(t)\| \le \zeta(\|x(t_0) - z(t_0)\|, t - t_0) + \theta\big(\|v - w\|_{L^\infty(t_0,t)}\big) \qquad \forall\, (v, x) \in \mathcal{B} \ \text{ with } v \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n),\ \forall\, t \ge t_0\,. \tag{3.13}$$

In particular, if $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ *and* $(v, x) \in \mathcal{B}$*, then* x *is bounded.*

Proof. As $t \mapsto f(t, y^*)$ is essentially bounded, it follows from (A1) that $t \mapsto f(t, y)$ is essentially bounded for every $y \in \mathbb{R}^p$. Set $\tilde{f}(t, y) := f(t, y) - f(t, 0)$ for all $(t, y) \in \mathbb{R}_+ \times \mathbb{R}^p$. Appealing to statement (1) of Lemma 3.2 (with $\Gamma = \{0\}$), we see that there exists $\alpha_0 \in \mathcal{K}_\infty$ such that

$$\sup_{t \ge 0} \|\tilde{f}(t, y) - Ky\| \le r^K \|y\| - \alpha_0(\|y\|) \quad \forall\, y \in \mathbb{R}^p.$$

Setting $F(y) := \operatorname{cl} \mathbb{B}_{\mathbb{R}}\big(Ky, r^K\|y\| - \alpha_0(\|y\|)\big)$ for $y \in \mathbb{R}^p$, every trajectory of the Lur'e equation $(A, B, C, \tilde{f})$ is also a trajectory of the Lur'e inclusion (A, B, C, F). Thus, Proposition 2.8 guarantees that the Lur'e equation $(A, B, C, \tilde{f})$ is ISS. Noting that if $(w, z) \in \mathcal{B}$, then $(w + Bf(\,\cdot\,, 0), z)$ is a trajectory of $(A, B, C, \tilde{f})$, it follows that for given $b > 0$, there exists $\rho > 0$, such that $\|z\|_{L^\infty} \le \rho$ for all $(w, z) \in \mathcal{B}$ satisfying $\|w\|_{L^\infty} + \|z(0)\| \le b$ (where we have used that $f(\,\cdot\,, 0)$ is essentially bounded).

Define Y_ρ and f_z by (3.6) and (3.8), respectively. Using once again statement (1) of Lemma 3.2 (this time with $\Gamma = Y_\rho$), we obtain that there exists $\alpha \in \mathcal{K}_\infty$ such that (3.7) is satisfied. Letting $(w, z) \in \mathcal{B}$ with $\|w\|_{L^\infty} + \|z(0)\| \le \sigma$, we have that $\|z\|_{L^\infty} \le \rho$ and thus (3.9) holds (with $\alpha \in \mathcal{K}_\infty$). For every $(v, x) \in \mathcal{B}$, $(v - w, x - z)$ is a trajectory of the Lur'e equation (A, B, C, f_z) and hence of the Lur'e inclusion (A, B, C, F) with $F(y) := \operatorname{cl} \mathbb{B}_{\mathbb{R}}\big(Ky, r^K\|y\| - \alpha(\|y\|)\big)$. Since $\alpha \in \mathcal{K}_\infty$, the assumptions of Proposition 2.8 are satisfied, and, for $t_0 = 0$, the claim now follows from Proposition 2.8. Using the left-shift invariance of the behaviour of the Lur'e inclusion (A, B, C, F), the claim can be easily obtained for arbitrary $t_0 \ge 0$.

Finally, let $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ and $(v, x) \in \mathcal{B}$. Defining w by $w(t) = -Bf(t, 0)$ for $t \ge 0$, it is obvious that $(w, 0) \in \mathcal{B}$ and thus (3.13) guarantees that x is bounded. □

4 Response to Stepanov almost periodic inputs

We start this section by recalling some relevant definitions and properties from the theory of almost periodic functions. Let $R = \mathbb{R}$ or $\mathbb{R}_+$ and let X be a Banach space. A set $S \subseteq R$ is said to be *relatively dense* (in R) if there exists $l > 0$ such that

$$[a, a + l] \cap S \ne \emptyset \quad \forall\, a \in R\,.$$

For $\varepsilon > 0$, we say that $\tau \in R$ is an *ε-period* of $v \in C(R, X)$ if

$$\|v(t) - v(t + \tau)\| \le \varepsilon \quad \forall\, t \in R\,.$$

We denote by $P(v, \varepsilon) \subseteq R$ the set of ε-periods of v and we say that $v \in C(R, X)$ is almost periodic (in the sense of Bohr) if $P(v, \varepsilon)$ is relatively dense in R for every $\varepsilon > 0$. Let $AP(R, X)$ denote the set of all almost periodic functions $v \in C(R, X)$. We note that $AP(R, X)$ is a closed subspace of $BUC(R, X)$, the space of bounded uniformly continuous functions $R \to X$ endowed with the sup-norm.

The straightforward proof of the following lemma is left to the reader.

Lemma 4.1. *If $v \in AP(\mathbb{R}_+, X)$, then, for every $\tau \in \mathbb{R}_+$,*

$$\sup_{t\in\mathbb{R}_+,\, t\geq\tau} \|v(t)\| = \|v\|_{L^\infty} = \limsup_{t\to\infty} \|v(t)\| .$$

Furthermore, if $v \in AP(\mathbb{R}, X)$, then, for every $\tau \in \mathbb{R}$,

$$\sup_{t\in\mathbb{R},\, t\geq\tau} \|v(t)\| = \|v\|_{L^\infty} \quad \text{and} \quad \sup_{t\in\mathbb{R},\, t\leq\tau} \|v(t)\| = \|v\|_{L^\infty} .$$

The above lemma shows that almost periodic functions are completely determined by their "infinite tails": if $v, w \in AP(\mathbb{R}_+, X)$ and there exists $\tau \in \mathbb{R}_+$ such that $v(t) = w(t)$ for all $t \geq \tau$, then $v = w$; similarly, if $v, w \in AP(\mathbb{R}, X)$ and there exists $\tau \in \mathbb{R}$ such that $v(t) = w(t)$ for all $t \geq \tau$, or for all $t \leq \tau$, then $v = w$.

We say that a function $v \in C(\mathbb{R}_+, X)$ is *asymptotically almost periodic* if it is of the form $v = v^{\mathrm{ap}} + w$ with $v^{\mathrm{ap}} \in AP(\mathbb{R}_+, X)$ and $w \in C_0(\mathbb{R}_+, X)$, where $C_0(\mathbb{R}_+, X)$ is the space of functions $u \in C(\mathbb{R}_+, X)$ such that $\lim_{t\to\infty} u(t) = 0$. The space of all asymptotically almost periodic functions $v \in C(\mathbb{R}_+, X)$ is denoted by $AAP(\mathbb{R}_+, X)$, that is,

$$AAP(\mathbb{R}_+, X) = AP(\mathbb{R}_+, X) + C_0(\mathbb{R}_+, X) .$$

Noting that, by Lemma 4.1,

$$\|v + w\|_{L^\infty} \geq \|v\|_{L^\infty} \quad \forall\, v \in AP(\mathbb{R}_+, X),\ \forall\, w \in C_0(\mathbb{R}_+, X) ,$$

it is easy to see that $AAP(\mathbb{R}_+, X)$ is a closed subspace of $BUC(\mathbb{R}_+, X)$.

As an immediate consequence of Lemma 4.1, we obtain the following result.

Lemma 4.2. *The following statements hold.*

(1) $AP(\mathbb{R}_+, X) \cap C_0(\mathbb{R}_+, X) = \{0\}$.

(2) *If $v \in AAP(\mathbb{R}_+, X)$, then the decomposition $v = v^{\mathrm{ap}} + w$, where $v^{\mathrm{ap}} \in AP(\mathbb{R}_+, X)$ and $w \in C_0(\mathbb{R}_+, X)$, is unique.*

It is well-known that $v \in C(\mathbb{R}, X)$ is almost periodic if, and only if, the set of translates $\{\mathbf{S}_\tau v : \tau \in \mathbb{R}\}$ is relatively compact in $BUC(\mathbb{R}, X)$, see, for example, [9, Theorem 6.6]. Since, for any $v \in C_0(\mathbb{R}_+, X)$, the set of left-translates $\{\mathbf{S}_\tau v : \tau \in \mathbb{R}_+\}$ is relatively compact in $BUC(\mathbb{R}_+, X)$, it is clear that the above characterisation of almost periodicity on $\mathbb{R}$ is not valid for functions in $C(\mathbb{R}_+, X)$. Interestingly, the elements of $AAP(\mathbb{R}_+, X)$ are precisely the functions for which the set $\{\mathbf{S}_\tau v : \tau \in \mathbb{R}_+\}$ is relatively compact in $BUC(\mathbb{R}_+, X)$ (if $\dim X < \infty$, this has been shown in [27], for the general case see [31]). For more information on, and further characterisations of, almost periodicity, we refer the reader to the literature, for instance, [7, 9].

There exists a close relationship between the spaces $AP(\mathbb{R}_+, X)$ and $AP(\mathbb{R}, X)$ which we now briefly explain. Following an idea in [5, Remark on p. 318], for every $v \in AP(\mathbb{R}_+, X)$, we define a function $v_{\mathrm{e}} : \mathbb{R} \to X$ by

$$v_{\mathrm{e}}(t) := \lim_{k\to\infty} v(t + \tau_k) \quad \forall\, t \in \mathbb{R} , \tag{4.1}$$

where $\tau_k \in P(v, 1/k)$ for each $k \in \mathbb{N}$ and $\tau_k \to \infty$ as $k \to \infty$. For given $t \in \mathbb{R}$, we have

$$\|v(t + \tau_k) - v(t + \tau_l)\| \leq \|v(t + \tau_k) - v(t + \tau_k + \tau_l)\| + \|v(t + \tau_k + \tau_l) - v(t + \tau_l)\| \leq \frac{1}{l} + \frac{1}{k} ,$$

for all $k, l \in \mathbb{N}$ sufficiently large, and so $(v(t + \tau_k))_k$ is a Cauchy sequence. Hence $v_{\mathrm{e}}(t)$ is well-defined for each $t \in \mathbb{R}$. By construction, $v_{\mathrm{e}}(t) = v(t)$ for all $t \geq 0$, that is, v_{e} extends v to $\mathbb{R}$. Furthermore, it is not difficult to show that $P(v_{\mathrm{e}},\varepsilon) = \{\pm\tau : \tau \in P(v,\varepsilon)\}$. In particular, $v_{\mathrm{e}} \in AP(\mathbb{R},X)$. Moreover, there is no other function in $AP(\mathbb{R},X)$ which extends v to $\mathbb{R}$, and Lemma 4.1 guarantees that

$$\|v_{\mathrm{e}}\|_{L^\infty} = \sup_{t\in\mathbb{R}} \|v_{\mathrm{e}}(t)\| = \sup_{t\in\mathbb{R}_+} \|v(t)\| = \|v\|_{L^\infty}.$$

We conclude that the map $AP(\mathbb{R}_+,X) \to AP(\mathbb{R},X)$, $v \mapsto v_{\mathrm{e}}$ is an isometric isomorphism.

For a function $v \in AP(\mathbb{R},X)$, we define the generalized Fourier coefficients of v by

$$\hat{v}(\lambda) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-i\lambda t} v(t)\mathrm{d}t \quad \forall\, \lambda \in \mathbb{R}.$$

It is well-known that the above limit exists for all $\lambda \in \mathbb{R}$ and the frequency spectrum

$$\sigma_{\mathrm{f}}(v) := \{\lambda \in \mathbb{R} : \hat{v}(\lambda) \neq 0\}$$

of v is countable, see for example, [9, Section VI.3]. The module $\mathrm{mod}(v)$ of $v \in AP(\mathbb{R},X)$ is the set of all numbers of the form $\sum_{\lambda\in\sigma_{\mathrm{f}}(v)} m(\lambda)\lambda$, where $m : \sigma_{\mathrm{f}}(v) \to \mathbb{Z}$ has finite support, that is, $m(\lambda) \neq 0$ for at most finitely many $\lambda \in \sigma_{\mathrm{f}}(v)$. Note that $\mathrm{mod}(v)$ carries the structure of a $\mathbb{Z}$-module and is the smallest subgroup of $\mathbb{R}$ containing $\sigma_{\mathrm{f}}(v)$.

We now recall another concept of almost periodicity which is weaker than that we have just defined. To this end, let $v \in L^1_{\mathrm{loc}}(R,\mathbb{R}^n)$ and $\varepsilon > 0$. We say that $\tau \in R$ is an ε-period of v (in the sense of Stepanov) if

$$\sup_{a\in R} \int_a^{a+1} \|v(s+\tau) - v(s)\|\mathrm{d}s \leq \varepsilon.$$

The set of ε-periods of v (in the sense of Stepanov) is denoted by $P_1(v,\varepsilon)$. We say that v is almost periodic in the sense of Stepanov if, for every $\varepsilon > 0$, the set $P_1(v,\varepsilon)$ is relatively dense in R. The set of all functions in $L^1_{\mathrm{loc}}(R,\mathbb{R}^n)$ which are almost periodic in the sense of Stepanov is denoted by $S^1(R,\mathbb{R}^n)$. We remark that $AP(R,\mathbb{R}^n) \subset S^1(R,\mathbb{R}^n)$, where the inclusion is strict (as, for example, $S^1(R,\mathbb{R}^n)$ contains certain discontinuous functions). A routine argument shows that $S^1(R,\mathbb{R}^n)$ is a closed subspace of the Banach space of uniformly locally integrable functions

$$UL^1_{\mathrm{loc}}(R,\mathbb{R}^n) := \Big\{ f \in L^1_{\mathrm{loc}}(R,\mathbb{R}^n) : \sup_{a\in R} \int_a^{a+1} \|v(s)\|\mathrm{d}s < \infty \Big\},$$

endowed with the Stepanov norm

$$\|v\|_{\mathrm{S}} := \sup_{a\in R} \int_a^{a+1} \|v(s)\|\mathrm{d}s.$$

Sometimes it will be convenient to associate with a function $v \in L^1_{\mathrm{loc}}(R,\mathbb{R}^n)$ another function $\tilde{v} : R \to L^1([0,1],\mathbb{R}^n)$, the so-called Bochner transform of v, which is defined by

$$\big(\tilde{v}(t)\big)(s) := v(t+s) \quad \forall\, t \in R,\ \forall\, s \in [0,1].$$

Then $\tilde{v} \in C(R, L^1([0,1],\mathbb{R}^n))$ and, furthermore, $v \in S^1(R,\mathbb{R}^n)$ if, and only if, $\tilde{v} \in AP(R, L^1([0,1],\mathbb{R}^n))$.

Let $v \in S^1(\mathbb{R}_+,\mathbb{R}^n)$ and let $\tau_k \in P_1(v,1/k)$ for all $k \in \mathbb{N}$ and $\tau_k \to \infty$ as $k \to \infty$. Then it can be proved that, for each $\tau > 0$, $\big(v(\cdot+\tau_k)\big)_k$ is a Cauchy sequence in $L^1([-\tau,\tau],\mathbb{R}^n)$ and hence defines a function $v_{\mathrm{e}} \in L^1_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^n)$. We note that if $v \in AP(\mathbb{R}_+,\mathbb{R}^n) \subset S^1(\mathbb{R}_+,\mathbb{R}^n)$, then this extension coincides with the extension defined via (4.1). It can be shown that $v_{\mathrm{e}}|_{\mathbb{R}_+} = v$, that is, v_{e} extends v to $\mathbb{R}$, $v_{\mathrm{e}} \in S^1(\mathbb{R},\mathbb{R}^n)$, $P_1(v_{\mathrm{e}},\varepsilon) = \{\pm\tau : \tau \in P_1(v,\varepsilon)\}$ for every $\varepsilon > 0$, and the map $S^1(\mathbb{R}_+,\mathbb{R}^n) \to S^1(\mathbb{R},\mathbb{R}^n)$, $v \mapsto v_{\mathrm{e}}$ is an isometric isomorphism.[1]

[1] These properties of v_{e} are not difficult to prove and should be well known, but we were not able to find them in the published literature. Details can be found in the first author's PhD thesis [13, Appendix C.2].

We are now in the position to use the results of Section 3 to prove the following theorem which describes the behaviour of (3.1) (with time-independent f) under forcing in $S^1(\mathbb{R}_+,\mathbb{R}^n)$. It is assumed throughout that $\mathbf{G}(s)\not\equiv 0$ and $K\in\mathbb{S}_{\mathbb{R}}(A,B,C)$.

Theorem 4.3. *Assume that f in (3.1) does not depend on t and (A1) and (A3) hold. Let $w\in S^1(\mathbb{R}_+,\mathbb{R}^n)$ and assume that there exists a trajectory $(w,z)\in\mathcal{B}$ with bounded z. Then there exists a unique $z^{\rm ap}\in AP(\mathbb{R}_+,\mathbb{R}^n)$ such that $(w,z^{\rm ap})\in\mathcal{B}$, for every $\varepsilon>0$, there exists $\delta>0$ such that $P_1(w,\delta)\subset P(z^{\rm ap},\varepsilon)$, and the following statements hold.*

(1) *For every $(v,x)\in\mathcal{B}$ with $v-w\in L^1(\mathbb{R}_+,\mathbb{R}^n)$, we have that*

$$\lim_{t\to\infty}\big(x(t)-z^{\rm ap}(t)\big)=0\,,$$

in particular, $x\in AAP(\mathbb{R}_+,\mathbb{R}^n)$.

(2) *If w is periodic with period τ, then $z^{\rm ap}$ is τ-periodic.*

(3) *$(w_{\rm e},z_{\rm e}^{\rm ap})\in\mathcal{BB}$ and there is no other bounded function $x:\mathbb{R}\to\mathbb{R}^n$ such that $(w_{\rm e},x)\in\mathcal{BB}$.*

(4) *$\mathrm{mod}(z_{\rm e}^{\rm ap})\subset\mathrm{mod}(\tilde w_{\rm e})$.*

As for statement (4), we recall that $\tilde w_{\rm e}$ is the function in $AP(\mathbb{R},L^1([0,1],\mathbb{R}^n))$ defined by $\big(\tilde w_{\rm e}(t)\big)(s)=w_{\rm e}(t+s)$ for all $t\in\mathbb{R}$ and all $s\in[0,1]$.

Proof of Theorem 4.3. Let $w\in S^1(\mathbb{R}_+,\mathbb{R}^n)$ and assume that there exists $(w,z)\in\mathcal{B}$ with bounded z. Choose $\rho\geq\|z\|_{L^\infty}$ and set $\mathcal{B}_\rho:=\{(v,x)\in\mathcal{B}:\|x\|_{L^\infty}\leq\rho\}$. By statement (1) of Theorem 3.3, there exist $\psi\in\mathcal{KL}$ and $\varphi\in\mathcal{K}$ such that

$$\|x_1(t)-x_2(t)\|\leq\psi\big(\|x_1(t_0)-x_2(t_0)\|,t-t_0\big)+\varphi\Big(\int_{t_0}^t\|v_1(s)-v_2(s)\|\,{\rm d}s\Big)\quad\forall\,(x_1,v_1)\in\mathcal{B},\ \forall\,(x_2,v_2)\in\mathcal{B}_\rho,\ \forall\,t\geq t_0\,.\tag{4.2}$$

We proceed in several steps.

Step 1: Construction of $z^{\rm ap}$. Choose a non-decreasing sequence $(\tau_k)_k$ such that

$$\tau_k\in P_1(w,1/k^2)\quad\text{and}\quad\tau_k>k\qquad\forall\,k\in\mathbb{N}.\tag{4.3}$$

We are going to show that $(\mathbf{S}_{\tau_k}z)_k$ is a Cauchy sequence in $BUC(\mathbb{R}_+,\mathbb{R}^n)$. To this end, we note that

$$\int_a^{a+k}\|w(t+\tau_k)-w(t)\|\,{\rm d}t=\sum_{j=1}^k\int_{a+j-1}^{a+j}\|w(t+\tau_k)-w(t)\|\,{\rm d}t\leq\frac{1}{k}\quad\forall\,a\geq 0,\ \forall\,k\in\mathbb{N}\,.\tag{4.4}$$

Note that $(\mathbf{S}_\tau w,\mathbf{S}_\tau z)\in\mathcal{B}$ for all $\tau\geq 0$ by (2.2). Obviously, $\|\mathbf{S}_\tau z\|_{L^\infty}\leq\rho$ for every $\tau\geq 0$, and it follows from (4.2) that

$$\begin{aligned}\|(\mathbf{S}_\sigma z)(s)-(\mathbf{S}_{\sigma+\tau}z)(s)\|\leq{}&\psi\big(\|z(\sigma+s_0)-z(\sigma+\tau+s_0)\|,s-s_0\big)\\&+\varphi\Big(\int_{s_0}^s\|(\mathbf{S}_\sigma w)(\eta)-(\mathbf{S}_{\sigma+\tau}w)(\eta)\|\,{\rm d}\eta\Big)\quad\forall\,s\geq s_0\geq 0,\forall\,\sigma,\tau\geq 0\,.\end{aligned}\tag{4.5}$$

Trivially, for $k,\ell\in\mathbb{N}$ with $k\geq\ell$,

$$(\mathbf{S}_{\tau_\ell}z)(t)-(\mathbf{S}_{\tau_k}z)(t)=(\mathbf{S}_tz)(\tau_\ell)-(\mathbf{S}_{t+\tau_k-\tau_\ell}z)(\tau_\ell)\quad\forall\,t\geq 0\,,$$

and so, setting

$$I(t;k,\ell) := \int_{\tau_\ell-\ell}^{\tau_\ell} \|(\mathbf{S}_t w)(\eta) - (\mathbf{S}_{t+\tau_k-\tau_\ell} w)(\eta)\|\,\mathrm{d}\eta \quad \forall\, t \geq 0\,,$$

and invoking (4.5) with $s = \tau_\ell$, $s_0 = \tau_\ell - \ell$, $\sigma = t$ and $\tau = \tau_k - \tau_\ell$, we arrive at

$$\|(\mathbf{S}_{\tau_\ell} z)(t) - (\mathbf{S}_{\tau_k} z)(t)\| \leq \psi\big(\|z(t+\tau_\ell-\ell) - z(t+\tau_k-\ell)\|, \ell\big) + \varphi(I(t;k,\ell)) \\ \forall\, t \geq 0,\ \forall\, k,\ell \in \mathbb{N} \text{ s.t. } k \geq \ell. \tag{4.6}$$

Now

$$I(t;k,\ell) \leq \int_{\tau_\ell-\ell}^{\tau_\ell} \|(\mathbf{S}_t w)(\eta) - (\mathbf{S}_{t+\tau_k} w)(\eta)\|\mathrm{d}\eta + \int_{\tau_\ell-\ell}^{\tau_\ell} \|(\mathbf{S}_{t+\tau_k} w)(\eta) - (\mathbf{S}_{t+\tau_k-\tau_\ell} w)(\eta)\|\,\mathrm{d}\eta,$$

and so, for all $t \geq 0$ and all $k, \ell \in \mathbb{N}$ such that $k \geq \ell$,

$$I(t;k,\ell) \leq \int_{t+\tau_\ell-\ell}^{t+\tau_\ell-\ell+k} \|w(\eta) - (\mathbf{S}_{\tau_k} w)(\eta)\|\,\mathrm{d}\eta + \int_{t+\tau_k-\ell}^{t+\tau_k} \|(\mathbf{S}_{\tau_\ell} w)(\eta) - w(\eta)\|\,\mathrm{d}\eta.$$

Consequently, by (4.4),

$$I(t;k,\ell) \leq \frac{1}{k} + \frac{1}{\ell} \quad \forall\, t \geq 0\,,\ \forall\, k,\ell \in \mathbb{N} \text{ s.t. } k \geq \ell,$$

and it follows from (4.6) that

$$\|(\mathbf{S}_{\tau_\ell} z)(t) - (\mathbf{S}_{\tau_k} z)(t)\| \leq \psi(2\rho,\ell) + \varphi(1/k + 1/\ell) \quad \forall\, t \geq 0,\ \forall\, k,\ell \in \mathbb{N} \text{ s.t. } k \geq \ell.$$

This shows that $(\mathbf{S}_{\tau_k} z)_k$ is a Cauchy sequence in $BUC(\mathbb{R}_+, \mathbb{R}^n)$, the limit of which we denote by z^{ap}.

Step 2: Almost periodicity of z^{ap}. To show that $z^{\mathrm{ap}} \in AP(\mathbb{R}_+, \mathbb{R}^n)$, let $\varepsilon > 0$ and choose $T \in \mathbb{N}$ and $a > 0$ such that

$$\psi(2\rho, T) \leq \frac{\varepsilon}{2} \quad \text{and} \quad \varphi(a) \leq \frac{\varepsilon}{2}\,.$$

Furthermore, let $\tau \in P_1(w, a/T)$. Obviously,

$$(\mathbf{S}_{\tau_\ell} z)(t+\tau) - (\mathbf{S}_{\tau_\ell} z)(t) = (\mathbf{S}_{t+\tau} z)(\tau_\ell) - (\mathbf{S}_t z)(\tau_\ell), \quad \forall\, \ell \in \mathbb{N},\ \forall\, t \geq 0\,,$$

and so, invoking (4.5) with $s = \tau_\ell$, $s_0 = \tau_\ell - T$, and $\sigma = t$,

$$\|(\mathbf{S}_{\tau_\ell} z)(t+\tau) - (\mathbf{S}_{\tau_\ell} z)(t)\| \leq \psi\big(\|z(t+\tau_\ell-T) - z(t+\tau+\tau_\ell-T)\|, T\big) \\ + \varphi\Big(\int_{\tau_\ell-T}^{\tau_\ell} \|(\mathbf{S}_t w)(\eta) - (\mathbf{S}_{t+\tau} w)(\eta)\|\,\mathrm{d}\eta\Big) \quad \forall\, t \geq 0,\ \forall\, \ell \in \mathbb{N} \text{ s.t. } \tau_\ell \geq T.$$

Now,

$$\begin{aligned} \int_{\tau_\ell-T}^{\tau_\ell} \|(\mathbf{S}_t w)(\eta) - (\mathbf{S}_{t+\tau} w)(\eta)\|\,\mathrm{d}\eta &= \int_{t+\tau_\ell-T}^{t+\tau_\ell} \|w(\eta) - (\mathbf{S}_\tau w)(\eta)\|\,\mathrm{d}\eta \\ &= \sum_{j=1}^{T} \int_{t+\tau_\ell-T+j-1}^{t+\tau_\ell-T+j} \|w(\eta) - w(\eta+\tau)\|\,\mathrm{d}\eta \\ &\leq a \quad \forall\, t \geq 0,\ \forall\, \ell \in \mathbb{N} \text{ s.t. } \tau_\ell \geq T, \end{aligned}$$

where the last inequality follows from the choice of τ. Thus,

$$\|(\mathbf{S}_{\tau_\ell} z)(t+\tau) - (\mathbf{S}_{\tau_\ell} z)(t)\| \leq \psi(2\rho, T) + \varphi(a) \leq \varepsilon \quad \forall\, t \geq 0,\ \forall\, \ell \in \mathbb{N} \text{ s.t. } \tau_\ell \geq T.$$

Letting $\ell \to \infty$ shows that

$$\|z^{\mathrm{ap}}(t+\tau) - z^{\mathrm{ap}}(t)\| \leq \varepsilon \quad \forall\, t \geq 0\,,$$

and so $\tau \in P(z^{\rm ap},\varepsilon)$. Since $\tau \in P_1(w,a/T)$ was arbitrary, we conclude that

$$P_1(w,a/T) \subset P(z^{\rm ap},\varepsilon)\,,$$

showing that $P(z^{\rm ap},\varepsilon)$ is relatively dense in $\mathbb{R}_+$. Consequently, $z^{\rm ap} \in AP(\mathbb{R}_+,\mathbb{R}^n)$. We note that by construction and choice of ρ we have that $\|z^{\rm ap}\|_\infty \le \rho$.

Step 3: Trajectory property of $(w, z^{\rm ap})$**.** To show that $(w,z^{\rm ap}) \in \mathcal{B}$, let $T \in \mathbb{N}$ be arbitrary and note that, by (4.3),

$$\int_0^T \|(\mathbf{S}_{\tau_\ell} w)(s) - w(s)\|{\rm d}s \le \frac{T}{\ell^2} \quad \forall\, \ell \in \mathbb{N}.$$

Hence, $\mathbf{S}_{\tau_\ell} w \to w$ in $L^1([0,T],\mathbb{R}^n)$ as $\ell \to \infty$. Furthermore, invoking (A1) and the fact that $\big(\mathbf{S}_{\tau_\ell} z\big)_\ell$ converges uniformly to $z^{\rm ap}$ on $\mathbb{R}_+$, we see that the sequence $\big(f(C\mathbf{S}_{\tau_\ell} z)\big)_\ell$ converges uniformly to $f(Cz^{\rm ap})$ on $\mathbb{R}_+$, and consequently, $f(C\mathbf{S}_{\tau_\ell} z) \to f(Cz^{\rm ap})$ in $L^1([0,T],\mathbb{R}^n)$ as $\ell \to \infty$. Therefore, since,

$$\begin{aligned}(\mathbf{S}_{\tau_\ell} z)(t) = (\mathbf{S}_{\tau_\ell} z)(0) + A\int_0^t (\mathbf{S}_{\tau_\ell} z)(s){\rm d}s + B\int_0^t f(C(\mathbf{S}_{\tau_\ell} z)(s)){\rm d}s \\ + \int_0^t (\mathbf{S}_{\tau_\ell} w)(s){\rm d}s \quad \forall\, \ell \in \mathbb{N},\ \forall\, t \in [0,T],\end{aligned}$$

it follows that in the limit, as $\ell \to \infty$,

$$z^{\rm ap}(t) = z^{\rm ap}(0) + A\int_0^t z^{\rm ap}(s){\rm d}s + B\int_0^t f(Cz^{\rm ap}(s)){\rm d}s + \int_0^t w(s){\rm d}s \quad \forall\, t \in [0,T].$$

As this holds for every $T \in \mathbb{N}$, we have that $z^{\rm ap} \in W^{1,1}_{\rm loc}(\mathbb{R}_+,\mathbb{R}^n)$ and

$$\dot z^{\rm ap}(t) = Az^{\rm ap}(t) + Bf(Cz^{\rm ap}(t)) + w(t) \quad \text{for a.e. } t \ge 0,$$

showing that $(w,z^{\rm ap}) \in \mathcal{B}$.

Step 4: Uniqueness of $z^{\rm ap}$ **within** $AP(\mathbb{R}_+,\mathbb{R}^n)$**.** Assume that $z^* \in AP(\mathbb{R}_+,\mathbb{R}^n)$ with $(w,z^*) \in \mathcal{B}$. To establish uniqueness, we have to show that $z^* = z^{\rm ap}$. But this follows easily: an application of statement (1) of Theorem 3.3 shows that $(z^*(t) - z^{\rm ap}(t)) \to 0$ as $t \to \infty$, and therefore, as $z^* - z^{\rm ap}$ is almost periodic, we conclude that $z^* = z^{\rm ap}$.

Step 5: Proof of statement (1). Let $(v,x) \in \mathcal{B}$ with $v - w \in L^1(\mathbb{R}_+,\mathbb{R}^n)$. As $\|z^{\rm ap}\|_{L^\infty} \le \rho$, it follows from (4.2) that

$$\|z^{\rm ap}(t) - x(t)\| \le \psi\big(\|z^{\rm ap}(t_0) - x(t_0)\|, t - t_0\big) \;\; + \varphi\Big(\int_{t_0}^t \|w(s) - v(s)\|{\rm d}s\Big) \;\; \forall\, t \ge t_0 \ge 0\,.$$

Let $\varepsilon > 0$ and choose $t_0, t_1 \ge 0$ such that

$$\varphi\Big(\int_{t_0}^\infty \|w(s) - v(s)\|{\rm d}s\Big) \le \frac{\varepsilon}{2} \quad \text{and} \quad \psi\big(\|z^{\rm ap}(t_0) - x(t_0)\|, t_1\big) \le \frac{\varepsilon}{2}\,,$$

where the existence of a suitable t_0 follows from the assumption that $v - w \in L^1(\mathbb{R}_+,\mathbb{R}^n)$. Then, $\|z^{\rm ap}(t) - x(t)\| \le \varepsilon$ for all $t \ge t_0 + t_1$, showing that $\|z^{\rm ap}(t) - x(t)\| \to 0$ as $t \to \infty$.

Step 6: Proof of statement (2). Assume that w is τ-periodic. Then, $\tau \in P_1(w,\delta)$ for every $\delta > 0$. Since, by Step 2, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $P_1(w,\delta) \subset P(z^{\rm ap},\varepsilon)$, it follows that $\tau \in P(z^{\rm ap},\varepsilon)$ for every $\varepsilon > 0$, showing that $z^{\rm ap}$ is τ-periodic.

Step 7: Proof of statement (3). We know from what has already been proved that, for each $k \in \mathbb{N}$, there exists $\delta_k \in (0,1/k)$ such that

$$P_1(w,\delta_k) \subset P(z^{\rm ap},1/k) \quad \forall\, k \in \mathbb{N}\,. \tag{4.7}$$

For the following, it is convenient to set

$$g(\xi) := A\xi + Bf(C\xi) \quad \forall\, \xi \in \mathbb{R}^n\,.$$

By assumption (A1) there exists $\kappa > 0$ such that $P(z^{\rm ap}, 1/k) \subset P(g\circ z^{\rm ap}, \kappa/k)$ for all $k \in \mathbb{N}$, and so,

$$P_1(w, \delta_k) \subset P(g\circ z^{\rm ap}, \kappa/k) \subset P_1(g\circ z^{\rm ap}, \kappa/k) \quad \forall\, k \in \mathbb{N}\,. \tag{4.8}$$

Let $a < 0$ be fixed, but arbitrary, let $\tau_k \in P_1(w, \delta_k) \cap [-a, \infty)$, and note that

$$z_{\rm e}^{\rm ap}(t+\tau_k) - z_{\rm e}^{\rm ap}(a+\tau_k) = z^{\rm ap}(t+\tau_k) - z^{\rm ap}(a+\tau_k) = \int_{a+\tau_k}^{t+\tau_k} \big(g(z^{\rm ap}(s)) + w(s)\big)\,{\rm d}s \quad \forall\, t \in [a, 0]\,.$$

Therefore,

$$\begin{aligned} z_{\rm e}^{\rm ap}(t+\tau_k) - z_{\rm e}^{\rm ap}(a+\tau_k) &= \int_a^t \big(g(z^{\rm ap}(s+\tau_k)) + w(s+\tau_k)\big){\rm d}s \\ &= \int_a^t \big(g(z_{\rm e}^{\rm ap}(s+\tau_k)) + w_{\rm e}(s+\tau_k)\big)\,{\rm d}s \;\; \forall\, t \in [a, 0]\,. \end{aligned} \tag{4.9}$$

Now, for all $k \in \mathbb{N}$,

$$P_1(w, \delta_k) \subset P_1(w_{\rm e}, \delta_k),\;\; P(z^{\rm ap}, 1/k) \subset P(z_{\rm e}^{\rm ap}, 1/k),\;\; P_1(g\circ z^{\rm ap}, \kappa/k) \subset P_1(g\circ z_{\rm e}^{\rm ap}, \kappa/k)\,,$$

and thus, by (4.7) and (4.8),

$$\tau_k \in P_1(w_{\rm e}, \delta_k) \cap P(z_{\rm e}^{\rm ap}, 1/k) \cap P_1(g\circ z_{\rm e}^{\rm ap}, \kappa/k) \quad \forall\, k \in \mathbb{N}\,.$$

Therefore, letting $k \to \infty$ in (4.9) yields

$$z_{\rm e}^{\rm ap}(t) - z_{\rm e}^{\rm ap}(a) = \int_a^t \big(g(z_{\rm e}^{\rm ap}(s)) + w_{\rm e}(s)\big){\rm d}s \quad \forall\, t \in [a, 0]\,.$$

Since $a < 0$ was arbitrary, we conclude that

$$\dot z_{\rm e}^{\rm ap}(t) = g(z_{\rm e}^{\rm ap}(t)) + w_{\rm e}(t) = Az_{\rm e}^{\rm ap}(t) + Bf(Cz_{\rm e}^{\rm ap}(t)) + w_{\rm e}(t) \quad \text{for a.e. } t \le 0,$$

establishing that $(w_{\rm e}, z_{\rm e}^{\rm ap}) \in \mathcal{BB}$.

To prove that $z_{\rm e}^{\rm ap}$ is the unique bounded function defined on $\mathbb{R}$ such that $(w_{\rm e}, z_{\rm e}^{\rm ap}) \in \mathcal{BB}$, let $x : \mathbb{R} \to \mathbb{R}^n$ be bounded and assume that $(w_{\rm e}, x) \in \mathcal{BB}$. We will show that $x = z_{\rm e}^{\rm ap}$. Obviously, by (3.2), for any $\tau \in \mathbb{R}$, the restrictions of the pairs $(\mathbf{S}_\tau w_{\rm e}, \mathbf{S}_\tau z_{\rm e}^{\rm ap})$ and $(\mathbf{S}_\tau w_{\rm e}, \mathbf{S}_\tau x)$ to $\mathbb{R}_+$ are in $\mathcal{B}$. Consequently, invoking statement (1) of Theorem 3.3, there exists $\psi \in \mathcal{KL}$ such that

$$\|(\mathbf{S}_\tau x)(s) - (\mathbf{S}_\tau z_{\rm e}^{\rm ap})(s)\| \le \psi\big((\mathbf{S}_\tau x)(0) - (\mathbf{S}_\tau z_{\rm e}^{\rm ap})(0), s\big) \quad \forall\, s \ge 0,\; \forall\, \tau \in \mathbb{R}.$$

Now let $t \in \mathbb{R}$ and $\varepsilon > 0$. Choosing $\tau \le t$ such that $\psi\big(\|z_{\rm e}^{\rm ap}\|_{L^\infty} + \|x\|_{L^\infty}, t-\tau\big) \le \varepsilon$ and applying the above inequality with $s = t - \tau$ leads to

$$\|x(t) - z_{\rm e}^{\rm ap}(t)\| = \|(\mathbf{S}_\tau x)(t-\tau) - (\mathbf{S}_\tau z_{\rm e}^{\rm ap})(t-\tau)\| \le \psi\big(x(\tau) - z_{\rm e}^{\rm ap}(\tau), t-\tau\big)\,,$$

whence,

$$\|x(t) - z_{\rm e}^{\rm ap}(t)\| \le \psi\big(\|z_{\rm e}^{\rm ap}\|_{L^\infty} + \|x\|_{L^\infty}, t-\tau\big) \le \varepsilon\,.$$

As $\varepsilon > 0$ was arbitrary, it follows that $x(t) = z_{\rm e}^{\rm ap}(t)$. Finally, since $t \in \mathbb{R}$ was arbitrary, we obtain that $x = z_{\rm e}^{\rm ap}$, completing the proof.

Step 8: Proof of statement (4). For every $\tau \in \mathbb{R}$, the trajectory $(\mathbf{S}_\tau w_\mathrm{e}, \mathbf{S}_\tau z_\mathrm{e}^{\mathrm{ap}})$ is in $\mathcal{BB}$, and hence $(\mathbf{S}_\tau w_\mathrm{e}, \mathbf{S}_\tau z_\mathrm{e}^{\mathrm{ap}})|_{\mathbb{R}_+} \in \mathcal{B}$. As $\|z_\mathrm{e}^{\mathrm{ap}}\|_{L^\infty} = \|z^{\mathrm{ap}}\|_{L^\infty} \le \rho$, it is an immediate consequence of (4.2) that

$$\|(\mathbf{S}_\tau z_\mathrm{e}^{\mathrm{ap}})(t) - (\mathbf{S}_\sigma z_\mathrm{e}^{\mathrm{ap}})(t)\| \le \psi\big(\|(\mathbf{S}_\tau z_\mathrm{e}^{\mathrm{ap}})(t_0) - (\mathbf{S}_\sigma z_\mathrm{e}^{\mathrm{ap}})(t_0)\|, t - t_0\big) + \varphi\Big(\int_{t_0}^t \|(\mathbf{S}_\tau w_\mathrm{e})(s) - (\mathbf{S}_\sigma w_\mathrm{e})(s)\|\mathrm{d}s\Big) \quad \forall\, t \ge t_0 \ge 0,\ \forall\, \tau, \sigma \in \mathbb{R}. \tag{4.10}$$

Let $(\tau_k)_k$ be a sequence in $\mathbb{R}$ such that $(\mathbf{S}_{\tau_k} \tilde{w}_\mathrm{e})_k$ converges in $AP(\mathbb{R}, L^1([0,1], \mathbb{R}^n))$. By [1, Statement X on p. 34], it is sufficient to prove that the sequence $(\mathbf{S}_{\tau_k} z_\mathrm{e}^{\mathrm{ap}})_k$ converges in $AP(\mathbb{R}, \mathbb{R}^n)$, or, equivalently, that $(\mathbf{S}_{\tau_k} z_\mathrm{e}^{\mathrm{ap}})_k$ is a Cauchy sequence in $AP(\mathbb{R}, \mathbb{R}^n)$. To this end, let $\varepsilon > 0$, set $\rho := \|z_\mathrm{e}^{\mathrm{ap}}\|_{L^\infty}$ and choose $T \in \mathbb{N}$ and $\delta > 0$ such that $\psi(2\rho, T) \le \varepsilon/2$ and $\varphi(\delta) \le \varepsilon/2$, in which case

$$\psi(2\rho, s) \le \frac{\varepsilon}{2} \quad \forall\, s \ge T \qquad \text{and} \qquad \varphi(s) \le \frac{\varepsilon}{2} \quad \forall\, s \in [0, \delta].$$

Since $(\mathbf{S}_{\tau_k} \tilde{w}_\mathrm{e})_k$ converges in $AP(\mathbb{R}, L^1([0,1], \mathbb{R}^n))$, it follows that $(\mathbf{S}_{\tau_k} w_\mathrm{e})_k$ converges in $S^1(\mathbb{R}, \mathbb{R}^n)$, and hence is a Cauchy sequence in $S^1(\mathbb{R}, \mathbb{R}^n)$. Consequently, there exists $N \in \mathbb{N}$ such that

$$\|\mathbf{S}_{\tau_k} w_\mathrm{e} - \mathbf{S}_{\tau_\ell} w_\mathrm{e}\|_\mathrm{S} \le \frac{\delta}{2T} \quad \forall\, k, \ell \ge N,$$

and thus,

$$\begin{aligned}\int_a^{a+2T} \|(\mathbf{S}_{\tau_k} w_\mathrm{e})(s) - (\mathbf{S}_{\tau_\ell} w_\mathrm{e})(s)\|\mathrm{d}s &= \sum_{j=0}^{2T-1} \int_{a+j}^{a+j+1} \|(\mathbf{S}_{\tau_k} w_\mathrm{e})(s) - (\mathbf{S}_{\tau_\ell} w_\mathrm{e})(s)\|\mathrm{d}s \\ &\le 2T \|\mathbf{S}_{\tau_k} w_\mathrm{e} - \mathbf{S}_{\tau_\ell} w_\mathrm{e}\|_\mathrm{S} \\ &\le \delta \quad \forall\, a \in \mathbb{R},\ \forall\, k, \ell \ge N.\end{aligned} \tag{4.11}$$

For $t \ge 0$, let p_t denote the unique integer such that $t \in [p_t T, (p_t + 1)T)$. By (4.10),

$$\|(\mathbf{S}_{\tau_k} z_\mathrm{e}^{\mathrm{ap}})(t) - (\mathbf{S}_{\tau_\ell} z_\mathrm{e}^{\mathrm{ap}})(t)\| \le \psi(2\rho, t - (p_t - 1)T) + \varphi\Big(\int_{(p_t-1)T}^t \|(\mathbf{S}_{\tau_k} w_\mathrm{e})(s) - (\mathbf{S}_{\tau_\ell} w_\mathrm{e})(s)\|\mathrm{d}s\Big) \quad \forall\, t \ge T. \tag{4.12}$$

Appealing to (4.11), we obtain

$$\begin{aligned}\int_{(p_t-1)T}^t \|(\mathbf{S}_{\tau_k} w_\mathrm{e})(s) - (\mathbf{S}_{\tau_\ell} w_\mathrm{e})(s)\|\mathrm{d}s &\le \int_{(p_t-1)T}^{(p_t-1)T+2T} \|(\mathbf{S}_{\tau_k} w_\mathrm{e})(s) - (\mathbf{S}_{\tau_\ell} w_\mathrm{e})(s)\|\mathrm{d}s \\ &\le \delta \quad \forall\, t \ge T,\ \forall\, k, \ell \ge N.\end{aligned}$$

Since, for any $t \ge T$, we have $(p_t - 1)T \ge 0$ and $t - (p_t - 1)T \ge T$, it follows from (4.12) that

$$\|(\mathbf{S}_{\tau_k} z_\mathrm{e}^{\mathrm{ap}})(t) - (\mathbf{S}_{\tau_\ell} z_\mathrm{e}^{\mathrm{ap}})(t)\| \le \frac{\varepsilon}{2} + \varphi(\delta) \le \varepsilon \quad \forall\, t \ge T,\ \forall\, k, \ell \ge N.$$

Consequently, by almost periodicity of the function $\mathbf{S}_{\tau_k} z_\mathrm{e}^{\mathrm{ap}} - \mathbf{S}_{\tau_\ell} z_\mathrm{e}^{\mathrm{ap}}$, we may use Lemma 4.1 to conclude

$$\|\mathbf{S}_{\tau_k} z_\mathrm{e}^{\mathrm{ap}} - \mathbf{S}_{\tau_\ell} z_\mathrm{e}^{\mathrm{ap}}\|_{L^\infty} = \sup_{t \ge T} \|(\mathbf{S}_{\tau_k} z_\mathrm{e}^{\mathrm{ap}})(t) - (\mathbf{S}_{\tau_\ell} z_\mathrm{e}^{\mathrm{ap}})(t)\| \le \varepsilon \quad \forall\, k, \ell \ge N,$$

showing that $(\mathbf{S}_{\tau_k} z_\mathrm{e}^{\mathrm{ap}})_k$ is a Cauchy sequence in $AP(\mathbb{R}, \mathbb{R}^n)$ and completing the proof. □

In the light of the above proof and Corollary 3.5, it is clear that the following circle-criterion version of Theorem 4.3 holds.

Corollary 4.4. *Let $K_1, K_2 \in \mathbb{R}^{m\times p}$. Assume that f in* (3.1) *does not depend on t and (A, B, C) is controllable and detectable or, alternatively, stabilizable and observable. Let $w \in S^1(\mathbb{R}_+, \mathbb{R}^n)$ and assume that there exists a trajectory $(w, z) \in \mathcal{B}$ with bounded z. If $(I - K_2\mathbf{G})(I - K_1\mathbf{G})^{-1}$ is positive real and*

$$\langle f(y+\xi) - f(\xi) - K_1 y, f(y+\xi) - f(\xi) - K_2 y\rangle < 0 \quad \forall\, y, \xi \in \mathbb{R}^p,\; y \neq 0\,,$$

then the conclusions of Theorem 4.3 *hold.*

Before presenting the final result of this section, we pause to compare Theorem 4.3 to related results in the literature. The most relevant results in this context are [13, Theorem 3.2.9], [15, Theorem 4.3], [16, Theorem 4.5], [32, Theorem 2] and [41, Theorem 1]. The papers [32, 41] are restricted to scalar nonlinearities, that is, $m = p = 1$) and in [15, 32, 41] the forcing functions are assumed to be almost periodic in the sense of Bohr. A Lyapunov approach is used in [13, 15, 41], whilst the analyses in [16, 32] are based on input-output methods. The paper [16] considers a large class of infinite-dimensional continuous-time systems with the underlying linear system being well-posed in the sense of [36, 38], whilst [15] considers finite-dimensional discrete-time systems. An inspection of the assumptions on the nonlinearity f imposed in the relevant results in [13, 15, 16, 32, 41] shows that, in each case, they are equivalent to the existence, for each ξ, of a function $\gamma_\xi \in \mathcal{K}_\infty$ such that

$$\|f(y+\xi) - f(\xi) - Ky\| \le r^K\|y\| - \gamma_\xi(\|y\|) \quad \forall\, y \in \mathbb{R}^p, \tag{4.13}$$

where in [16, 32, 41] it is assumed that there exists $\varepsilon > 0$ such that $\gamma_\xi(s) = \varepsilon s$ for every $\xi \in \mathbb{R}^p$. We emphasize that condition (4.13), with $\gamma_\xi \in \mathcal{K}_\infty$ for every ξ, is considerably stronger than (A1) (in fact, it is equivalent to (A1) and (A2) holding simultaneously). Furthermore, we note that if there exists $\varepsilon > 0$ such that (4.13) is satisfied with $\gamma_\xi(s) = \varepsilon s$ for every $\xi \in \mathbb{R}^p$, then the Lur'e system enjoys a much stronger stability property, namely global exponential incremental ISS [16, 19], in which case the convergence in statement (1) of Theorem 4.3 is exponentially fast, see [16].

If, in the assumptions of Theorem 4.3, (A3) is replaced by (A2) and w is not only in $S^1(\mathbb{R}_+, \mathbb{R}^n)$, but also essentially bounded, then the conclusions can be strengthened, as the following the result shows.

Theorem 4.5. *Assume that f in* (3.1) *does not depend on t,* (A1) *and* (A2) *hold, and let $w \in S^1(\mathbb{R}_+, \mathbb{R}^n) \cap L^\infty(\mathbb{R}_+, \mathbb{R}^n)$. Then there exists a unique $z^{\rm ap} \in AP(\mathbb{R}_+, \mathbb{R}^n)$ such that $(w, z^{\rm ap}) \in \mathcal{B}$, and, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $P_1(w, \delta) \subset P(z^{\rm ap}, \varepsilon)$. Furthermore, the following assertions are true.*

(1) *Statements* (1)–(4) *of* Theorem 4.3 *hold.*

(2) *For every $(v, x) \in \mathcal{B}$ with $v \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ and such that $\operatorname{ess\,sup}_{s\ge t} \|v(s) - w(s)\| \to 0$ as $t \to \infty$*

$$\lim_{t\to\infty} \big(x(t) - z^{\rm ap}(t)\big) = 0\,.$$

(3) *There exists $\theta \in \mathcal{K}$ such that, for every pair $(v, x^{\rm ap}) \in \mathcal{B}$ with $v \in S^1(\mathbb{R}_+, \mathbb{R}^n) \cap L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ and $x^{\rm ap} \in AP(\mathbb{R}_+, \mathbb{R}^n)$,*

$$\|x^{\rm ap} - z^{\rm ap}\|_{L^\infty} \le \theta(\|v - w\|_{L^\infty})\,. \tag{4.14}$$

Commenting on statement (3), we note that the earlier part of Theorem 4.5 (preceding statement (1)) guarantees that, for every $v \in S^1(\mathbb{R}_+, \mathbb{R}^n) \cap L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, there exists a unique $x^{\rm ap} \in AP(\mathbb{R}_+, \mathbb{R}^n)$ such that $(v, x^{\rm ap}) \in \mathcal{B}$.

Proof of Theorem 4.5. Theorem 3.7 guarantees that, for every $(w, z) \in \mathcal{B}$, the state z is bounded. Invoking now Theorem 3.7 instead of Theorem 3.3, statement (1) can be proved by arguments identical to those used in the proof of Theorem 4.3. Statement (2) can be established by an argument similar to that used in Step 5 of the proof of Theorem 4.3 by appealing to (3.13) instead of (3.5). To prove statement (3), let $\varepsilon > 0$ and note, that by Theorem 3.7, there exist $\theta \in \mathcal{K}$ and such that, for every pair $(v, x^{\mathrm{ap}}) \in \mathcal{B}$ with $v \in S^1(\mathbb{R}_+, \mathbb{R}^n) \cap L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ and $x^{\mathrm{ap}} \in AP(\mathbb{R}_+, \mathbb{R}^n)$,

$$\limsup_{t \to \infty} \|x^{\mathrm{ap}}(t) - z^{\mathrm{ap}}(t)\| \leq \varepsilon + \theta(\|v - w\|_{L^\infty}).$$

By Lemma 4.1 it follows that, for every pair $(v, x^{\mathrm{ap}}) \in \mathcal{B}$ with $v \in S^1(\mathbb{R}_+, \mathbb{R}^n) \cap L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ and $x^{\mathrm{ap}} \in AP(\mathbb{R}_+, \mathbb{R}^n)$,

$$\|x^{\mathrm{ap}} - z^{\mathrm{ap}}\|_{L^\infty} \leq \varepsilon + \theta(\|v - w\|_{L^\infty}),$$

establishing (4.14) as $\varepsilon > 0$ was arbitrary. □

5 An example

As an illustrative example, we consider the system of forced nonlinear differential equations modelling a linked sequence of chemical reactions, namely

$$\left.\begin{aligned} \dot{z}_1 &= -a_1 z_1 + g(z_3) + v_1, \quad z_1(0) = z_1^0, \\ \dot{z}_2 &= z_1 - a_2 z_2 + v_2, \quad z_2(0) = z_2^0, \\ \dot{z}_3 &= z_2 - a_3 z_3 + v_3, \quad z_3(0) = z_3^0. \end{aligned}\right\} \tag{5.1}$$

Here z_i denotes the concentration of the i-th reagent, a_i are positive constants, v_i are forcing terms, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonlinear function modelling activation or inhibition of reagent z_1 by z_3. These equations were considered in [33, Example 3.5], and are inspired by the metabolic control mechanisms from [30, Section 7.2]. We refer the reader to these reference for more details on the interpretation of the model.

To express (5.1) as a forced Lur'e differential equation (3.1), we first define the linear data

$$A := \begin{pmatrix} -a_1 & 0 & 0 \\ 1 & -a_2 & 0 \\ 0 & 1 & -a_3 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad C := \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}. \tag{5.2}$$

Evidently, the matrix A is Metzler and Hurwitz. Furthermore, (A, B, C) is controllable and observable, so that (A3) holds, and

$$\|\mathbf{G}\|_{H^\infty} = \mathbf{G}(0) = \frac{1}{a_1 a_2 a_3} > 0.$$

The paper [33] considers the situation wherein g models inhibition of z_3 on z_1. Here we assume that z_3 activates z_1, and assume that g is locally Lipschitz with $g(0) > 0$ and is strictly increasing. Consequently, $(0, 0)$ is not an equilibrium pair of (5.1), but we assume that there exists a unique positive solution $y^+ > 0$ of the equation

$$\mathbf{G}(0) g(y) = y, \tag{5.3}$$

in which case $y^+ = Cz^+$, where $z^+ := -A^{-1} B g(y^+)$ is the unique vector in $\mathbb{R}^3$ such that $(z^+, 0)$ is an equilibrium pair of (5.1).

To express the system (5.1) as a forced Lur'e differential equation of the form (3.1), we write $z := (z_1, z_2, z_3)^*$, $z^0 := (z_1^0, z_2^0, z_3^0)^*$, $v := (v_1, v_2, v_3)^*$ and set

$$x := z - z^+, \quad x^0 := z^0 - z^+, \quad f(y) := \begin{cases} g(y + y^+) - g(y^+), & y \geq -y^+ \\ g(0) - g(y^+), & y < -y^+. \end{cases} \tag{5.4}$$

Since A is Metzler and B, C and g are non-negative, it follows that z^+ is non-negative, and, for all $z^0 \in \mathbb{R}^3_+$ and $v \in L^1_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^3_+)$, the forward solution $z(\cdot\,; z^0, v)$ of (5.1) is also non-negative. Consequently, the corresponding solution $x(\cdot\,; x^0, v)$ of (3.1) satisfies

$$x(t; x^0, v) = z(t; z^0, v) - z^+ \geq -z^+ \quad \forall\, t \geq 0\,,$$

and, therefore, $Cx(t; x^0, v) \geq -Cz^+ = -y^+$ for all $t \geq 0$. The definition of $f(y)$ for $y < -y^+$ is to fit the model (5.1) into the framework of Section 3 and, although artefactual, the extension is not seen in physically motivated situations.

In the following, we will provide illustrations of Theorems 3.3 and 4.3. As A is Hurwitz, we may take $K = 0$ and set

$$r := r^0 = \frac{1}{\|\mathbf{G}\|_{H^\infty}} = \frac{1}{\mathbf{G}(0)} = a_1 a_2 a_3.$$

Specifically, we consider the situation wherein g is a so-called smoothed rectifier function, given by

$$g(y) = \frac{1}{k} \ln\big(1 + e^{kry}\big) - b \quad \forall\, y \geq 0\,, \tag{5.5}$$

where $b, k > 0$ and $b < (\ln 2)/k$. Since $\ln\big(1 + e^{kry}\big) = kry + \ln\big(1 + e^{-kry}\big)$, we have that

$$ry - g(y) \to b \quad \text{as } y \to \infty. \tag{5.6}$$

If $g(z_3)$ in (5.1) is replaced by $rz_3 - b$, then, as 0 is an eigenvalue of $A + rBC$, there exist non-negative forcing functions v of arbitrarily small positive L^∞-norm which generate unbounded solutions. However, (5.1) has better stability properties: indeed, it will be shown below that, for all non-negative initial conditions and all forcing functions v of sufficiently small L^∞-norm, the solutions of (5.1) are bounded.

Under the stated assumptions, it is routine to verify that

$$y^+ := -\frac{\mathbf{G}(0)}{k} \ln(e^{kb} - 1) = \frac{\mathbf{G}(0)}{k} \ln\Big(\frac{1}{e^{kb} - 1}\Big) > 0\,, \tag{5.7}$$

satisfies (5.3). As $K = 0$, the nonlinearity f, as defined in (5.4), satisfies assumption (A1) if, and only, if

$$|f(y + \xi) - f(\xi)| < r|y| \quad \forall\, y \neq 0,\ \forall\, \xi \in \mathbb{R}\,, \tag{5.8}$$

To establish (5.8) note that

$$f'(y) = \frac{r}{1 + e^{-kr(y + y^+)}} < r \quad \forall\, y > -y^+ \quad \text{and} \quad f'(y) = 0 \quad \forall\, y < -y^+,$$

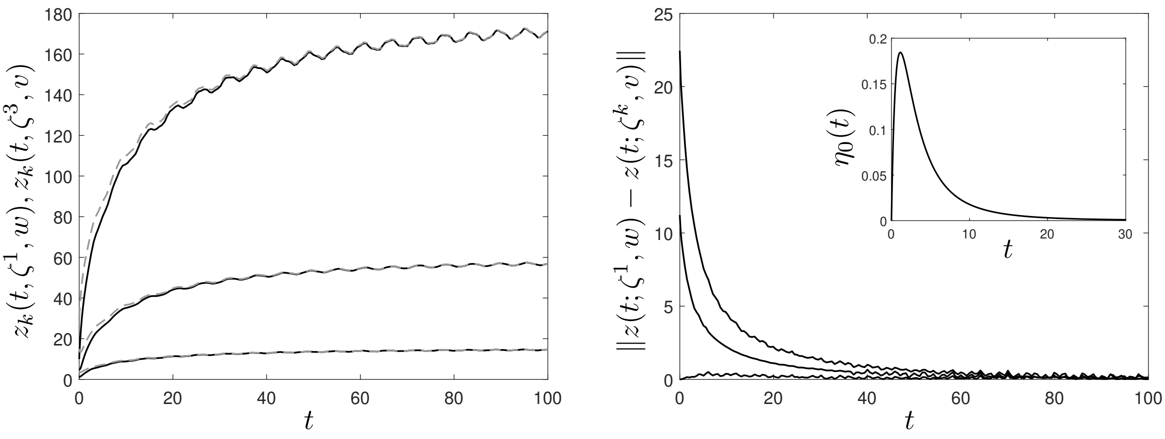
and so $|f'(y)| < r$ for all $y \neq -y^+$. An application of the mean-value theorem then shows that $|f(y + \xi) - f(\xi)| < r|y|$ for all $y \neq 0$ and all ξ, establishing (5.8) and hence (A1). Furthermore, using (5.6) and the fact that f is non-decreasing, it is a routine exercise to show that, for all $\xi \in \mathbb{R}$,

$$r|y| - |f(y + \xi) - f(\xi)| = ry - \big(f(y + \xi) - f(\xi)\big) \to b + c \quad \text{as } y \to \infty,$$

where $c = g(\xi + y^+) - g(y^+) - r\xi$ if $\xi \geq -y^+$ and $c = g(y^+) - g(0) - r\xi$ if $\xi < -y^+$. Consequently, (A2) does not hold, and so, there does not exist $\xi \in \mathbb{R}$ and $\gamma_\xi \in \mathcal{K}_\infty$ such that (4.13) holds (the latter is a key assumption in [33]).

Lemma 3.6 guarantees that the solutions of (5.1) generated by sufficiently small forcing functions are bounded provided that condition (3.11) holds with $y^\dagger = 0$. To show this, we use the fact that g is increasing and (5.6) to obtain,

$$r|y| - |f(y)| = ry - g(y + y^+) + g(y^+) = r(y + y^+) - g(y + y^+) \to b > 0 \quad \text{as } y \to \infty.$$



(a) (b)

Figure 5.1: (a) Components of $z(t,\zeta^1,w)$ (black solid lines) and $z(t,\zeta^3,v)$ (grey dashed lines). (b) Norms of differences of solutions of (5.1) plotted against time t. The inset shows the graph of η_0.

The function f is constant for $y < -y^+$, and thus, $r|y| - |f(y)| \to \infty$ as $y \to -\infty$. As $f(0) = 0$, we may now conclude that (3.11) is satisfied with $y^\dagger = 0$ and $K = 0$ (recall that $r = r^0$).

For a numerical simulation, we take

$$a_1 = 2, \quad a_2 = 3, \quad a_3 = 4, \quad k = 0.01 \quad \text{and} \quad b = 10\,. \tag{5.9}$$

With these choices, we compute

$$y^+ \approx 9.384 \quad \text{and} \quad z^+ = -A^{-1}Bg(y^+) \approx \begin{pmatrix} 113.0 \\ 37.5 \\ 9.38 \end{pmatrix}.$$

To define the forcing functions we are going to consider, it is convenient to set

$$B_{\mathrm{e}} := 0.4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \omega_1 := \frac{1}{2}, \quad \omega_2 := \frac{\sqrt{3}}{10} \quad \text{and} \quad \eta_0(t) := \frac{t}{2\big(1+\frac{1}{4}t\big)^{\frac{9}{2}}} \quad \forall\, t \geq 0\,,$$

and, furthermore, for $t \geq 0$,

$$\eta^{\mathrm{ap}}(t) := 1 + \sin(2\pi\omega_1 t) + 2\big(1 + \sin\big(2\pi\omega_2 t\big)\big), \qquad \eta^{\mathrm{aap}} := \eta^{\mathrm{ap}} + \eta_0\,,$$

$$\eta^{\mathrm{s}}(t) := 2 + \sin\big(\mathrm{mod}(t, 3\pi/2)\big) + \sin\big(\sqrt{2}\,\mathrm{mod}(t, 3\pi/(2\sqrt{2}))\big), \qquad \eta^{\mathrm{as}} := \eta^{\mathrm{s}} + 4\eta_0\,.$$

Here, for $t \geq 0$ and $\tau > 0$,

$$\mathrm{mod}(t,\tau) := t - k\tau \in [0,\tau), \quad \text{where } k \text{ is the largest integer in } \mathbb{Z}_+ \text{ such that } t \geq k\tau.$$

The functions η^{ap}, η^{aap}, η^{s}, η^{as} are, respectively, almost periodic (in the sense of Bohr), asymptotically almost periodic, Stepanov almost periodic, and asymptotically Stepanov almost periodic. We will consider the forcing functions

$$w := B_{\mathrm{e}}\eta^{\mathrm{ap}}, \quad v := B_{\mathrm{e}}\eta^{\mathrm{aap}} \quad \tilde{w} := B_{\mathrm{e}}\eta^{\mathrm{s}} \quad \text{and} \quad \tilde{v} := B_{\mathrm{e}}\eta^{\mathrm{as}},$$

and use the following three initial conditions

$$z^0 = \zeta^k := k \begin{pmatrix} 10 \\ 5 \\ 1 \end{pmatrix} \qquad k = 1, 2, 3\,.$$

Note that the functions $w - v$ and $\tilde{w} - \tilde{v} = 4(w - v)$ are in $L^1(\mathbb{R}_+, \mathbb{R}^3)$.

Figures 5.1 and 5.2 show numerical simulations of (3.1) with model data (5.2), (5.4), (5.5) and (5.9). All simulations were performed in Mathworks MATLAB 2020a using the command `ode45` to numerically solve the differential equations. Figure 5.1(a) plots the three components of the solution $z(t; \zeta^1, w)$ of (5.1) against t (black solid lines). In particular, we see that this solution is bounded and, therefore, Theorem 3.3, Corollary 3.4 and Theorem 4.3 can be applied. Further, Figure 5.1(a) seems to show that $z(t; \zeta^1, w) - z^{\rm ap}(t) \to 0$ as $t \to \infty$, where $z^{\rm ap}$ is an almost periodic function, as predicted by Theorem 4.3. Figure 5.1(b) displays plots of $\|z(t; \zeta^1, w) - z(t; \zeta^k, v)\|$ against t for $k = 1, 2, 3$. In each case, we see that the norms of the differences increase as $\|\zeta^1 - \zeta^k\|$ increases, but, in each case, they decrease to zero over time, as predicted by Theorem 3.3 and Corollary 3.4. The inset in Figure 5.1(b) shows the graph of η_0. Moreover, by statement (1) of Theorem 4.3, $\big(z(t; \zeta^3, v) - z^{\rm ap}(t)\big) \to 0$ as $t \to \infty$ which is observed by eye in Figure 5.1(a) (grey dashed lines).

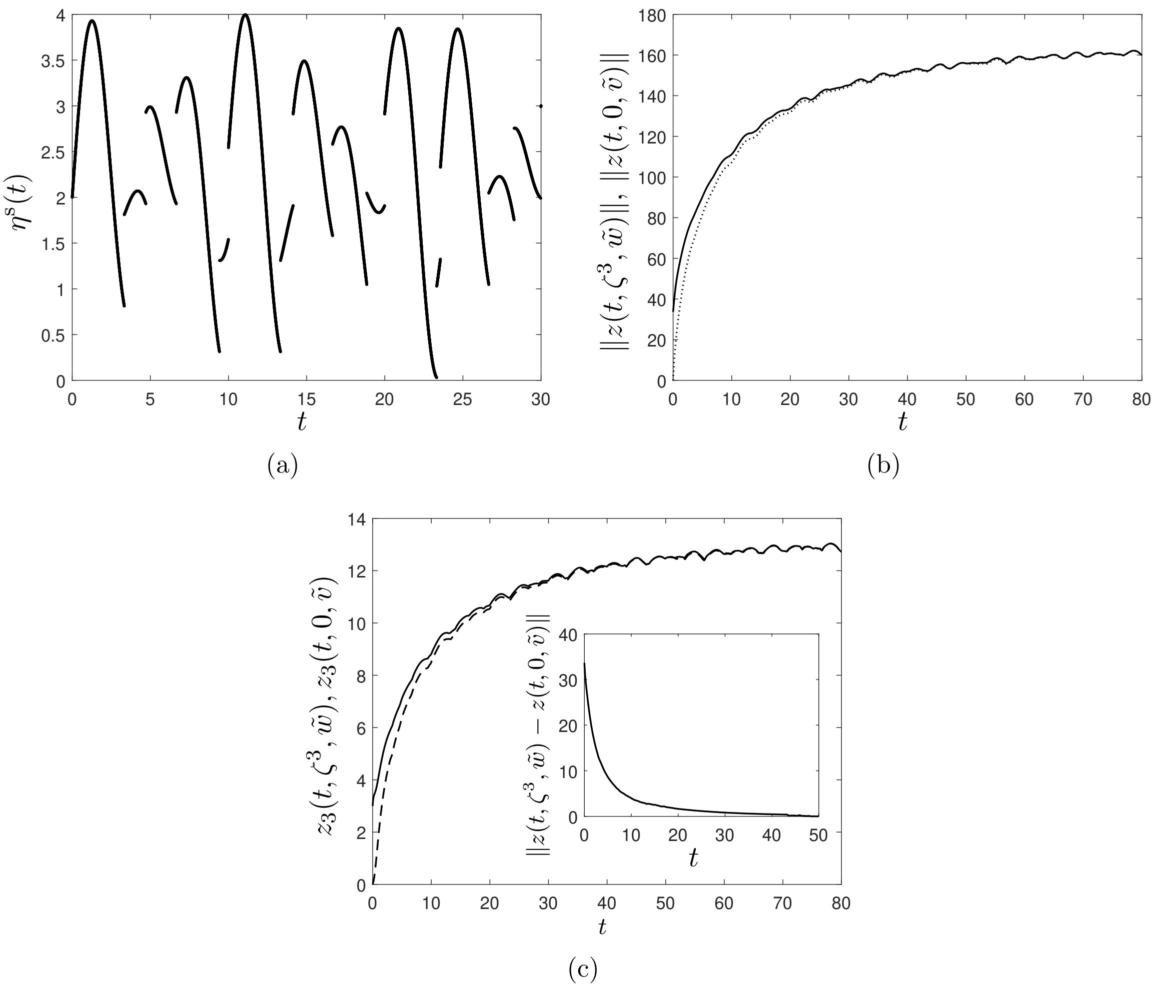


Figure 5.2: (a) Graph of $\eta^{\rm s}$. (b) Norms of $z(\cdot\,; \zeta^3, \tilde{w})$ (solid line) and $z(\cdot\,; 0, \tilde{v})$ (dotted-line). (c) Third components of $z(\cdot\,; \zeta^3, \tilde{w})$ (solid line) and $z(\cdot\,; 0, \tilde{v})$ (dashed line). The inset shows the graph of $\|z(\cdot\,; \zeta^3, \tilde{w}) - z(\cdot\,; 0, \tilde{v})\|$.

Figure 5.2(a) shows the graph of the Stepanov almost periodic function η^{s}. Figure 5.2(b) plots the norms $\|z(t;\zeta^3,\tilde{w})\|$ and $\|z(t;0,\tilde{v})\|$, both against t, in solid and dotted lines, respectively. We see that $\|z(t;\zeta^3,\tilde{w})\|$ appears bounded, and so Corollary 3.4 and Theorem 4.3 are applicable. Figure 5.2(c) plots the third components of $z(t;\zeta^3,\tilde{w})$ and $z(t;0,\tilde{v})$ against t in solid and dashed lines, respectively. We see by eye convergence of the components to one another, and they appear to be asymptotically almost periodic, thereby illustrating Theorem 4.3. We have considered only one (the third) component as the asymptotic oscillations are small compared to the absolute size of each component, and the fluctuations are obscured when all three components are plotted in the same co-ordinate system. The inset in Figure 5.2(c) plots the norm of the difference $z(t;\zeta^3,\tilde{w}) - z(t;0,\tilde{v})$ against t, which is seen to converge to zero as t increases as predicted by Corollary 3.4.

6 Appendix

In this Appendix, we provide a proofs of Lemmas 2.2, 3.1 and 3.2.

Proof of Lemma 2.2. Define a function $g:\mathbb{R}^p\to\mathbb{R}_+$ by setting $g(y):=r\|y\|-|F(y)-Ky|_{\mathrm{m}}$ for all $y\in\mathbb{R}^p$. By (2.3), $g(y)>0$ for all $y\neq 0$. As F is an upper-semicontinuous set-valued map, the function g is lower semicontinuous. Next, let us define $h:\mathbb{R}^p\to\mathbb{R}_+$ by

$$h(y):=\inf_{z\in\mathbb{R}^p}\big(g(z)+\|z-y\|\big)\quad\forall\, y\in\mathbb{R}^p\,.$$

Obviously, $h(y)\le g(y)$ for all $y\in\mathbb{R}^p$, and, invoking the lower semi-continuity and positivity of g, we see that $h(y)>0$ for all $y\neq 0$. Furthermore, as $g(z)+\|z-y\|\le g(z)+\|z-x\|+\|x-y\|$ for all $x,y,z\in\mathbb{R}^p$, we obtain

$$h(y)-h(x)\le\|x-y\|\quad\forall\, x,y\in\mathbb{R}^p\,.$$

Interchanging the roles of x and y, we may conclude that

$$|h(y)-h(x)|\le\|x-y\|\quad\forall\, x,y\in\mathbb{R}^p\,,$$

showing that h is globally Lipschitz. Consequently, the function $\beta:\mathbb{R}_+\to\mathbb{R}_+$ given by

$$\beta(s):=\inf_{\|y\|=s}h(y)\quad\forall\, s\ge 0\,,$$

is continuous. Moreover, $\beta(0)=0$ and $\beta(s)>0$ for $s>0$, whence $\beta\in\mathcal{P}$. As

$$\beta(\|y\|)\le h(y)\le g(y)\le r\|y\|-|F(y)-Ky|_{\mathrm{m}}\quad\forall\, y\in\mathbb{R}^p,$$

it follows that

$$|F(y)-Ky|_{\mathrm{m}}\le r\|y\|-\beta(\|y\|)\quad\forall\, y\in\mathbb{R}^p.$$

Furthermore, if the divergence condition (2.5) holds, then $g(y)\to\infty$ as $\|y\|\to\infty$, and so $h(y)\to\infty$ as $\|y\|\to\infty$, which in turn implies that $\lim_{s\to\infty}\beta(s)=\infty$. Consequently, the function $\tilde{\gamma}:\mathbb{R}_+\to\mathbb{R}_+$ given by $\tilde{\gamma}(s)=(1-e^{-s})\inf_{t\ge s}\beta(t)$ is in $\mathcal{K}_\infty$ and satisfies $\tilde{\gamma}(s)\le\beta(s)$ for all $s\ge 0$. It follows now from [25, Lemma 1] that there exists a continuously differentiable function $\gamma\in\mathcal{K}_\infty$ such that $\gamma(s)\le\tilde{\gamma}(s)\le\beta(s)$ for all $s\ge 0$.

In the absence of the divergence condition (2.5), we invoke [25, Lemma 18] which guarantees that there exist $\beta_1\in\mathcal{K}_\infty$ and $\beta_2\in\mathcal{L}$ such that $\beta(s)\ge\beta_1(s)\beta_2(s)$ for all $s\ge 0$. The existence of a continuously differentiable function $\gamma\in\mathcal{P}$ such that $\gamma(s)\le\beta(s)$ for all $s\ge 0$ follows now from Lemmas 6.1 and 6.2 below. □

The following lemmas are not surprising. For completeness we provide proofs which can be found at the end of the Appendix.

Lemma 6.1. *For each $\varphi \in \mathcal{K}$, there exists a continuously differentiable $\theta \in \mathcal{K}$ such that $\theta(s) \le \varphi(s)$ for all $s \ge 0$ and $\theta'(s) = \mathrm{O}(1/s^3)$ as $s \to \infty$.*

Lemma 6.2. *For each $\varphi \in \mathcal{L}$, there exists a continuously differentiable $\theta \in \mathcal{L}$ such that $\theta(s) \le \varphi(s)$ for all $s \ge 0$ and $\theta'(0) = 0$.*

Proof of Lemma 3.1. Defining $g : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ by

$$g(t, y) := f(t, y) - Ky \quad \forall\, t \ge 0,\ \forall\, y \in \mathbb{R}^p, \tag{6.1}$$

the claim can be written in the form

$$\lim_{\|y\|\to\infty} \big(r^K \|y\| - \sup_{t\ge 0} \|g(t, y + \xi) - g(t, \xi)\|\big) = \infty \quad \forall\, \xi \in \mathbb{R}^p. \tag{6.2}$$

To establish (6.2), let $\xi \in \mathbb{R}^p$ be fixed but arbitrary, set $\zeta := y + \xi - \xi_0$, where $y \in \mathbb{R}^p$, and note that,

$$r^K\|y\| - \|g(t, y+\xi) - g(t, \xi)\| \ge r^K(\|\zeta\| - \|\xi_0 - \xi\|) - \|g(t, \zeta+\xi_0) - g(t, \xi_0)\| - \|g(t, \xi_0) - g(t, \xi)\| \quad \forall t \ge 0.$$

By (A1), it follows that $\|g(t, \xi_0) - g(t, \xi)\| \le r^K\|\xi - \xi_0\|$ for all $t \ge 0$, and thus,

$$r^K\|y\| - \|g(t, y + \xi) - g(t, \xi)\| \ge r^K(\|\zeta\| - 2\|\xi - \xi_0\|) - \sup_{\tau\ge 0} \|g(\tau, \zeta + \xi_0) - g(\tau, \xi_0)\| \quad \forall\, t \ge 0.$$

This in turn implies that

$$r^K\|y\| - \sup_{t\ge 0} \|g(t, y + \xi) - g(t, \xi)\| \ge r^K\|\zeta\| - \sup_{\tau\ge 0} \|g(\tau, \zeta + \xi_0) - g(\tau, \xi_0)\| - 2r^K\|\xi - \xi_0\|. \tag{6.3}$$

Since $\|\zeta\| \to \infty$ as $\|y\| \to \infty$, it follows from (A2) that the RHS of (6.3) goes to ∞ as $\|y\| \to \infty$, and, *a fortiori*, the same applies to the LHS of (6.3), establishing (6.2). □

Proof of Lemma 3.2. Defining $g : \mathbb{R}_+ \times \mathbb{R}^p \to \mathbb{R}^m$ by (6.1), the function β_Γ can be written in the form

$$\beta_\Gamma(s) := \sup_{t\ge 0,\, \xi\in\Gamma,\, \|y\|\le s} \|g(t, y + \xi) - g(t, \xi)\|.$$

To prove statement (1), we note that, by (A1),

$$\big\|g(t, y_1 + \xi) - g(t, \xi) - \big(g(t, y_2 + \xi) - g(t, \xi)\big)\big\| \le r^K\|y_1 - y_2\| \quad \forall\, t \ge 0,\ \forall\, y_1, y_2, \xi \in \mathbb{R}^p,$$

and hence,

$$\|g(t, y_1 + \xi) - g(t, \xi)\| \le \|g(t, y_2 + \xi) - g(t, \xi)\| + r^K\|y_1 - y_2\| \quad \forall\, t \ge 0,\ \forall\, y_1, y_2, \xi \in \mathbb{R}^p.$$

Setting

$$h(y) := \sup_{t\ge 0,\, \xi\in\Gamma} \|g(t, y + \xi) - g(t, \xi)\| \quad \forall\, y \in \mathbb{R}^p,$$

it follows that

$$h(y_1) \le h(y_2) + r^K\|y_1 - y_2\| \quad \forall\, y_1, y_2 \in \mathbb{R}^p.$$

Interchanging the roles of y_1 and y_2 in the above argument gives that h is globally Lipschitz with Lipschitz constant r^K. Furthermore, we note that

$$\beta_\Gamma(s) = \max_{\|y\|\le s} h(y) \quad \forall\, s \ge 0\,. \tag{6.4}$$

Let $s_1, s_2 \ge 0$, $s_1 \ne s_2$. Without loss of generality we assume that $s_2 > s_1$. Let $y_2 \in \mathbb{R}^p$ be such that $\|y_2\| \le s_2$ and $\beta_\Gamma(s_2) = h(y_2)$. Setting $y_1 := (s_1/s_2)y_2$, it follows that $\|y_1\| \le s_1$, $\|y_2 - y_1\| \le s_2 - s_1$ and

$$0 \le \beta_\Gamma(s_2) - \beta_\Gamma(s_1) \le h(y_2) - h(y_1) \le r^K\|y_2 - y_1\| \le r^K(s_2 - s_1)\,,$$

showing that β_Γ is globally Lipschitz with Lipschitz constant r^K.

We proceed to show that the function α defined by $\alpha(s) = r^K s - \beta_\Gamma(s)$ for all $s \geq 0$ is in $\mathcal{P}$. Evidently, $\alpha(0) = 0$, $\alpha(s) \geq 0$ for all $s \geq 0$ and α is continuous. Therefore, it remains to show that $\alpha(s) > 0$ for $s > 0$. To this end, note that, as a consequence of (A1),

$$\big\|g(t, y + \xi_1) - g(t, \xi_1) - \big(g(t, y + \xi_2) - g(t, \xi_2)\big)\big\| \leq 2r^K \|\xi_1 - \xi_2\| \quad \forall\, t \geq 0,\ \forall\, \xi_1, \xi_2, y \in \mathbb{R}^p .$$

Therefore, for each fixed $y \in \mathbb{R}^p$, the function

$$g_y : \mathbb{R}^p \to \mathbb{R}_+,\ \xi \mapsto \sup_{t\geq 0} \|g(t, y + \xi) - g(t, \xi)\| ,$$

is globally Lipschitz. Invoking assumption (A1) once more, we have that

$$\sup_{t\geq 0} \|g(t, y + \xi) - g(t, \xi)\| = g_y(\xi) < r^K \|y\| \quad \text{for all } y, \xi \in \mathbb{R}^p,\ y \neq 0 ,$$

and it follows from continuity of g_y and compactness of Γ that

$$\sup_{\xi\in\Gamma} \sup_{t\geq 0} \|g(t, y + \xi) - g(t, \xi)\| = \sup_{\xi\in\Gamma} g_y(\xi) < r^K \|y\| \quad \text{for all } y \in \mathbb{R}^p,\ y \neq 0 .$$

Since the LHS is equal to $h(y)$, we conclude that $h(y) < r^K\|y\|$ for all non-zero $y \in \mathbb{R}^p$, and so, the continuity of h guarantees that $\beta_\Gamma(s) < r^K s$ for all $s > 0$, implying that $\alpha(s) > 0$ for all $s > 0$.

Moreover, by construction, it holds that, for all $y \in \mathbb{R}^p$ and all $\xi \in \Gamma$,

$$\sup_{t\geq 0} \|f(t, y + \xi) - f(t, \xi) - Ky\| = \sup_{t\geq 0} \|g(t, y + \xi) - g(t, \xi)\| \leq \beta_\Gamma(\|y\|) = r^K \|y\| - \alpha(\|y\|) ,$$

which is (3.4).

To complete the proof of statement (1), assume that (A2) holds. It is sufficient to prove that $\alpha(s) \to \infty$ as $s \to \infty$, in which case the function α_0 defined by

$$\alpha_0(s) = (1 - e^{-s}) \inf_{t\geq s} \alpha(t) \quad \forall\, s \geq 0 \tag{6.5}$$

is a $\mathcal{K}_\infty$-function satisfying $\alpha_0(s) \leq \alpha(s)$ for all $s \geq 0$. Seeking a contradiction, suppose that there exists a sequence $(s_j)_j$ in $\mathbb{R}_+$ such that $s_j \to \infty$ as $j \to \infty$ and $(\alpha(s_j))_j$ is bounded. By (6.4), there exist vectors $y_j \in \mathbb{R}^p$ such that $\|y_j\| \leq s_j$ and $h(y_j) = \beta_\Gamma(s_j)$. As $\alpha(s_j) = r^K s_j - h(y_j)$, it follows that the sequence $(r^K s_j - h(y_j))_j$ is bounded. By (A1),

$$r^K s_j - h(y_j) \geq r^K (s_j - \|y_j\|) \geq 0 \quad \forall\, j \in \mathbb{N},$$

showing that the sequence $(s_j - \|y_j\|)_j$ is bounded. Consequently, choosing sequences $(t_j)_j$ in $\mathbb{R}_+$ and $(\xi_j)_j$ in Γ such that

$$\lim_{j\to\infty} \big(h(y_j) - \|g(t_j, y_j + \xi_j) - g(t_j, \xi_j)\|\big) = 0,$$

we conclude that

$$0 \leq \sup_{j\in\mathbb{N}} \big(r^K \|y_j\| - \|g(t_j, y_j + \xi_j) - g(t_j, \xi_j)\|\big) < \infty \tag{6.6}$$

As Γ is compact, we may assume, without loss of generality, that $(\xi_j)_j$ is convergent with limit $\zeta \in \Gamma$. Routine estimates show that

$$r^K \|y_j\| - \|g(t_j, y_j + \zeta) - g(t_j, \zeta)\| \leq r^K \|y_j\| - \|g(t_j, y_j + \xi_j) - g(t_j, \xi_j)\| + 2r^K \|\zeta - \xi_j\| \quad \forall\, j \in \mathbb{N} . \tag{6.7}$$

By (6.6), the RHS of (6.7) is bounded. On the other hand, since the sequence $(s_j - \|y_j\|)_j$ is bounded, we have that $\|y_j\| \to \infty$ as $j \to \infty$, and so, invoking Lemma 3.1, the LHS of (6.7) converges to ∞ as $j \to \infty$, yielding the desired contradiction.

To prove statement (2), set $\alpha(s) := r^K s - \beta_{\{\xi_0\}}(s)$ for all $s \geq 0$. By statement (1), $\alpha \in \mathcal{P}$, and arguments very similar to those used in the proof of statement (1) show that $\liminf_{s\to\infty} \alpha(s) > 0$ (we leave the details to the reader). Consequently, the function α_0 defined in (6.5) is in $\mathcal{K}$ and satisfies $\alpha_0(s) \leq \alpha(s)$ for all $s \geq 0$. □

Proof of Lemma 6.1. Let $\varphi \in \mathcal{K}$. As in the proof of [29, Lemma 5.42] it can be shown that there exists continuously differentiable $\varphi_1 \in \mathcal{K}$ such that $\varphi_1(s) \leq \varphi(s)$ for all $s \geq 0$. Choose $s_* > 0$ such that $\varphi_1'(s_*) > 0$ and define $\varphi_2 : [s_*, \infty) \to \mathbb{R}_+$ by

$$\varphi_2(s) := \min\big(\varphi_1'(s), s_*^3\varphi_1'(s_*)/s^3\big) \quad \forall\, s \geq s_*\,.$$

Obviously, $\varphi_2(s_*) = \varphi_1'(s_*) > 0$ and $\varphi_2(s) = \mathrm{O}(1/s^3)$ as $s \to \infty$. Defining the function θ by

$$\theta(s) := \begin{cases} \varphi_1(s)\,, & 0 \leq s \leq s_*\,,\\ \varphi_1(s_*) + \displaystyle\int_{s_*}^{s} \varphi_2(t)\mathrm{d}t\,, & s > s_*\,, \end{cases}$$

we see that θ is continuously differentiable, $\theta \in \mathcal{K}$, $\theta(s) \leq \varphi_1(s) \leq \varphi(s)$ for all $s \geq 0$ and

$$\theta'(s) = \varphi_2(s) = \mathrm{O}(1/s^3) \quad \text{as } s \to \infty,$$

completing the proof. □

Proof of Lemma 6.2. Let $\varphi \in \mathcal{L}$ and define $\varphi_1 : \mathbb{R}_+ \to \mathbb{R}_+$ by $\varphi_1(0) := 0$ and $\varphi_1(s) := (1-e^{-s})\varphi(1/s)$ for $s > 0$. Then φ_1 is a $\mathcal{K}$-function with $\varphi_1(s) \to \varphi(0)$ as $s \to \infty$. It follows from Lemma 6.1 that there exists a continuously differentiable $\varphi_2 \in \mathcal{K}$ such that $\varphi_2(s) \leq \varphi_1(s)$ for all $s \geq 0$ and $\varphi_2'(s) = \mathrm{O}(1/s^3)$ as $s \to \infty$. Define $\theta : \mathbb{R}_+ \to (0, \infty)$ by $\theta(0) := \lim_{s\to\infty} \varphi_2(s)$ and $\theta(s) := \varphi_2(1/s)$ for $s > 0$. The function θ is in $\mathcal{L}$, $0 < \theta(0) \leq \varphi(0)$ and

$$\theta(s) = \varphi_2(1/s) \leq \varphi_1(1/s) = (1 - e^{-1/s})\varphi(s) < \varphi(s) \quad \forall\, s > 0.$$

Moreover, θ is continuously differentiable on $(0, \infty)$ with

$$\theta'(s) = -\varphi'(1/s)/s^2 \quad \forall\, s > 0\,,$$

and so, as $\varphi_2'(s) = \mathrm{O}(1/s^3)$ as $s \to \infty$,

$$\lim_{s\to 0} \theta'(s) = -\lim_{s\to\infty} \big(s^2\varphi_2'(s)\big) = 0\,. \tag{6.8}$$

Finally, let $c > 0$ be such that $\varphi_2'(s) \leq c/s^3$ for all $s > 0$, and note that

$$\left|\frac{\theta(s) - \theta(0)}{s}\right| = \frac{1}{s}\int_{1/s}^{\infty} \varphi_2'(t)\mathrm{d}t \leq \frac{c}{s}\int_{1/s}^{\infty} \frac{\mathrm{d}t}{t^3} = \frac{c}{2}s\,.$$

Consequently, $\theta'(0) = 0$, and thus, by (6.8) and the fact that θ is continuously differentiable on $(0, \infty)$, we conclude that θ is continuously differentiable on $[0, \infty)$, completing the proof. □

References

[1] L. Amerio & G. Prouse. *Almost Periodic Functions and Functional Equations*, Springer Verlag, New York, 1971.

[2] D. Angeli. A Lyapunov approach to the incremental stability properties. *IEEE Trans. Automat. Control*, **47** (2002), 410–421.

[3] D. Angeli, E.D. Sontag & Y. Wang. A characterization of integral input-to-state stability, *IEEE Trans. Automat. Control*, **45** (2000), 1082–1097.

[4] M. Arcak & A. Teel. Input-to-state stability for a class of Lurie systems, *Automatica*, **38** (2002), 1945–1949.

[5] A. Bellow & V. Losert. The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences, *Trans. Am. Math. Soc.*, **288** (1985), 307–345.

[6] A. Bill, C. Guiver, H. Logemann & S. Townley. The converging-input converging-state property for Lur'e systems, *Math. Control, Signals Systems*, **29:4** (2017), DOI 10.1007/s00498-016-0184-3.

[7] H. Bohr. *Almost Periodic Functions*. Chelsea Publishing Company, New York, 1947 (Dover edition, 2018).

[8] D. Burago, Y. Burago & S. Ivanov. *A Course in Metric Geometry*, American Mathematical Society, Providence, Rhode Island, 2001.

[9] C. Corduneanu. *Almost Periodic Functions*, 2nd edition, Chelsea Publsihing Company, New York, 1989.

[10] S.N. Dashkovskiy, D.V. Efimov & E.D. Sontag. Input-to-state stability and related properties of systems, *Autom. Remote Control*, **72** (2011), 1579–1614.

[11] K. Deimling. *Multivalued Differential Equations*. De Gruyter, Berlin, 1992.

[12] C.A. Desoer & M. Vidyasagar. *Feedback Systems: Input-Output Properties*. Academic Press, New York, 1975.

[13] M.E. Gilmore. *Stability and Convergence Properties of Forced Lur'e Systems* (Ph.D. thesis), University of Bath, 2020.

[14] M.E. Gilmore, C. Guiver & H. Logemann. Stability and convergence properties of forced infinite-dimensional discrete-time Lur'e systems, *Int. J. Control* (2019) https://doi.org/10.1080/00207179.2019.1575528.

[15] M.E. Gilmore, C. Guiver & H. Logemann. Semi-global incremental input-to-state stability of discrete-time Lur'e systems, *Systems & Control Letters*, **136** (2020), 104593.

[16] M.E. Gilmore, C. Guiver & H. Logemann. Infinite-dimensional Lur'e systems with almost periodic forcing, *Mathematics of Control, Signals and Systems*, **32** (2020), 327–360.

[17] C. Guiver & H. Logemann. A circle criterion for strong integral input-to-state stability, *Automatica*, **111** (2020), 108641.

[18] C. Guiver, H. Logemann & M.R. Opmeer. Transfer functions of infinite-dimensional systems: positive realness and stabilization, *Math. Control, Signals Systems*, **29** (2017), article no. 20, https://doi.org/10.1007/s00498-017-0203-z.

[19] C. Guiver, H. Logemann & M.R. Opmeer. Infinite-dimensional Lur'e systems: input-to-state stability and convergence properties, *SIAM J. Control Optim.*, **57** (2019) 334–365.

[20] D. Hinrichsen & A.J. Pritchard. Destabilization by output feedback, *Differ. Integral Eqn.*, **5** (1992), 357–386.

[21] D. Hinrichsen & A.J. Pritchard. *Mathematical Systems Theory I,* Springer-Verlag, Berlin, 2005.

[22] B. Jayawardhana, H. Logemann & E.P. Ryan. Input-to-state stability of differential inclusions with applications to hysteretic and quantized feedback systems, *SIAM J. Control Optim.*, **48** (2009), 1031–1054.

[23] B. Jayawardhana, H. Logemann & E.P. Ryan. The circle criterion and input-to-state stability: new perspectives on a classical result, *IEEE Control Systems Magazine*, **31** (August 2011), 32–67.

[24] B. Jayawardhana, E.P. Ryan & A.T. Teel. Bounded-energy convergent-state property of dissipative nonlinear systems: an iISS approach, *IEEE Trans. Automat. Control*, **55** (2010), 159–164.

[25] C.M. Kellet. A compendium of comparison function results, *Mathematics of Control, Signals, and Systems*, **26** (2014), 339–374.

[26] H.K. Khalil. *Nonlinear Systems*, 3rd ed., Prentice-Hall, Upper Saddle River, NJ, 2002.

[27] K. de Leeuw & I. Glicksberg. Almost periodic functions on semigroups, *Acta Math.*, **105** (1961), 99–140.

[28] G. Leoni. *A First Course in Sobolev Spaces*, 2nd ed., American Mathematical Society, Providence, Rhode Island, 2017.

[29] H. Logemann & E.P. Ryan. *Ordinary Differential Equations: Analysis, Qualitative Theory and Control*, Springer Verlag, London, 2014.

[30] J.D. Murray. *Mathematical Biology I*, 3rd edn. Springer-Verlag, New York, 2002.

[31] W.M. Ruess & W.H. Summers. Compactness in spaces of vector-valued continuous functions and asymptotic almost periodicity, *Math. Nachr.*, **135** (1988), 7–33.

[32] I.W. Sandberg. The circle criterion and almost periodic inputs, *IEEE Trans. Circuits Syst. I Fund. Theory Appl.*, **47** (2000), 825–829.

[33] E. Sarkans & H. Logemann. Input-to-state stability of Lur'e systems, *Math. Control, Signals, and Systems*, **27** (2015), 439–465.

[34] E.D. Sontag. Smooth stabilization implies coprime factorization, *IEEE Trans. Automat. Control*, **34** (1989), 435–443.

[35] E.D. Sontag. Input to state stability: basic concepts and results, in P. Nistri and G. Stefani (eds.) *Nonlinear and Optimal Control Theory*, pp. 163–220, Springer Verlag, Berlin, 2006.

[36] O.J. Staffans. *Well-Posed Linear Systems*, Cambridge University Press, Cambridge, 2005.

[37] A.R. Teel & L. Praly. A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semdefinite functions, *ESAIM Control Optim. Calc. Var.*, **5** (2000), 313–367.

[38] M. Tucsnak & G. Weiss. Well-posed systems – the LTI case and beyond, *Automatica*, **50** (2014), 1757–1779.

[39] M. Vidyasagar. *Nonlinear Systems Analysis*, 2nd edition, Prentice-Hall, Englewood Cliffs, NJ, 1993.

[40] R. Vinter. *Optimal Control*, Boston, Birkhäuser, 2000.

[41] V.A. Yakubovich. Matrix inequalities method in stability theory for nonlinear control systems: I. Absolute stability of forced vibrations, *Autom. Remote Control*, **7** (1964), 905–917.