

**DEVELOPMENT OF RISK ANALYSIS MODELS  
FOR DECISION-MAKING  
IN PROJECT MANAGEMENT**

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## ABSTRACT

Risks and uncertainties are inherent in construction projects and if neglected these risks often lead to project cost and time overruns. Traditional methods of forecasting risks rely upon intuition and ‘feel’ which has proved inadequate for the needs of investors in modern construction projects. To cope with these recognised risks, a risk management framework, which consists four components (risk identification, risk classification, risk analysis and risk response), has been developed. The present research focuses on financial risks in construction management, and in particular, the development of enhanced quantitative, probabilistic methods for risk analysis.

A comprehensive review of the treatment of risk and uncertainty in the construction industry is undertaken. Background knowledge of probability theory and Monte Carlo simulation is reviewed, as is previous investigations into construction network analysis and project economics.

A comparison of the Programme Evaluation and Review Technique (PERT) and the Monte Carlo Simulation (*MCS*) methods in construction networks risk analysis is carried out. Two example projects are analysed by both methods. When applying the MCS method, a sensitivity analysis is carried out by investigating the effect of different probability distributions (Normal, Log-Normal, Beta, Triangular and Uniform) for individual activity durations, the number of simulations used and the effect of the manner of how the mean and standard deviations are set for the different probability distributions.

A new analytical method, the Modified Stochastic Assignment Model (MSAM), is proposed for the prediction of project duration. Five example projects are used to demonstrate the validity of the MSAM and to illustrate its application in construction project evaluations. The accuracy of the MSAM method is assessed by comparison to the MCS method. A comparison of the MSAM with other analytical methods commonly used in construction network analysis, such as PERT and the Probabilistic Network Evaluation Technique (PNET), is also presented.

The First Order Second Moment (FOSM) method, a methodology previously used solely in system reliability analysis is applied to project economics. The definition of the FOSM method is given and detailed mathematical treatments of these methods are described. The methodology of using the FOSM in construction economics is explained and ten examples are analysed using both the FOSM method and the MCS to show the applicability and the degree of accuracy of these methods.

The current research shows that the MSAM method yields the probability of project completion within a prescribed target time, or the required project time at a specific probability. The research also shows that it is possible to use the FOSM methods for risk analysis in decision-making in construction economics in such areas as selection of project, elemental cost analysis, cash flow streams and setting of plant hire rates. Both methods require computational time that is significantly less than an equivalent MCS.

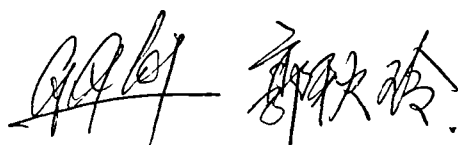
## DECLARATION

This thesis is submitted to Napier University for the Degree of Doctor of Philosophy. The work described in this thesis was carried out under the supervision of Dr. Sam Wamuziri and Prof. Alan Sibbald. The work was undertaken in The School of the Built Environment (formerly The Department of Civil and Transportation Engineering), Napier University, Edinburgh.

In accordance with the Regulations of Napier University governing the requirements for the Degree of Doctor of Philosophy, the candidate submits that the work presented in this thesis is original unless otherwise referenced within the text.

During this period of research the following papers have been published and/or presented:

1. Qiu Ling Guo, Mike Maher and Sam Wamuziri, "Risk analysis in construction networks using a modified stochastic assignment model". Journal of Civil Engineering and Environmental Systems, Vol. 18, Issue 3, June 2001, pp. 215-241.
2. Qiu Ling Guo, Sam Wamuziri, "The application of FOSM methods for risk analysis in construction economics". Proceedings of the First International Conference on New Information Technologies for Decision Making in Civil Engineering. University of Quebec, Montreal, Canada, 11-13 October 1998, pp. 1163-1173.
3. Qiu Ling Guo, "Development of risk analysis models for decision-making in project economics and management". The American Real Estate Society (ARES) Fourteenth Annual Doctoral Seminar. Santa Barbara, California, USA, 29 March 2000.



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## NOTATION

$f_X(x)$	probability density function
$F_X(x)$	probability distribution function
$X, x$	variable
$\mu$	mean
$\sigma$	standard deviation
$N(\mu, \sigma)$	Normal distribution with mean, $\mu$ , and standard deviation, $\sigma$ .
$N(0,1)$	standard Normal distribution
$\Phi(s)$	probability distribution function of the standard Normal variable
$\phi(s)$	probability density function of the standard Normal variable
$a, b$	real number
$\ln X$	the natural logarithm of $X$
$P, p, q$	probability
$\lambda$	mean of $\ln X$
$\zeta$	standard deviation of $\ln X$
$\delta$	coefficient of variation (COV)
$q, r (\alpha_1, \alpha_2)$	shape parameters of the Beta distribution
$B(q, r)$	Beta function
$a, b, c$	minimum, most likely and maximum values in Triangular distribution
$\alpha, \beta$	minimum and maximum values in Uniform distribution
@Risk	a commercial simulation computer package

In PERT network:

$t_e, E(t)$	expected duration
$s$	standard deviation
$a$	optimistic duration
$M, m$	most likely duration
$b$	pessimistic duration
$T$	actual completion time of the project
$t$	time
$\cap$	intersection
$\cup$	union
$\equiv$	equivalent to
$\in$	belongs to
$\subset$	is contained in
$\prod_{i=m}^n X$	the product as $i$ goes from $m$ to $n$ of the expression $X$
$\rho$	correlation
$\pi$	path
$Z_k$	path cost variables
$n$	link, node
$B_n$	(the “before” links) the set of links which enter node $n$

$A_n$	(the “after” links) the set of links which exit from node $n$
$W_n$	the random variable cost (duration) from the origin to node $n$
$\beta$	the ratio of variance to mean (reliability index in Chapter 5)
$\forall$	including all
$\xi_1, \xi_2$	non-Normal variables
$\eta_1, \eta_2$	Normal variables
$p_s$	reliability of the system
$p_F$	failure probability of the system
$g(X)$	performance function

## ABBREVIATIONS

CRMS	Construction Risk Management System
RMP	Risk Management Process
CPM	Critical Path Method
PERT	Programme Evaluation and Review Technique
PNET	Probabilistic Network Evaluation Technique
MCS	Monte Carlo Simulation
SAM	Stochastic Assignment Model
MSAM	Modified Stochastic Assignment Model
FOSM	First Order Second Moment
CLT	Central Limit Theorem
CDF	Cumulative Distribution Function
PDF	Probability Density Function
COV	Coefficient Of Variation
PNT	Project Network Techniques
AoA	Activity-on-Arrow
AoN	Activity-on-Node
CPA	Critical Path Analysis
O.R.	Operational Research
SPO	Special Projects Office
MoP	Method of Potentials
TPT	Total Project Time
EST	Earliest Starting Time
EFT	Earliest Finishing Time
LFT	Latest Finishing Time
LST	Latest Starting Time
EET	Earliest Event Time
LET	Latest Event Time
NRB	Narrow Reliability Bounds Method
LST	Linear Scheduling Technique
O-D	Origin-Destination

# CHAPTER 1 INTRODUCTION

## 1.1 Nature of construction projects

Projects in the construction industry are inherently subject to more risk and uncertainty than those in most other industries. For example, the required duration for completion of an activity will normally depend on the availability of resources (labour and equipment), on weather conditions and on the availability of materials. Quite often, construction projects fail to achieve their time, quality, and budget goals. The purpose of risk analysis and management is to help managers to avoid or minimise the economic impact of these failures.

Each construction project is, by its nature, different. Based on the ‘uniqueness’ of the project, there are many special problems in construction. As opposed to the view that construction is a high risk business and that the future can not be forecast, some writers <sup>[1,2]</sup> do not agree with these rather negative viewpoints. Raftery <sup>[1]</sup> suggests: “In fact, viewed systematically, there are many features common to a wide range of construction projects. Most projects will have a start date and a finish date. Most projects will be designed and built by *teams of people and firms drawn together for that particular project*. Most projects will require contractors to marshal labour, equipment, materials and components to a specific site. Many physical elements of construction will recur across projects. For example, all bridges need a deck and a supporting structure. All buildings need a substructure, cladding and services.”

Indeed, looking at risk in a systematic way, Flanagan and Norman <sup>[2]</sup> state that the typical risks on a construction project include:

- Failure to achieve completion within the stipulated design and construction time.
- Failure to obtain the expected outline planning, detailed planning or building code/regulation approvals within the time allowed in the design program.
- Unforeseen adverse ground condition.
- Exceptionally inclement weather.
- Strike by the labour force.

- Unexpected price rises for labour and materials.
- Failure to let to a tenant upon completion.
- An accident to an operative on site causing physical injury.
- Latent defects occurring in the structure through poor workmanship.
- Natural disasters (flood, earthquake, etc.).
- A claim from the contractor for loss and expense caused by the late production of design details by the design team.
- Failure to complete the project within the client's budget allowance.

Traditional methods of forecasting risks rely upon intuition and 'feel'. This is clearly inadequate in modern construction project management. Risk analysis and management provides a systematic framework in which management can pay greater attention to risk and thus improve project performance.

## **1.2 A risk management framework**

The managerial techniques used to identify, analyse and respond to risk have been applied in construction only during the last two decades <sup>[3,4,5,6,7,8,9]</sup>. Healy <sup>[10]</sup> after reviewing various authors, developed one approach which is suitable for risk management in large projects. Wideman <sup>[11]</sup> proposed a theoretical framework of a construction risk management model. Al-Bahar and Crandall <sup>[12]</sup> based on Wideman's <sup>[11]</sup> conceptual ideas converted the model into a completely defined, quantitative management model.

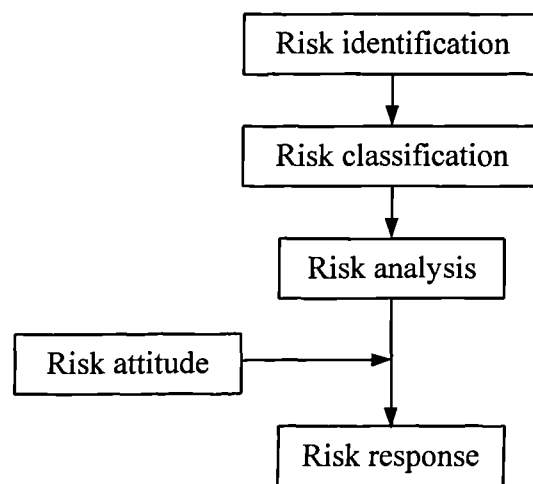
Al-bahar and Crandall's model, entitled Construction Risk Management System (CRMS), consists of the following four processes:

- Risk identification.
- Risk analysis and evaluation.
- Response management.
- System administration.

The models' particular emphasis is placed on how to identify and manage risks before, rather than after, they materialise into losses or claims. The CRMS has the features of

a systematic framework of risk management which is methodical, objective, analytical, has quantitative measurement, and is self-contained.

Risk management framework is sometimes called Risk Management Process (RMP) [7,9] and can be described in terms of phases (stages), which are decomposed in a variety of ways. Some models are more detailed than others, but all show similar characteristics as the model by Flanagan and Norman [2] who have described a framework as shown in Figure 1.1.



**Figure 1.1 The risk management framework**

The risk management framework, sometimes called the risk management cycle in engineering construction, consists of four components.

#### 1. Risk identification:

In this stage, a clear view of the event is the first requirement, focusing on the sources of risk and the effect of the event. One can do it in a structured and systematic way by distinguishing between controllable and uncontrollable risks, dependent and independent risks and total dependence or partial dependence. Alternatively, one can work with an experienced team to consider explicitly three separate areas. These are: risks internal to the project, risks external to the project and anticipation of sources of claims. A formal brainstorming session can be used at this stage in order to list possible risk sources. Alternatively, tick lists of similar past projects can be reviewed as a guide to identify sources of risk on new projects.

## 2. Risk classification:

The three ways of classifying risk are by identifying the type, impact and consequence of risk. The types of risk in the construction industry can be classified as:

- 'Pure' risk (no potential gain).
- 'Speculative' risk (possibility of loss or gain).

The impact of risk can be viewed in a risk hierarchy such as:

- Environment.
- Market/industry.
- Company.
- Project/individual.

The consequence of risk can also be classified into:

- Maximum probable loss.
- Most likely cost of the loss.
- Likely cost of servicing the loss if no insurance has been effected.
- Cost of insuring against the event occurring.
- Reliability of the prediction about the event.

## 3. Risk analysis:

The essence of risk analysis is that it attempts to encompass all feasible options and to analyse the various outcomes of any decision. The use of risk analysis gives an insight into what happens if the project does not proceed according to plan. The risk analysis can be applied in following steps:

- All the various options should be considered.
- Consider the risk attitude of the decision-maker.
- Consider what risks have been identified, which are controllable and what the impact is likely to be.
- Measurement, both quantitative and qualitative.
- Interpretation of the results of the analysis and development of a strategy to deal with the risk.
- Decide what risks to retain and what risks to allocate to other parties.

#### 4. Risk response:

The purpose of the identification and the analysis is to enable the decision maker to make a considered response in advance of the problem occurring. The response to risk can take any of four basic forms: risk retention, risk reduction, risk transfer and risk avoidance.

### **1.3 Research objectives**

The primary goal of this PhD is to develop enhanced methods for risk analysis to aid decision-making in project management. Techniques for risk analysis have been theoretically established for a number of years and the most relevant to construction can be summarised as follows:

1. Sensitivity analysis.
2. Probability analysis and Monte Carlo simulation.
3. Decision tree analysis.
4. Utility theory.
5. Portfolio theory.

These techniques have been applied to project management in different situations to different degrees and are well documented in numerous sources [2,4,9,13,14].

Probability theory concepts provide the theoretical framework for analysing risks and various methods have been suggested. Some sophisticated methods such as Monte Carlo simulation are well understood but there is still a gap between the theory and the techniques applied in practice. The quantitative techniques appropriate to the analysis of risk require further development and a step-by-step procedure for estimating the impact of risk has not been provided.

The reasons for this may be due firstly to the necessity and importance of quantified risk analysis not being recognised. Most people in practice believe that it is possible to deal with risk competently, consistently, and comprehensively with the use of very little mathematics. Also, there are quite strong opinions that there is no such thing as a software-only solution to the problem of risk management. Risk analysis software is



neither necessary nor sufficient for risk management. Rigorous, comprehensive and competent risk analysis is primarily dependent on the attitude of the appropriate decision maker and their advisers. Secondly, to analyse risk quantitatively may be too costly. The people who perform the analysis need to be trained and the analysis may be limited by computer resources.

However, due to the rapid developments in the construction industry, no one can ignore the tools and techniques that can improve the quality and accuracy of risk analysis. Also the dramatic growth of computer technology, especially the development of cheaper and more powerful desktop personal computers, implies the need for the appropriate techniques is more urgent than ever.

Based on the viewpoints above, the present research focuses on risk analysis, emphasising financial risks in construction management, especially the quantitative aspects of risk analysis. A review of the treatment of risk and uncertainty in construction project management has been undertaken to identify the main areas of study and the overall objective of this investigation is to develop enhanced quantitative, probabilistic methods for risk analysis. The enhanced risk analysis methods can then be used to improve decision-making and encourage appropriate risk allocation, risk modification, mitigation or avoidance for highly uncertain investment projects.

In particular, the specific research objectives in this study are as follows:

1. To review probability theory and the common probability distributions which are used to model risk in project management.
2. To review the various methods which have been developed for network analysis in construction projects.
3. To compare the project completion time predictions given by the Programme Evaluation and Review Technique (PERT) with those from Monte Carlo Simulation (MCS). Furthermore, to assess the influence of the various continuous probability distributions and different numbers of simulations for activity duration on predictions of overall project completion time when using MCS.

4. To develop a computer program based on a Stochastic Assignment Model (SAM) for a probabilistic network analysis of project completion time and to compare its predictions with those from MCS. This developed method is referred to as the Modified Stochastic Assignment Model (MSAM) and can be used in the planning and execution stages of a project.
5. To show how First Order Second Moment (FOSM) methods used previously only in the analysis of structural safety can be used for the assessment of various forms of risk in project management. This method is especially suited to the conceptual stage of a project but can be used to assess the failure probability throughout a project life cycle.

#### **1.4 Outline of thesis**

Chapter one presents an introduction to the nature of risk in construction projects and the scope and aims of the present research are described.

Chapter two provides a review of the mathematical foundations for risk analysis, such as probability theory and principles of MCS. Some common probability distributions for modeling risk and uncertainty that are used in the present study are detailed. Previous investigations and methods of construction network analysis are also reviewed.

In Chapter three, a comparison of the PERT and the MCS methods in construction networks risk analysis is carried out and two example projects are analysed by both methods. When applying the MCS method, a sensitivity analysis is carried out by investigating the effect of different probability distributions for individual activity durations, the number of iterations used and the effect of the manner of how the mean and standard deviations are set for the different probability distributions.

In Chapter four, a new analytical method, the MSAM, is proposed for the prediction of project duration. The MSAM algorithm is described and five example projects are used to demonstrate the validity of the MSAM and to illustrate its application in construction project evaluations. A comparison of the MSAM with other analytical

methods commonly used in construction network analysis, such as PERT, the Probabilistic Network Evaluation Technique (PNET) and MCS is presented.

Chapter five provides a description of the principles of FOSM methods previously only used in the area of structural safety. FOSM methods are approximate analytical methods used to analyse risk. A number of example problems have been analysed in this chapter using FOSM methods to show the applicability of these methods and the degree of accuracy of the results obtained has been assessed by comparison with results from large sample MCS. The FOSM methods are also improved by giving a set of concise algorithms that can be easily coded to a computer program for practical application.

A summary of the conclusions of the study is given in Chapter six including recommendations for further research.

## **CHAPTER 2 LITERATURE REVIEW**

### **2.1 Introduction**

Management of risks and uncertainties in construction projects is only possible if risks have been identified and the potential impacts have been analysed. Principles of probability theory offer the mathematical basis for modelling risks and uncertainty and the analysis of its effect. This chapter begins with a review of the basic concepts of probability and some useful probability distribution functions which are commonly used in the construction industry. Such probability distributions include the Normal, Log-Normal, Beta, Triangular and Uniform distribution. Monte Carlo Simulation (MCS) methods and the application of these concepts in construction engineering are also reviewed. A literature review and historical development of construction Project Network Techniques (PNT) is carried out, including Critical Path Method (CPM), Program Evaluation and Review Technique (PERT), Probabilistic Network Evaluation Technique (PNET) and MCS.

### **2.2 Probability theory**

#### **2.2.1 Basic probability concepts**

Probabilities are used when considering future events with more than one possible outcome. In a given situation only one of these outcomes will occur but in advance we cannot say which. Such situations are called stochastic, as opposed to deterministic situations where the outcome is determined in advance. The probability of an event is a measure of the chance that it will occur and is measured as a value in the interval between 0 and 1. Something that is almost certain to happen has a probability close to 1, while an event that is extremely unlikely has a probability close 0. Probabilities are usually assessed (estimated) by experience based data.

In any case, an outcome or event may be identified through the value(s) of a function and such a function is a random variable. Since the value of a random variable represents an event, it can assume a numerical value only with an associated

probability or probability measure. The rule for describing the probability measures associated with all of the values of a random variable is a probability distribution.

Some probability distribution functions especially useful in the construction industry are presented and their special properties are described in the Section 2.2.2.

## 2.2.2 Useful probability distributions

### 2.2.2.1 Normal distribution

The best-known and most important probability distribution is the Normal distribution, also known as the Gaussian distribution. Historically, the Normal distribution has played a central role in the development of probability and statistics. The reasons for this pre-eminence are both practical and theoretical. The numerical measurements of many diverse phenomena can be modelled by the Normal distribution - for instance, the quantitative analysis of errors. The Central Limit Theorem, to be discussed in Section 2.4.4, shows that the Normal distribution holds a special place among all probability distributions.

The Normal distribution has a probability density function given by:

$$f_x(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \quad -\infty < x < \infty \quad (2.1)$$

where  $\mu$  and  $\sigma$  are the parameters of the distribution, which are also the mean and standard deviation, respectively, of the variate. A short notation for this distribution is  $N(\mu, \sigma)$ .

A Normal distribution with parameters  $\mu = 0$  and  $\sigma = 1$  is known as the standard Normal distribution and is denoted appropriately as  $N(0, 1)$ . The density function, accordingly, is:

$$f_s(s) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(s)^2\right] \quad -\infty < s < \infty \quad (2.2)$$

The Normal cumulative probability distribution function is given by:

$$F_X(x) = \int_{-\infty}^x \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2\right] dy \quad (2.3)$$

Especially important in calculations is the probability distribution function of the standard Normal variable, denoted by a special notation  $\Phi(s)$ , that is,  $\Phi(s) = F_S(s)$ , where  $S$  has  $N(0,1)$  distribution.

$$\Phi(s) = F_S(s) = \int_{-\infty}^s \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}x^2\right] dx \quad (2.4)$$

The above integral is not soluble analytically but has been numerically evaluated and is widely tabulated in the literature <sup>[15,16]</sup>.

With the table of  $\Phi(s)$ , probabilities of any other Normal distributions can then be determined readily as follows. Suppose a Normal variate  $X$  with distribution  $N(\mu, \sigma)$ , the probability:

$$P(a < X \leq b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] dx \quad (2.5)$$

Theoretically, this required probability can be obtained by evaluating the preceding integral directly; however, this can be done also by making the following change of variable:

$$s = \frac{x-\mu}{\sigma} \quad \text{and} \quad dx = \sigma \cdot ds \quad (2.6)$$

Then:

$$\begin{aligned} P(a < X \leq b) &= \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}(s)^2\right] \sigma ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{(a-\mu)/\sigma}^{(b-\mu)/\sigma} \exp\left[-\frac{1}{2}(s)^2\right] ds \end{aligned} \quad (2.7)$$

Thus:

$$P(a < X \leq b) = \Phi\left(\frac{b - \mu}{\sigma}\right) - \Phi\left(\frac{a - \mu}{\sigma}\right) \quad (2.8)$$

Empirical evidence <sup>[17]</sup> indicates that the Normal distribution provides a good representation for many physical variables. Examples include measurements on weight, length and strength of material, instrumentation error and the financial rate of return.

The Normal distribution is also often adopted as a convenient approximation to other distributions which are less widely tabulated. For example, to find approximations to probabilities associated with other distributions, such as the Gamma, the Binomial and the Poisson.

#### 2.2.2.2 Logarithmic Normal distribution

A random variable  $X$  has a Logarithmic Normal (or simply Log-Normal) probability distribution if  $\ln X$  (the natural logarithm of  $X$ ) is Normal. The Log-Normal distribution has a probability density function of  $X$  is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\zeta_X} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right] \quad 0 \leq x < \infty \quad (2.9)$$

where  $\lambda = E(\ln X)$  and  $\zeta = \sqrt{\text{Var}(\ln X)}$  are, respectively, the mean and standard deviation of  $\ln X$ , and are parameters of the distribution.

Because of its relationship with the Normal distribution (that is involving a logarithmic transformation), probabilities associated with a Log-Normal variate can also be determined using the table of standard Normal probabilities. This is shown as follows:

$$P(a < X \leq b) = \int_a^b \frac{1}{\zeta_X \sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln x - \lambda}{\zeta}\right)^2\right] dx \quad (2.10)$$

Let:

$$s = \frac{\ln x - \lambda}{\zeta} \quad \text{and} \quad dx = x\zeta \cdot ds \quad (2.11)$$

Then:

$$P(a < X \leq b) = \frac{1}{\sqrt{2\pi}} \int_{(\ln a - \lambda)/\zeta}^{(\ln b - \lambda)/\zeta} \exp\left[-\frac{1}{2}(s)^2\right] ds \quad (2.12)$$

Thus:

$$P(a < X \leq b) = \Phi\left(\frac{\ln b - \lambda}{\zeta}\right) - \Phi\left(\frac{\ln a - \lambda}{\zeta}\right) \quad (2.13)$$

In view of this convenient facility for calculating probabilities of Log-Normal variates and also because the values of the random variable are always positive, the Log-Normal distribution is useful in those applications where the values of the variate are known to be strictly positive.

It can be seen that from Equation 2.9, the probability is a function of the parameters  $\lambda$  and  $\zeta$ . These parameters are related to the mean  $\mu$  and standard deviation  $\sigma$  of the variate as follows:

Let  $Y = \ln X$ , which is  $N(\lambda, \zeta)$ . It follows that  $X = e^Y$  and

$$\mu = E(X) = E(e^Y) = \exp\left(\lambda + \frac{1}{2}\zeta^2\right) \quad (2.14)$$

Thus:

$$\lambda = \ln \mu - \frac{1}{2}\zeta^2 \quad (2.15)$$

Similarly, the mean-square value of  $X$ ,  $E(X^2)$  is:

$$E(X^2) = \exp[2(\lambda + \zeta^2)] \quad (2.16)$$

$$\text{Var}(X) = E(X^2) - \mu^2 = \mu^2(e^{\zeta^2} - 1) \quad (2.17)$$

Thus:



$$\zeta^2 = \ln\left(1 + \frac{\sigma^2}{\mu^2}\right) \quad (2.18)$$

If  $\sigma/\mu$  is not large, say  $\leq 0.30$ ,  $\ln\left[1 + \left(\sigma^2/\mu^2\right)\right] \approx \sigma^2/\mu^2$ . In such cases, therefore:

$$\zeta \approx \frac{\sigma}{\mu} = \delta \quad (2.19)$$

where  $\delta$  is the coefficient of variation (COV).

The Log-Normal distribution has the theoretical advantage of precluding negative values and has been used in a variety of applications. For example, the time for project completion, the strength and fatigue life of material, and the volume of air traffic. The Log-Normal distribution is also used very widely in reliability studies.

### 2.2.2.3 Beta distribution

A probability distribution appropriate for a random variable whose values are bounded, say between finite limits  $a$  and  $b$ , is the Beta distribution. The density function of such a distribution is:

$$f_x(x) = \begin{cases} \frac{1}{B(q,r)} \frac{(x-a)^{q-1} (b-x)^{r-1}}{(b-a)^{q+r-1}} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases} \quad (2.20)$$

in which  $q$  and  $r$  are parameters of the distribution, and  $B(q, r)$  is the Beta function:

$$B(q, r) = \int_0^1 x^{q-1} (1-x)^{r-1} dx \quad (2.21)$$

which is related to the Gamma function as follows:

$$B(q, r) = \frac{\Gamma(q)\Gamma(r)}{\Gamma(q+r)} \quad (2.22)$$

Depending on the parameters  $q$  and  $r$ , the density function of the Beta distribution will have different shape.

If the values of the variate are limited between 0 and 1 (that is,  $a = 0$  and  $b = 1$ ), Equation 2.20 becomes:

$$f_x(x) = \begin{cases} \frac{1}{B(q,r)} x^{q-1} (1-x)^{r-1} & 0 \leq x \leq 1 \\ 0 & \text{elsewhere} \end{cases} \quad (2.23)$$

which is called the standard Beta distribution.

The probability associated with a Beta distribution can be evaluated in terms of the incomplete Beta distribution, which is defined as:

$$B_x(q,r) = \int_0^x y^{q-1} (1-y)^{r-1} dy \quad 0 < x < 1 \quad (2.24)$$

Values of the incomplete Beta function ratio have been tabulated; for example, by Pearson <sup>[18]</sup> and Pearson and Johnson <sup>[19]</sup>. Therefore, probabilities involving the Beta distribution can be evaluated conveniently using tables of the incomplete Beta function ratio.

By virtue of Equation 2.24, it is observe that the cumulative distribution function (CDF) of the standard Beta distribution, Equation 2.23, with parameters  $q$  and  $r$ , is given by:

$$\beta(x|q,r) = \frac{B_x(q,r)}{B(q,r)} \quad (2.25)$$

Effectively, therefore, tables of the incomplete Beta function ratio are also the tables for the CDF of the standard Beta distribution.

The mean and variance of the Beta distribution are:

$$\mu = a + \frac{q}{q+r}(b-a) \quad (2.26)$$

$$\sigma^2 = \frac{qr}{(q+r)^2(q+r+1)}(b-a)^2 \quad (2.27)$$

The Beta distribution is the basis for the approximation of activity time in a PERT network. It can also be used as a rough model in the absence of data.

#### 2.2.2.4 Triangular distribution

The Triangular distribution describes a situation where one can estimate the minimum, maximum and most likely value. Values near the minimum and maximum are less likely to occur than those near the most likely. The density function of the Triangular distribution is:

$$f(x) = \begin{cases} \frac{2(x-a)}{(b-a)(c-a)} & \text{if } a \leq x \leq b \\ \frac{2(c-x)}{(c-a)(c-b)} & \text{if } b < x \leq c \end{cases} \quad (2.28)$$

where  $a$  is minimum,  $b$  is most likely and  $c$  is maximum.

The probability distribution function of the Triangular distribution is:

$$F(x) = \begin{cases} \frac{(x-a)^2}{(c-a)(b-a)} & a \leq x \leq b \\ 1 - \frac{(c-x)^2}{(c-a)(c-b)} & b \leq x \leq c \end{cases} \quad (2.29)$$

The mean and variance of the Triangular distribution are:

$$\mu = \frac{a+b+c}{3} \quad (2.30)$$

$$\sigma^2 = \frac{a^2 + b^2 + c^2 - ab - ac - bc}{18} \quad (2.31)$$

The Triangular distribution is widely used because of its ease of use and can be used when actual data is absent. However, its drawback is that it is an approximation.

#### 2.2.2.5 Uniform distribution

The Uniform distribution with the parameters  $\alpha$  and  $\beta$  is defined by the equation:

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \alpha \leq x \leq \beta \\ 0 & \text{elsewhere} \end{cases} \quad (2.32)$$

The probability distribution function of a Uniform distribution:

$$F(x) = \begin{cases} 1 & \text{if } x \geq \beta \\ \frac{x - \alpha}{\beta - \alpha} & \text{if } \alpha \leq x \leq \beta \\ 0 & \text{otherwise} \end{cases} \quad (2.33)$$

The mean and the variance of Uniform distribution are:

$$\mu = \frac{\alpha + \beta}{2} \quad (2.34)$$

$$\sigma^2 = \frac{1}{12}(\beta - \alpha)^2 \quad (2.35)$$

The parameters  $\alpha$  and  $\beta$  are often defined as the minimum and maximum value, and all values between these two values are equally likely to occur. The Uniform distribution can be applied in situations where quantities vary uniformly between two values. For example, if there has been no information about the existing utilities on the site the value for any connections is equally likely to occur. However, the Uniform distribution is only used in the case of low sensitivity variables or to be on the safe side to overestimate the probability of the extremes of the variables' range<sup>[20]</sup>.

## **2.3 Monte Carlo Simulation (MCS) techniques**

### **2.3.1 Introduction**

Simulation is the process of replicating the real world based on a set of assumptions and conceived models of reality. It may be performed theoretically or experimentally. In practice, theoretical simulation is usually performed numerically and this has become a much more practical tool since the advent of computers. As with experimental methods, numerical simulation may be used to obtain (simulated) data, either in lieu of or in addition to actual real-world data. In effect, theoretical simulation is a method of “numerical or computer experimentation.”

The term Monte Carlo was introduced during World War II as a code name for the simulation of problems associated with the development of the atomic bomb. Monte Carlo Simulation (MCS) techniques are used for problems involving random variables with known (or assumed) probability distributions. This involves repeating a simulation process, using in each simulation a particular set of values of the random variables generated in accordance with the corresponding probability distributions. There are a wide variety of algorithms available for generating random samples from different types of probability distributions. For a given set of generated random numbers, the simulation process is deterministic. By repeating the process, a sample of solutions, each corresponding to a different set of values of the random variables, is obtained.

A sample from a Monte Carlo simulation is similar to a sample of experimental observations. Therefore, the results of Monte Carlo simulations may be treated statistically; such results may also be presented in the form of histograms, and methods of statistical estimation and inference are applicable. For these reasons, Monte Carlo simulation is also a sampling technique, and as such shares the same problems of sampling theory; namely the results are also subject to sampling errors. The accuracy of the Monte Carlo simulations is largely dependent on the randomness of the sample. Generally, therefore, Monte Carlo solutions from finite samples are not “exact” (unless the sample size is infinitely large).

Apart from statistical errors which can be made as small as desired, in principle the MCS yields exact information on model systems which are precisely characterised. In contrast, the information provided by analytic theory is exact only in rare cases, while in most other cases uncontrolled approximations are required. Thus computer simulations are often designed to check the accuracy of some approximations made in the analytical treatment of a model. In the present research, MCS is used to check the accuracy of the proposed methods.

### **2.3.2 MCS applications in construction management**

Newton's review <sup>[21]</sup> of cost modelling activity in construction revealed how popular the MCS has become as a construction management tool. The use of the MCS is very straightforward and was described clearly in four main steps by Macaluso <sup>[22]</sup>:

- Formulate a model.
- Distribute appropriate data in the model.
- Sample from the model data.
- Analyse the sample.

The PERT (Program Evaluation Review Technique) and the CPM (Critical Path Method), widely used in construction project planning, have been developed using MCS to account for uncertainty in the duration of construction operations. Karni's <sup>[23]</sup> stochastic project network uses samples from discrete probability states and not continuous distributions at the heart of other applications described. It is of interest that Karni concentrates on the development of the model rather than the data needed to fit it, and uses a hypothetical (non-construction) project as an illustrative example. Attempts to incorporate the effects of variable site productivity and interference from external sources were undertaken by Bennet and Ormerod <sup>[24]</sup>. Where possible, the risk elements were addressed using historical data. In other cases, a library of distributions, from which the user selected the most appropriate to model variability, were provided. Detailed data was easy to obtain from the meteorological office for the weather simulation model of their programme.

In contrast attempts to quantify the variability of different site activities was plagued by a lack of information, highlighting the extent of the problems in construction management. In the opinion of Baxendale <sup>[25]</sup> the available data was limited and of dubious accuracy. Baxendale modelled project durations by recording the actual duration of certain major site tasks rather than trying to model the factors which influence them. Cost significant tasks which are repetitive in nature (such as pouring floor slabs in Wilson's <sup>[26]</sup> example) make the best subjects for this type of modelling. Reliable data can only be obtained by direct observation, for example by studying actual concreting observations to tabulate a frequency distribution. The components of the model represent known behaviour, and data can be collected from similar processes. This provides probability distributions with a good description of the range of possible values. The obvious drawbacks associated with this method are similar to those of Wilson's <sup>[26]</sup> cost model, namely the great time and costs involved.

### **2.3.3 Advantages and disadvantages of MCS**

It has been claimed that the introduction of simulation methods for construction management is likely to have as great an impact on the construction industry as did the introduction of network planning and scheduling methods some four decades ago <sup>[27]</sup>. Some of the advantages claimed for the technique can be summarised as <sup>[24]</sup>:

1. MCS provides an almost unlimited capacity to model construction operations, and permits the construction manager to quickly evaluate many different combinations of equipment and methods under varying conditions of operation at moderate cost.
2. MCS can give the manager an insight into which factors are important and hence where to concentrate his effort and how they interact.
3. Additionally if a person can interact with the computer simulation in a gaming environment, experience can be gained under realistic conditions before the work is started. This should lead to better management through a deeper knowledge of the problem.

4. MCS models often predict things which are not specifically incorporated into the model. Simulation of repetitive processes has shown that when uncertainty exists there are large penalties rather than benefits of scale. There is some evidence that in the construction industry it is the larger projects which go wrong most frequently. Also, most work-study experts talk about the benefits of specialisation. MCS shows that for large projects subject to uncertainty there are penalties of specialisation.
5. Further, most models of construction processes assume that the cost of a project is the sum of the costs of the activities. MCS of a repetitive process shows that costs are largely generated by the uncertainties that exist, and that simple additive models like the Bill of Quantities seriously under-estimate cost.

However, MCS does have several disadvantages which include:

1. In theory, MCS can be applied to large and complex systems. However, in practice, it may be limited by constraints of economy and computer capability. This has been true in the past when the only computers with the required capacity were large main-frame computers with high operating costs. Presently, however, due to the rapid developments of computer software and hardware, the objection that the process is time-consuming has been overcome.
2. MCS is a technique for solving random variable problems but it is prudent to regard the present state of the art as incomplete when it is applied to construction management <sup>[4]</sup>. For example, the difficulties of accommodating a correlation (or interdependency) between two or more variables have long been recognised but adequate techniques to manage these difficulties appear to be lacking. Other concerns of considerable importance to a user, include the choice of distribution, the number of variables, the range of variation and the number of iterations for the statistical analysis.
3. Another limitation of MCS is that solutions obtained may not be amenable to generalisation or extrapolation. Therefore, as a general rule, MCS should be used only as a last resort <sup>[28,29]</sup>, that is, when and if analytical solution methods are not



available or are ineffective (e.g. because of the need for gross idealisations). Monte Carlo solutions, however, are often the only means of verifying or validating approximate analytical solution methods.

#### **2.3.4 @Risk MCS computer package**

In the present research, the MCS were carried out using the @Risk (pronounced “at risk”) simulation computer package <sup>[30,31]</sup>. @Risk is a commercial package which allows analysis of technical construction situations impacted by risk. It is a software which “adds-in” to Microsoft Project & Excel or Lotus 1-2-3. @Risk uses a technique of ‘simulation’ to combine all of the uncertainties identified in a model. Both Monte Carlo and Latin Hypercube sampling are supported by the software, but only Monte Carlo simulation was used in the present investigation.

@Risk provides over thirty probability distribution functions that allow the specification of nearly any type of uncertainty. The common distribution types such as Normal, Log-Normal, Beta, Triangular and Uniform, are all included.

During simulation the value of an operation is influenced by the effects of uncertainty by being chosen at random from a range of possibilities. The total project duration and cost are calculated from these randomly chosen values. This represents only one possible way in which the project may proceed. The whole process of choosing duration and cost under uncertain conditions is repeated and the result calculated to produce a different answer. Each calculation is known as an ‘iteration’. @Risk allows any number of iterations in a simulation.

The results generated in a simulation are presented in histograms, cumulative curves, summary graphs for cell ranges and zooming. Statistical reports on generated distributions and probability of occurrence for target values in a distribution can also be displayed.

All of the simulations in the present research were performed using an IBM-compatible 586 computer with 16 Megabytes of R.A.M. and a 133 MHz processing chip. A simulation of an activities network of about 60 activities (examples in

Chapters 3 and 4) with 100,000 iterations using a combination of @Risk, Microsoft Project and Excel takes approximately 4-5 hours to be completed while a simulation performed by @Risk and Excel only (examples in Chapter 5) with 100,000 iterations takes about 40 minutes to be completed.

## **2.4 Literature review of network analysis in construction**

### **2.4.1 Introduction**

When attempting to determine or predict the completion date for any project it is necessary to program all the activities that make up the project. In order to analyse risk affecting project duration, project network techniques (PNT) provide a basis for such an assessment. This section starts with the history of the PNT, then focuses on the quantitative methods used in analysis of project duration. Some most popular methods used in network analysis such as CPM, PERT, PNET and MCS methods are reviewed and discussed in detail.

### **2.4.2 History of Project Network Techniques (PNT)**

#### **2.4.2.1 Early development**

The Egyptians, Romans and ancient Chinese worked construction miracles in their day. Surviving ruins attest to the brilliance of their architecture, but of their construction planning and scheduling, we know little. It can be supposed that they solved many scheduling programs by the “use a bigger whip” philosophy. Project management has other roots reaching back into the days before the pyramids. Historical project managers included Noah, Solomon, and the unknown architect who designed the Tower of Babel. History records much about the construction details, but little about the methods of control. In the mid-nineteenth century, at least one writer discussed a work-versus-time graphical representation very similar to today’s bar charts. However, it remained for Henry L. Gantt and Frederick W. Taylor in the early 1900s to popularise their graphical representations of work versus time <sup>[32]</sup>. Their “Gantt charts” were the basis for today’s bar graphs or bar charts. The work of Taylor and Gantt was the first scientific consideration of the problem of work scheduling. Although this work was originally aimed at production scheduling, it was readily

accepted for planning construction and recording its progress. The bar graph was, and is, an excellent graphic representation of activity. It is easily read and understood by all levels of management and supervision.

#### 2.4.2.2 Gantt chart

In the Gantt chart, the time that an activity should take is represented by a horizontal line, the length of the line being proportional to the duration time of the activity. In order that several activities can be represented on the same chart, a framework or ruling is set up, giving time flowing from left to right, the activities being listed from top to bottom <sup>[33]</sup>. See Figure 2.1:

Activity	Week number													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
A														
B														
C														

**Figure 2.1 The Gantt chart**

Figure 2.1 reveals quite clearly how work should progress. Activities (A, B and C) must be carried out in sequence and by the end of week 8, the whole of activity A and B should be completed.

To show how work is actually progressing, a bar or line can be drawn within the uprights of the activity symbol, the length of the bar representing the amount of work completed. Thus, if 50 per cent of an activity is complete, then a bar half the length of the activity symbol is drawn.

Figure 2.2 shows a Gantt chart combining planning and recording progress. When it is viewed at the end of week 7 (denoted by two small arrows at the top and the bottom of the chart), then the following information is readily apparent:

Activity A should be complete and, in fact, is so.

Activity B should be 80 per cent complete, but, is only 20 per cent finished.

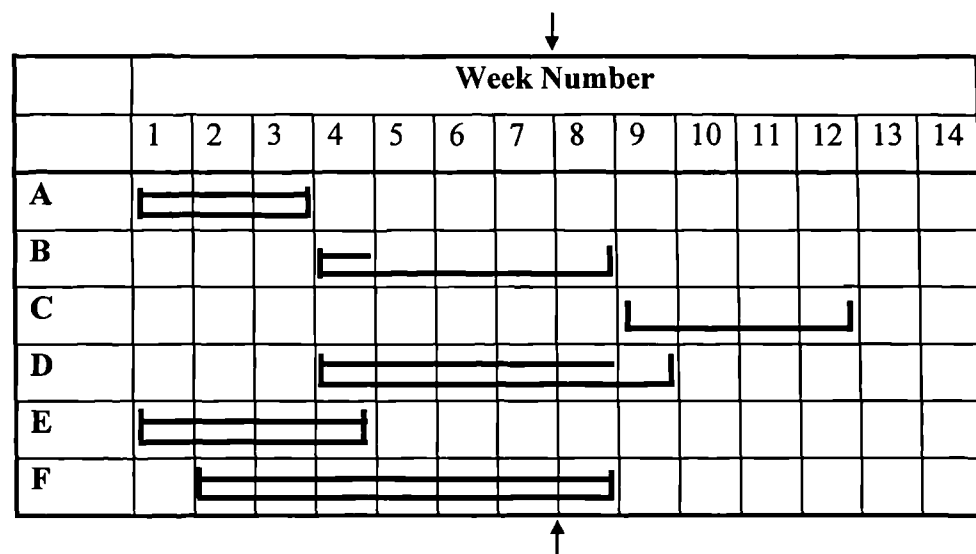
Activity C should not be started and, in fact, is not started.

Activity D should be 66 per cent complete and, in fact, is 83 per cent finished.

Activity E should be complete and, in fact, is so.

Activity F should be 85 per cent complete and, in fact, is already finished.

Thus we see that incomplete bars to the left of the cursor mean under-fulfilment, while those to the right mean over-fulfilment.



**Figure 2.2 Gantt chart combining planning and recording progress**

By the use of codes and/or symbols, the reasons for any delays can be displayed, and the whole chart can be very succinctly informative, combining both planning and recording progress. For many tasks the Gantt chart is unsurpassed, and its use has been very highly developed.

Though valuable, the Gantt chart presents two major difficulties, one concerning the problem of inter-relationships, the second that of needing to take several decisions simultaneously.

The problem of inter-relationships arises when for example, we consider activity E in Figure 2.2. It is shown here to start at the beginning of the project. However, it may be that there is another requirement, namely that it must be complete before activity C

can start. This means that activity E can in fact 'slide' 4 weeks without detriment to the whole project. There may also be another activity, K (not shown in Figure 2.2), which can start only when F is complete. How can these inter-relationships be displayed? It is possible, in small-scale work to 'tie' bars by dotted lines, but if more than a few activities are concerned the chart becomes so muddled as to be useless.

The problem of simultaneous decisions can be seen that when locating an activity on a Gantt chart, three simultaneous decisions have to be made:

1. Method (logic): activity C is shown to follow activity B, that is, a decision on the way the project is to be carried out has to be made - the logic has to be decided.
2. Time: any activity bar has a length, that is, a decision has been made on the time that each activity will occupy.
3. Resources: locating an activity in a position implies that resources are available to carry out the activity.

All projects have these three dimensions - logic, time, resources - and each is equally important. To require a planner to make decisions on these three features at one time is to set an impossible task, yet this is what is required when a Gantt chart is drawn. In practice the decisions must be taken serially.

The Gantt chart is now more generally referred to as a 'bar chart' - from which it was derived.

#### **2.4.2.3 Families of PNT**

The middle 1950s and the 1960s saw an explosion of interest in the problem of planning, and the family of methods, project network techniques (PNT), was born and very rapidly developed. Essentially, these techniques involve representing the proposed project by a diagram (or 'model') built up from a series of arrows and nodes (boxes or circles). The original structure of the model depends only upon the proposed method of proceeding and it is drawn in such a way that the logic is easily displayed and tested.

Broadly, there are two PNT families, the activity-on-arrow (AoA) family, where an activity is represented by an arrow, and the activity-on-node (AoN) family, where an activity is represented by a box or node. These two families each have their own advantages and disadvantages. The activity-on-arrow (AoA) system is more generally known as critical path analysis (CPA) or program evaluation and review technique (PERT). The activity-on-node (AoN) family are also termed as precedence networks.

In Great Britain Andrew <sup>[34]</sup> of I.C.I. was using the 'controlling sequence duration' for scheduling maintenance as early as 1955, and the Operational Research (O.R.) Section of the General Electricity Generating Board investigated the problems concerned with the overhaul of generating plant - a task of considerable complexity which was increasing in importance as new high-performance plant was being brought into service. By 1957 the O.R. Section had devised a technique which consisted essentially of identifying the "longest irreducible sequence of events," and using this technique they carried out in 1958 an experimental overhaul at a power station which reduced the overall time to 42 per cent of the previous average time for the same work. Continuing to work upon these lines the overhaul time was further reduced by 1960 to 32 per cent of the previous average time. the rather clumsy name, "longest irreducible sequence of events," was soon replaced by the name, "major sequence", and it was pointed out, for example, that delays in the "major sequence", would delay completion times, but that difficulties elsewhere need not necessarily involve extensions in total time. This work of the O.R. group was not made public, although comprehensive reports were circulated internally which foreshadowed much later work carried out elsewhere. In France, Roy <sup>[35]</sup>, began work on his 'method of potentials' in 1958 and had perfected a working method by 1960. Wille <sup>[36]</sup> had told how a military air base was constructed in Germany by Siemens & Halske under network control. Nevertheless, it is generally agreed that the main impetus was generated in the United States.

At much the same time similar developments were being undertaken in the U.S.A., the technique has developed along two parallel streams, one military, the other industrial.

In 1956, the E.I. du Pont de Nemours Company set up a group at its Newark, Delaware, facility to study the possible application of new management techniques to the company's engineering functions. One of the first areas considered was the

planning and scheduling of construction projects. The group had a UNIVAC I computer at its disposal and decided to evaluate the potential of computers in scheduling construction work. Mathematicians worked out a general approach, theorising that if the computer was fed with information on the sequence of work and the length of each activity, it could generate a schedule of work.

In early 1957, the UNIVAC Applications Research Centre, under the direction of Dr. John W. Mauchly, joined efforts with James E. Kelley, Jr., of Remington Rand (UNIVAC) and Morgan Walker of Du Pont in direct charge at network. The original conceptual work was revised and the resulting routines became the basic CPM. It is interesting that no fundamental changes have been made in this first work.

In December, 1957, a test group was set up to apply the new technique, then called the Kelley-Walker method. The test team (made up of six engineers, a process engineer, and an estimator) and a normal scheduling group were assigned to plan the construction of a \$10 million chemical plant in Louisville, Kentucky.

By March, 1958, both the normal and test planning groups had completed their work. The effort expended by each group was about equal. At this point, the company decided that the work would be done by contractors rather than by Du Pont forces. Also, a number of design changes were introduced into the schedule, for which the normal planning group expended about 40 percent of its original effort. The test (CPM) team needed to use only 10 percent of its original effort in re-planning.

Following the first test case, a \$2 million project was planned by the test team. This effort was completed in July, 1958, and management now acknowledged the potential of the new system. However, the first two projects would not be completed for some time. To get a quick field test, the method was applied to an equipment turnaround. In this particular turnaround, the product was a self-detonating neoprene intermediate. This meant that the equipment could not be maintained while the unit was shut down, purged, and maintained. The turnaround had been done many times before and the average time for the shutdown was 125 hours. Using the CPM plan, the overhaul was accomplished in 93 hours. This was starting, to say the least. CPM had been used on

subsequent turnarounds of the same unit and the shutdown time had been reduced to 74 hours.

These tests at Du Pont are the only examples of a controlled experimental application of CPM that have been published. (The 1959 Proceedings of the Eastern Joint Computer Conference included a complete report by Messrs. Kelley and Walker) <sup>[37]</sup>.

In 1958 Dr. Mauchly formed Mauchly Associates and was joined that year by Kelly and Walker. Du Pont continued to use and develop CPM for its own purposes. Through public and company-organised courses, particularly in the cost and resource allocation areas. Gradually other firms came into the CPM picture. Since the development of CPM has been funded by private industry on a competitive basis, paradoxical situations have existed and still do. Firms, universities and consultants have been retained to develop CPM - based routines which are already operational. None of this is by design. Rather, it illustrates the complexity of reaching the construction industry and the owners which it serves. The developers of CPM expected instant success because the system was logical, common sense, and non-complicated but the first years were uphill. No doubt Dr. Pasteur found it hard to sell (or give away) the first antirabies vaccines <sup>[32]</sup>. Even today, after literally thousands of key people have been directly exposed to CPM, it is difficult to understand why only a minority of the construction industry utilise the technique.

In January, 1958, development of PERT (Programme Evaluation Research Task as it was first called ) was originated by the Special Projects Office of the Navy Bureau of Ordinance, which was charged with the overall management of the Polaris missile program. At this time the Polaris program was already well under way and the problem facing the Special Projects Office (SPO) was monitoring and controlling the program. There were already more than 3,000 contractors and agencies working on the program. If all these people delivered on contractual schedule, the program was sure to be complete on time. To co-ordinate the wide-flung efforts of the contractors, the SPO had instituted weekly meetings of key personnel. However, the thought that a small contractor might be late with a minor but vital hardware assembly was a constant spectre. It was with this in mind that the SPO wanted an organised and thorough



method of keeping in constant touch with all components. The magnitude of the task suggested the usefulness of a computer-oriented approach.

The SPO-PERT originating team included personnel of Navy SPO, Lockheed, and the firm of Booz, Allen, and Hamilton. In July, 1958, a phase 1 report <sup>[38]</sup> was issued by this team. This report outlined the theoretical basis for the technique and proposed the method of application. In September, a phase 2 report <sup>[39]</sup> put forth detailed procedures for the use of PERT, and PERT was imposed upon the first Polaris contractors on October 16, 1958.

The first step in the application of PERT to Polaris was the identification of key events, or 'milestones', which had to be met to complete Polaris on schedule. Minor milestones within the major milestone areas were identified next. These milestones were monitored biweekly to evaluate the program status.

The Navy credits PERT with helping to complete Polaris ahead of schedule. This is particularly meaningful when we consider that the average weapons system contract exceeds the original schedule by 36 percent <sup>[32]</sup>.

On June 1, 1962, the Secretary of Defence noted, "Very striking improvements in time and cost control have been resulting from these procedures (i.e., critical path scheduling and PERT) and it is essential that we encourage their application in the future." Accordingly, the contractor working on government construction may encounter either PERT or CPM. National Aeronautics and Space Administration (NASA) in their description of NASA-PERT indicated that CPM is an equivalent <sup>[32]</sup>.

The term 'activity-on-node' (AoN) networking implies not a single system of networking but a family. The form of 'activity-on-node' is also known as precedence networks. The activity description is shown in a box or oval, with the sequence or flow still shown by interconnecting lines. In some cases, arrowheads are not used, although this leaves more opportunity for ambiguous network situations.

Of these AoN systems possibly the best known and most used is the ‘method of potentials (MoP)’ by Roy <sup>[35]</sup>, although these are indications that IBM’s ‘precedence diagramming’ is becoming popular. MoP is the simplest technique.

Professor John Fondahl of Stanford University, who was established in the early 1960s as an expert on non-computerised solutions to CPM and PERT networks, was one of the early supporters of the precedence method, terming it ‘circle and connecting arrow technique’. Professor Fondahl’s study <sup>[40]</sup> for the Navy’s Bureau of Yards and Docks included descriptive material and gave the technique early impetus, particularly on Navy projects.

An IBM brochure credits the H.B. Zachry Company of San Antonio with the development of the precedent form of CPM. In co-operation with IBM, Zachry developed computer programs which can handle precedence network computations on the IBM 1130 and IBM 360. This is particularly significant, since in 1964, Phillips and Moder <sup>[41]</sup> indicated the availability of only one computerised approach to precedence networks versus 60 for CPM and PERT.

The Precedence networks (AoN) actually was recast from CPM but with substantially different in appearance. Simplicity of form is one of the great advantages of precedence networks over CPM. Freedom from the need to introduce dummies is one of the most frequently cited advantages of AoN networking while accepting this considerable benefit it must be pointed out that an AoN diagram is likely to be larger, and appear more complex than the equivalent AoA network. Since in network analysis (calculating the duration of network) both AoA and AoN provided essentially the same calculated result, the next of this thesis is pinpointed to discuss AoA network analysis.

#### **2.4.2.4 AoA networks**

As mentioned in last section, project network techniques (PNT) can broadly be classed to two families: activity-on-arrow (AoA) family and activity-on-node (AoN) family. In the AoA system, sometimes is called arrow diagram, more generally known as critical path analysis (CPA) or program evaluation and review technique (PERT). Since the CPM and PERT become the most popular methods in network analysis

(calculating the duration of network), it is also that both AoA and AoN provided essentially the same calculated result, the AoA network system is discussed in this section: first the basic elements, then the dummy, and how to construct the networks are described in details.

An arrow diagram is built up from only three main symbols: full arrow which represent activities, nodes which correspond to events, and dummy arrows for logical sequences.

An activity, is an element of the work entailed in the project. In some instances the “work” is not real in the sense that neither energy nor money is consumed, and in some cases (see dummy activities below) no time is used. However, ignoring these last cases, an activity is a task which must be carried out. Thus, ‘waiting for delivery of component X’ is an activity just as much as is ‘making component Y’, since both are tasks that must be carried out. This ‘non-work’ aspect of some activities is sometimes found difficult to accept until the test of needfulness to the project is applied. Once this test is applied it is clear that waiting for delivery is an activity in the sense in which the word is used in drawing networks.

Activities are represented by arrows, the arrow-heads being at the completion of the activities. The length and orientation of the arrow are of no significance whatsoever, being chosen only for convenience of drawing. It is equally not essential that arrows should be straight, although it will be found that the appearance of the whole diagram will be improved if the main portion of each arrow is both straight and parallel to the main horizontal axis of the paper on which the diagram is drawn. This will often require that arrows are ‘bent’. The description of the activity should always be written upon the straight portion of the arrow.

An event, which is the start and/or finish of an activity or group of activities. The essential criterion is that a definite, unambiguous point in time can be isolated - a broad band of availability is of no use. The word ‘event’ may be misleading here, since there may in fact be a concurrence of a number of separate events, and for this reason some authorities prefer the terms ‘node’, ‘junction’, ‘mile-stone’ or ‘stage’. In general, ‘milestone’ is reserved for particularly significant events that require special

monitoring. 'Node' is possibly the most generally used term, and will be used subsequently.

Events are represented by labels, usually within convenient geometrical shapes - often circles. The node at the beginning of an activity is known as a 'tail' or 'preceding' node, while that at the conclusion of an activity is known as a 'head' or 'succeeding' node.

In some cases it is necessary to draw 'dummy' activities, *that is activities which do not* require resources but may in some cases take time. A dummy activity is always subject to the basic dependency rule that an activity emerging from the head node of another activity depends on that activity.

There are three occasions when dummies are used: identity dummies, logic dummies and transit time dummies. Dummies are drawn as broken or dotted arrows.

The arrows are arranged to show the plan or logical sequence in which the activities of the project are to be accomplished. This is done by answering the following questions with each arrow:

1. What arrows (activities) must precede this one?
2. What arrows (activities) can be concurrent with this one?
3. What arrows (activities) must follow this one?
4. What controls the start?
5. What controls the finish?

Basically, the representation of events and activities is governed by one, simple, dependency rule which requires that an activity which depends upon another activity is shown to emerge from the head of the activity upon which it depends, and that only dependent activities are drawn in this way.

This dependency rule gives rise to two fundamental properties of events and activities:

1. An event cannot be said to be realised (or 'be reached' or 'occur' ) until all activities leading into it are complete.
2. No activity can start until its tail event is realised.

These two statements can effectively be combined into a single comment, namely that 'No activity may start until all previous activities in the same chain are complete'. It must be understood, however, that this single statement has two facets as set out in 1 and 2 above.

There are two conventions usually adopted in drawing networks and, like all conventions, they may be ignored if circumstances warrant. The conventions are:

1. Time flows from left to right.
2. Head nodes always have a number higher than that of the tail node.

It is useful to realise that the head and tail labels of the activities effectively specify the logic of the diagram, and that from a list of head and tail labels the network can be constructed.

### **2.4.3 Critical Path Method (CPM)**

The most commonly used method in network analysis is the well-known Critical Path Method (CPM). Once an AoA network is constructed, the network analysis can be carried out. The systematic analysis of a network sorts out the individual activities into two main classes, critical and non-critical.

The main goal of network analysis is to find out the total project time (TPT). The TPT is the shortest time in which the project can be completed, and this is determined by a sequence (or sequences) of activities known as the critical path (or paths). Precisely, a critical path in a network is that continuous sequence of activities which produces both the maximum and the minimum duration time for the whole network.

The simplest way to calculate the TPT is to enumerate all the paths in a network and the critical path can then be found. However, this total path-enumeration analysis will not be useful in large networks and should be limited to small problems. The critical path in a small network can be found fairly easily by trial and error. For larger networks, a systematic procedure is needed.

The Critical Path Method is a methodology that through study the earliest and latest time of each activity or event in a systematic way that the TPT can be calculated, it also means that the critical path can be identified. In the other words, the project duration can be estimated.

#### **2.4.3.1 CPM calculations**

To determine the project duration, carry out a forward pass whereby the earliest starting times (EST) for each activity is calculated. In the calculation it will sometimes be necessary to refer to the earliest finishing time (EFT) of an activity, given by:

$$\text{Earliest finishing time} = \text{earliest starting time} + \text{duration}$$

The critical path is then identified by carrying out a backward pass whereby the latest finishing time (LFT) of an activity and its associated latest starting time (LST) are calculated, given by:

$$\text{Latest starting time} = \text{latest finishing time} - \text{duration}$$

The above activity times are indirectly derived from the forward and backward passes which directly give the event times for the nodes. A node has two times associated with it, one, from the forward pass, its earliest event time (EET), the earliest time the event can be realised, the other, from the backward pass, its latest event time (LET), the latest time by which the event must be realised if the total project time is to be achieved. The EET is the EST of all emerging activities, whilst the LET is the LFT of all entering activities.

The critical path in a network passes through all the events for which the EET and LET are the same. However, this is a necessary but not a sufficient condition<sup>[42]</sup> and float is the only means of identifying the critical path<sup>[33]</sup>. Detailed CPM calculations can be found in many text books such as by O'Brien<sup>[32]</sup>, Battersby<sup>[42]</sup> and Lockyer and Gordon<sup>[33]</sup>.

#### **2.4.3.2 Advantages and disadvantages of CPM**

The main advantage of the CPM is that it is very simple and effective. A variety of computer programs for CPM are available, such as “Microsoft Project”, in which the computer needs only the activity numbers, duration of each of activity and the dependencies of these activities. From this it can mathematically construct the network, the TPT can be calculated very quickly and also the floats, critical path (or paths) and time-scaled bar chart can be given as part of the output.

The main disadvantage of CPM is that it assumes that the duration of activities are deterministic and as such the project duration estimated from CPM using the activity network is invariably deterministic. In the construction industry, in most projects, the durations of the various activities are not always predictable. The activity durations should therefore be modelled as random variables, and evaluation of the project time considered as a problem of probabilistic network analysis. Some of the probabilistic scheduling methods for risk analyses are reviewed below.

#### **2.4.4 Program Evaluation and Review Technique (PERT)**

The Program Evaluation and Review Technique (PERT) <sup>[38,39,43]</sup> was developed in parallel with the CPM and they are essentially the same on arrow networks, with the only difference being a measure of the uncertainties involved. PERT relies on a formula for combining the estimates of three cases for an activity, namely:

- An optimistic time, which is considered to be the ‘best’ time given that all associated factors fall into place.
- A pessimistic time, which is the ‘worst-case’ scenario, with everything going wrong which could go wrong.
- A most likely duration, which is the ‘normal’ time for the activity, based upon expert judgement, experience or other factors.

The PERT group assumed that these three estimates would fall on a bell-shaped (Beta) curve but there was no proof available for this assumption. The choice of the Beta distribution is not justifiable on experimental grounds, but it is computationally easy to

handle, and its users state that it gives significantly useful answers. When PERT is applied to the three time estimates above, it can be shown that:

$$t_e = \frac{(a + 4M + b)}{6} \quad (2.36)$$

$$s = \frac{(b - a)}{6} \quad (2.37)$$

where  $t_e$  = expected duration,  $a$  = optimistic duration,  $M$  = most likely duration,  $b$  = pessimistic duration and  $s$  = standard deviation.

PERT uses the Central Limit Theorem (CLT) to find the expected project duration. The CLT indicates that for independent random variables:

$$E(T) = t_1 + t_2 + \dots + t_n \quad (2.38)$$

$$S^2 = s_1^2 + s_2^2 + \dots + s_n^2 \quad (2.39)$$

where  $E(T)$  = expected project duration,  $t_i$  = expected duration of  $i^{th}$  activity provided the activity lies on the critical path,  $S$  = standard deviation of the project duration and  $s_i$  = standard deviation of  $i^{th}$  activity duration.

The CLT theorem is valid irrespective of the distribution type for each of the activities provided the number of activities is large enough (typically  $n \geq 30$ ) and also provided none of the activities is dominant.

#### 2.4.4.1 Differences between PERT and CPM

It has been pointed out by Battersby <sup>[42]</sup> that PERT has two main features, both connected with probability, which set it apart from the simple CPM:

1. It provides a means of associating probabilities with the estimates of the duration of an activity.



2. It allows the probabilities of individual activities to be combined along with the durations so that the resultant estimate of the project duration is also qualified by probability assessments.

Each of these features demands certain assumptions, in addition to those generally associated with the employment of statistical methods such as the identification of probability with relative frequency. For the above point 2 independence is necessary: that is to say, the duration estimates for any one activity are assumed not to depend on those of any other activity or activities. Variances are assumed to be additive which, given independence, is reasonable. What is less reasonable is the assumption that the addition of variances along the critical path would remain free from interference by non-critical activities.

Assumptions about individual activity durations have provided a rich field for academic exploration. A likely candidate is the well known Beta distribution which has the desirable properties of being contained entirely inside a finite interval such as  $(a, b)$ , and can be symmetrical or skewed depending on the location of the mode,  $m$ , relative to  $a$  and  $b$ . Lacking an empirical basis for choosing a specific distribution, the Beta distribution was historically accepted as a mathematical model for activity duration times in the PERT.

#### **2.4.4.2 Advantages and disadvantages of PERT**

PERT was the first attempt to quantify the uncertainty in activity durations and the project network. Despite the well documented shortcomings of PERT, both in the estimation of uncertainty in activity durations<sup>[44,45,46]</sup> and in project networks<sup>[44,47,48]</sup> it is still suggested by many as the solution to uncertainty in activity duration networks.

The shortcomings of PERT in the estimation of activity durations are the simplifying assumptions in the approximation for expected value which restrict the shape of probability distribution to only one of three, namely those of skewness  $\pm 1/\sqrt{2}$  or 0. The contention of flexibility is therefore illusory<sup>[44]</sup> and Sasieni<sup>[46]</sup> questions the validity of the Beta distribution.

Another drawback of PERT is the assumption that project duration and its uncertainty can be determined by the longest (critical) path. While this assumption gives the maximum expected value for project duration it does not necessarily evaluate the maximum uncertainty because it ignores shorter but more uncertain paths <sup>[48]</sup>.

#### 2.4.5 Probabilistic Network Evaluation Technique (PNET)

The method of Probabilistic Network Evaluation Technique (PNET) was introduced by Ang, Abdelnour, and Chaker <sup>[48]</sup>. The algorithm used by PNET is based on the different modes of failure that a network can have. Failure, in this case, is the completion of a project in a time longer than the target duration. Each path in the network can become a mode of failure. Thus the completion of a project can be delayed by one or more paths in the network. PNET uses the simplified, approximate solution for the combination of modes of failure.

##### 2.4.5.1 Probability of project completion time

Consider a project network with a specified number of activities. Completion of the project obviously requires completion of all the activities, and thus of all the possible paths in the network. That is:

$$(T \leq t) = (T_1 \leq t) \cap (T_2 \leq t) \cap \dots \cap (T_n \leq t) \quad (2.40)$$

and by virtue of de Morgan's law:

$$(T > t) = (T_1 > t) \cup (T_2 > t) \cup \dots \cup (T_n > t) \quad (2.41)$$

in which  $T$  is the actual completion time of the project,  $T \leq t$  is the event of completion of a project in time  $t$  and  $T_1, T_2, \dots, T_n$  are the durations of the respective paths.  $\cap$  represents an intersection and  $\cup$  as a union of events.

From Equation 2.40 we see that completion of a project in time  $t$  will involve all the paths in the network. Theoretically, then, the probability of completing a project in time  $t$ , denoted  $p(t)$ , is:

$$p(t) \equiv P(T \leq t) = P(T_1 \leq t, T_2 \leq t, \dots, T_n \leq t) \quad (2.42)$$

whereas the probability that the project duration will exceed  $t$ , denoted  $q(t)$ , is:

$$q(t) \equiv P(T > t) = P(T_1 > t \cup T_2 > t \cup \dots \cup T_n > t) \quad (2.43)$$

Clearly:

$$p(t) + q(t) = 1 \quad (2.44)$$

The events  $(T_1 > t)$ ,  $(T_2 > t)$ , ...,  $(T_n > t)$  are not mutually exclusive. Moreover, the different paths in a network may have common activities and thus would be correlated (as shown subsequently). For this reason, the events  $(T_1 \leq t)$ ,  $(T_2 \leq t)$ , ...,  $(T_n \leq t)$  are not statistically independent and thus the determination of  $p(t)$  or  $q(t)$  requires the calculation of multiple joint probabilities. Equation 2.41 can be expressed as the union of the following mutually exclusive events:

$$(T > t) = (T_1 > t) \cup (T_1 \leq t, T_2 > t) \cup \dots \cup (T_1 \leq t, T_2 \leq t, \dots, T_{n-1} \leq t, T_n > t) \quad (2.45)$$

then:

$$p(t) = 1 - q(t) = 1 - [P(T_1 > t) + P(T_1 \leq t, T_2 > t) + \dots + P(T_1 \leq t, T_2 \leq t, \dots, T_{n-1} \leq t, T_n > t)] \quad (2.46)$$

When considering individual paths, it can be assumed that the duration of the individual activities in a network are random variables ( $\tau_k$ ,  $k = 1, 2, \dots$ , with means and standard deviations  $\mu_k$ ,  $\sigma_k$ , respectively) and that the activities are mutually statistically independent.

The duration  $T_i$  in  $\pi_i$  is the sum of the durations of the activities comprising path  $\pi_i$ , in which  $\pi_1, \pi_2, \dots, \pi_n$  are a set of all possible paths. Thus:

$$T_i = \sum_{k \in \pi_i} \tau_k \quad (2.47)$$

Accordingly, the mean duration of path  $\pi_i$  is:

$$\mu_{T_i} = \sum_{k \in \pi_i} \mu_k \quad (2.48)$$

and by virtue of statistical independence, the corresponding variance is:

$$\sigma_{T_i}^2 = \sum_{k \in \pi_i} \sigma_k^2 \quad (2.49)$$

Although the individual activities are statistically independent, two different paths may be correlated as a result of common activities. The correlation between any two paths  $\pi_i$  and  $\pi_j$  can be shown to be:

$$\rho_{ij} = \frac{\sum_{k \in (\pi_i \cap \pi_j)} \sigma_k^2}{\sigma_{T_i} \sigma_{T_j}} \quad (2.50)$$

in which  $k \in (\pi_i \cap \pi_j)$  denotes those activities common to paths  $\pi_i$  and  $\pi_j$ . According to Equation 2.50, the correlation between paths is always positive.

#### 2.4.5.2 Bounds and approximations

The completion-time probability,  $p(t)$ , of a project, or the complementary probability,  $q(t)$  is bounded. These bounds are important because, aside from defining the limits of  $p(t)$ , they also provide the basis for the development of the PNET, described subsequently.

From Equation 2.40, it can be seen that:

$$(T_1 \leq t) \cap (T_2 \leq t) \cap \dots \cap (T_n \leq t) \subset (T_i \leq t) \quad \text{for any } i \quad (2.51)$$

Therefore:

$$P(T_i \leq t, \dots, T_n \leq t) \leq P(T_i \leq t) \quad \text{for any } i \quad (2.52)$$

and in particular:

$$p(t) \leq \min_i [P(T_i \leq t)] \quad (2.53)$$

The equality in Equation 2.53 holds when all the paths in the network are perfectly correlated.

Alternatively, since the correlation between any two paths  $\pi_i$  and  $\pi_j$  is always positive,  $\rho_{ij} \geq 0$  we have:

$$P(T_i \leq t | T_j \leq t) \geq P(T_i \leq t) \quad (2.54)$$

Thus:

$$P(T_i \leq t, T_j \leq t) \geq P(T_i \leq t)P(T_j \leq t) \quad (2.55)$$

And by induction:

$$P(T_1 \leq t, T_2 \leq t, \dots, T_n \leq t) \geq P(T_1 \leq t)P(T_2 \leq t) \dots P(T_n \leq t) \quad (2.56)$$

which means:

$$p(t) \geq \prod_{i=1}^n P(T_i \leq t) \quad (2.57)$$

In this case, equality holds when all the paths are statistically independent. The right-hand sides of Equations. 2.53 and 2.57 therefore are, respectively, the upper and lower bounds of  $p(t)$ . Thus:

$$\prod_{i=1}^n P(T_i \leq t) \leq p(t) \leq \min_i P(T_i \leq t) \quad (2.58)$$

According to the previous bounds, therefore, the assumption of statistical independence among all the paths gives conservative estimate, whereas assuming perfect correlations among the paths would yield unconservative results. These may be called, respectively, the most pessimistic and most optimistic predictions of the completion-time. In this sense, PERT always gives the “most” optimistic estimate of a project time, i.e., the resulting  $p(t)$  is too high. This is also evident from Equation 2.46 if  $\pi_l$  is assumed to be the mean critical path.

### 2.4.5.3 Basis of PNET method

From Equation 2.46 we see that successively better estimates of the project completion-time probability can be obtained by considering more paths in the evaluation of  $p(t)$ . In particular, if the paths,  $\pi_1, \pi_2, \dots, \pi_n$ , are arranged in decreasing order of mean durations, i.e. such that:

$$\mu_{\tau_1} \geq \mu_{\tau_2} \geq \dots \geq \mu_{\tau_n} \quad (2.59)$$

then the additional paths in Equation 2.46 may be limited to the major paths (defined as paths with long mean durations). However, unless the major paths are all statistically independent (or perfectly correlated), the improvements suggested in Equation 2.46 still require evaluation of the joint probabilities which are generally not simple to perform.

PNET was developed as an approximate method for estimating  $p(t)$  in which all the major paths are considered, but evaluation of the joint probabilities is avoided. The method is an obvious improvement over the PERT method, and under certain conditions (when paths are/or assumed perfectly correlated) it reduces automatically to the PERT method.

PNET method is based on the following observations:

- 1 The paths with long mean durations and high coefficients of variation will have the greatest significance on  $p(t)$ .
- 2 According to Equation 2.53, if several paths are each highly correlated with a major path, then these paths are “represented” by the same major path, i.e., the

completion-time probability associated with these several paths is represented approximately by that of this major path alone.

- 3 On the other hand, according to Equation 2.57, if several paths have low mutual correlations, the completion-time probability associated with these paths can be approximated with the product of the respective path probabilities.

Therefore, on the basis of these observations, if all the major paths in a network can be divided into subgroups in accordance with their correlations, such that within each subgroup the paths are highly correlated and thus are “represented” by one path, whereas between the “representative” paths where the mutual correlation is low, then the completion-time probability of the project would be given approximately as the product of the completion-time probabilities of the individual “representative” paths. This can indeed be done conveniently for any network, resulting in the PNET algorithm for determining the project completion-time probability,  $p(t)$ .

#### 2.4.5.4 PNET Algorithm

1. Generate the major paths sequentially in order of decreasing mean path durations, starting with the mean critical path,  $\pi_l$ . This process may be done systematically and formally <sup>[49]</sup>. The major paths may be limited to those whose mean path durations are at least a certain percentage of the mean critical path duration.
2. Evaluate the standard deviations for each of the major paths, using Equation 2.49.
3. Identify the “representative” paths as follows:
  - Using Equation 2.50, calculate the correlations  $\rho_{li}$  between the critical path,  $\pi_l$ , and each of the other major paths,  $\pi_i$ . Those paths with  $\rho_{li} > 0.5$  (assuming that  $\rho = 0.5$  represents the transition between high and low correlation) are “represented” by  $\pi_l$ .
  - For those paths with  $\rho_{li} \leq 0.5$ , consider these to be a subset of paths  $\{\pi_j; j = 1, 2, \dots, k\}$  and designate the path within this subset with the longest mean duration to be  $\pi_{ll}$ . Calculate the correlations  $\rho_{llj}$  between  $\pi_{ll}$  and  $\pi_j$  within this subset. Again, those paths among  $\pi_j$  with  $\rho_{llj} > 0.5$  are “represented” by  $\pi_{ll}$ ,

whereas for those paths with  $\rho_{IIj} \leq 0.5$  the procedure is repeated to determine the subsequent “representative” paths.

- The process then leads to the set of “representative” paths,  $\pi_I, \pi_{II}, \pi_{III}, \dots, \pi_N$ .

4. Calculate the probability of the “representative” paths, i.e.:

$P(T_I \leq t), P(T_{II} \leq t), \dots, P(T_N \leq t)$ , assuming a convenient probability distribution for  $T_I, T_{II}, \dots, T_N$  and using the mean and variance evaluated in steps 1 and 2. The project completion-time probability,  $p(t)$ , then is given approximately by the product:

$$p(t) \approx P(T_I \leq t)P(T_{II} \leq t) \dots P(T_N \leq t) \quad (2.60)$$

Observe that if all the major paths are highly correlated with  $\pi_I$ , then Equation 2.60 becomes  $p(t) \approx P(T_I \leq t)$  and thus the PNET method reduces to the PERT method.

The probability of the individual paths  $P(T_i \leq t)$  may be calculated on the assumption of Normal distributions for  $T_i$ . If there are a large number of activities in a path, the path duration will tend toward a Normal variate (by virtue of the Central-Limit Theorem). In the range of probabilities of general interest, say  $0.05 \leq P(t) \leq 0.95$ , the distribution type would not make a significant difference on the calculated probabilities. The proposed PNET method, however, is independent of the distribution function, and thus is equally valid if other distributions for  $T_i$  are used.

#### 2.4.5.5 Advantages and disadvantages of PNET

It was claimed <sup>[48]</sup> that PNET has been verified by results of large-sample Monte Carlo simulations and very close agreement was obtained for the entire range of probabilities of interest. In contrast to the methods of CPM and PERT where only the main critical path is considered, PNET includes all major paths in a network and can be expected to generate better results <sup>[50]</sup>.

However, PNET is an approximate method and the accuracy of the method was found to be varying from “liberal or conservative” <sup>[51]</sup>. It was found that the variance depended on the correlation coefficient,  $\rho$ , with a higher  $\rho$  being more convenient in



some networks but giving more conservative results. As there is no evidence as to why the correlation coefficient should be 0.5, the parametric choice of  $\rho$  requires further study and it has been suggested that the problem may be solved by using two different values of  $\rho$  and using judgement to arrive at an appropriate answer.

Another disadvantage is that PNET requires path enumeration and in large network, the number of paths can be very large. A modification to the PNET was proposed by Oday Al-sadek and David G. Carmichael <sup>[50]</sup>, by introducing dimensionless factors ( $P_i$ ) whose values correspond to the correlation coefficient between the critical path and the  $i^{\text{th}}$  path.  $T$  in Equation 2.60 then becomes:

$$p(t) \approx \frac{P(T_1 \leq t)}{1 - \rho_1} \frac{P(T_{11} \leq t)}{1 - \rho_2} \dots \frac{P(T_N \leq t)}{1 - \rho_N} \quad (2.61)$$

where any of the terms on the right hand side are greater than 1, these terms are set equal to 1. The use of Equation 2.61 over Equation 2.60 reduces the number of computations.

#### 2.4.6 Monte Carlo Simulation (MCS)

Monte Carlo simulation in the analysis of networks appeared since the 1960s by authors such as Van Slyke <sup>[52]</sup>, McGowan <sup>[53]</sup> and others <sup>[54,55,56,57]</sup> and they all used a simulation procedure. Before running a simulation of the duration of a construction network the probability distribution function of each activity is determined or assumed. During each replication in the simulation, random values are assigned to the probability of completion of the activities. Once the probability of completion of the activities are known, their durations can be determined by the expected duration and the standard deviation of the critical path. Thereafter, the duration of each path is found by summing up the durations of all activities in the path. The network duration is the duration of the longest path <sup>[58]</sup>.

#### **2.4.6.1 Problems encountered in MCS method**

The problems encountered in Monte Carlo simulation mainly are the choice of the probability distribution function of the individual activity and the correlation between variables.

The choice of which probability distribution function to use for representing individual activities has provided rich field for researchers. Standard distributions are in favour since only a few parameters are needed to determine activity duration distribution. However, there is no consensus on which standard distribution should be used in construction project activities.

Normal and Log-Normal are the most common standard distributions being used in construction networks because if there is a large number of activities in a path, the path duration will tend toward a Normal variate due to the Central Limit Theorem. As was explained in detail by Sculli <sup>[59]</sup>, the normality assumption for activity durations is quite reasonable and another advantage is their simplicity. It is particularly valuable in view of potential practical implementations, where project network structure and evaluations of activity times may change often during project performance. This should also give occasion to acceptance of the biases arising from the approximations.

The Beta distribution is often recommended because of its flexibility. AbouRizk and Halpin <sup>[60]</sup> show, through their analysis of empirical construction activity duration data, that the Beta distribution is appropriate. AbouRizk, Halpin and Wilson <sup>[61,62]</sup> developed a procedure to fit Beta distributions to construction operations. Triangular and Uniform distributions are also common in practice <sup>[63,64]</sup>. Erlang distribution <sup>[65,50]</sup>, Pearson family <sup>[66]</sup>, Mixture distributions <sup>[67,68]</sup> have also been suggested.

The effects of the correlations between each activity have not been given adequate attention <sup>[20,69]</sup>. It was shown in the works by Pouliquen <sup>[20]</sup> and Wall <sup>[69]</sup> that the effect of correlations between variables is more significant than the effect of the choice of distribution for an individual activity. Their work was limited to showing that ‘with-correlations’ simulations produce results more like the distributions of actual data than ‘without-correlations’ simulations. They did not investigate whether one particular

method of assessing and building correlations into MCS was better than any other. A step by step procedure for estimating the impact of activity correlations on the total project completion time using a simulation approach has not been provided.

#### **2.4.6.2 Advantages and disadvantages of MCS**

The major advantages of MCS is that the results provide an unbiased estimate of the project completion distribution. MCS can provide an almost unlimited capacity to model networks as long as the computers and software are appropriate. In the absence of data, subjective judgement can be applied as different distributions for activities can be assumed, to evaluate the project completion time. Another advantage of MCS is that the method allows the calculation of a criticality index <sup>[50,52]</sup>. The criticality index is the probability that an activity will be on the critical path.

The disadvantages of the MCS are the correlation between activities are difficult to quantify and also that MCS is a time consuming method and expensive in computer time. Two decades ago, researchers had to develop algorithms to simplify the networks and thus reduced the time for the use of MCS <sup>[54,55]</sup>. This has been overcome as the rapid development of computer processing power in recent years and as such MCS has become a much more popular method.

#### **2.4.7 Other methods in network analysis**

Some less popular methods are discussed in this section. These are Narrow Reliability Bounds Method (NRB) <sup>[70]</sup>, Probabilistic Network Analysis by Putcha & Rao <sup>[71]</sup> and Linear Scheduling Technique (LST) <sup>[72,73]</sup>.

The Narrow Reliability Bounds Method (NRB) was developed for structural reliability analysis by Ditlevsen <sup>[70]</sup>, and was earlier applied for scheduling by Laferriere <sup>[74]</sup>.

Like PNET, the NRB model is based on the probability of failure of each path. Failure occurs when the network duration is longer than a predetermined target duration. A failure mode is equivalent to a network path. Each path is considered to be Normally

distribution with an expected duration and standard deviation. For a more detailed discussion of this approach see Ditlevsen <sup>[70]</sup> and Diaz <sup>[58]</sup>.

NRB finds two probabilities of failure for the combination of all existing paths: lower bound probability and upper bound probability. NRB can be used for calculating the probability of failure for target durations larger than the project expected duration. It was found that the results from the upper tail of the NRB are consistent with MCS and those from lower bound are at least as optimistic as those obtained with PNET or PERT <sup>[51]</sup>.

Unlike many authors who consider activities as a continuous probability distribution function Putcha & Rao <sup>[71]</sup> used Rosenblueth's 'two point estimates scheme' <sup>[75]</sup> for representing the randomness of the activities in a network. This is to replace a random variable with point estimates and according to Rosenblueth <sup>[76]</sup> and Lind <sup>[77]</sup>, if  $y = f(x)$  is a function of a random variable  $x$ , which has mean  $m_x$  and standard deviation  $\sigma_x$ , then the mean  $m_y$  and standard deviation  $\sigma_y$  can be calculated as:

$$m_y = \frac{1}{2}(y^+ + y^-) \quad (2.62)$$

$$\sigma_y = \frac{1}{2}(y^+ - y^-) \quad (2.63)$$

where  $y^+ = y(x^+) = y(m_x + \sigma_x)$  and  $y^- = y(x^-) = y(m_x - \sigma_x)$ . The term  $m_x$  is the expected value of  $x$  represented as  $m_x = E(x)$ .

This implicitly assumes that  $y$  is defined for  $x = m_x \pm \sigma_x$ . These are supposed to be exact for a particular distribution composed of two concentrated probability masses, the magnitude of each being 0.5 and are located at  $x^+$  and  $x^-$ . Such a distribution is called Symmetrical Rosenblueth Distribution. The same procedure can be applied for functions of several variables and the detailed procedure is discussed by Rosenblueth <sup>[75,76]</sup>.

In the paper by Putcha & Rao <sup>[71]</sup>, the Rosenblueth method is modified to calculate the final mean and standard deviation of project completion time, which is a random variable and is the maximum of the all possible sequences from start to finish with a given precedence. A scanning method was used and the last set of four points are then reduced to a final two Rosenblueth points corresponding to the maximum duration of all activities for the random variable  $T$ , representing project completion time. It has shown that the results from the method proposed by Putcha & Rao checked with Monte Carlo simulation and also a so call 'Exact' method. Excellent agreement was obtained between those methods <sup>[71]</sup>.

Linear Scheduling Technique (LST) has been developed from a graphical technique <sup>[72,78]</sup> to have the ability to determine a controlling activity path from activities on a linear schedule <sup>[73,79]</sup>. Unlike CPM, the location at which an activity changes from non-controlling to controlling can occur at any point along a linear activity. It was claimed that this capability provides a much more realistic controlling activity path for linear activities than can be obtained from CPM <sup>[79]</sup>.

However, LST can help with float identification, resource and cost allocation, schedule stating and updating but in the sense of predicting the project completion time, it is a deterministic method and risks and uncertainties can not be considered.

#### **2.4.8 Concluding remarks**

The techniques in construction network analysis have been reviewed. It can be seen that MCS has become increasingly popular but more work is needed to investigate the distribution choice for individual activities and the correlations between the variables. Further development of analytical methods can not be neglected as these methods provide approximate, yet acceptably accurate, probabilistic information and have the advantage of computational efficiency which saves computer time and resources.

## CHAPTER 3 COMPARISON OF THE PERT AND MCS METHODS

### 3.1 Introduction

In order to study network analysis in construction, the project duration of two example projects was analysed by using two existing methods, the PERT and MCS methods. As stated in Chapter 2, these two methods are the most common methods used in construction network analysis. When applying the Monte Carlo simulation method, a sensitivity analysis was carried out by investigating the effect of the different probability distributions for individual activity durations, the number of iterations used and the effect of the manner of how mean and standard deviations were set for the different probability distributions. The results from PERT have been compared to those of MCS method. Results and discussion are reported and from this conclusions are drawn.

### 3.2 The PERT method

Full details of the PERT method can be found in Chapter 2. The use of the PERT for calculating the total project duration is straightforward when the estimates of the optimistic time ( $a$ ), the most likely time ( $m$ ) and the pessimistic time ( $b$ ) of each activity in the project are known, which is the case of the two example projects.

The following equations of PERT were used to calculate the expected duration and standard deviation of each activity:

$$t_e = \frac{(a + 4m + b)}{6} \quad (3.1)$$

$$s = \frac{(b - a)}{6} \quad (3.2)$$

The total project duration is then assumed to follow a Normal probability distribution with expected duration  $E(T)$  and variance  $S^2$ :

$$E(T) = t_1 + t_2 + \dots + t_n \quad (3.3)$$

$$S^2 = s_1^2 + s_2^2 + \dots + s_n^2 \quad (3.4)$$

The range of project duration estimated using the PERT method is the mean project duration plus or minus 3 times the standard deviation. This is because it is assumed that the project duration lies within three standard deviations from the mean (99% confidence in a Normal distribution).

### **3.3 The MCS method**

#### **3.3.1 Introduction**

The Monte Carlo simulation methodology can be seen in Chapter 2. In both examples studied, five common standard distributions, namely the Normal, Log-Normal, Triangular, Uniform and Beta distributions were assumed for individual activity durations in order to compare the effects of different probability distributions for individual activity duration in a particular construction project. To simplify the problems, it was assumed that the activities are independent.

The number of iterations was investigated by running the simulation 100, 1000, 5000 and 10,000 times.

Further to this, the effect of the manner of how mean and standard deviations are set for different probability distributions was analysed.

#### **3.3.2 Assessment of the parameters for the different distributions**

Before running a simulation of the duration of a construction network, the probability distribution function of each activity has to be determined or assumed. For the purpose of the present research, five common standard distributions, namely the Normal, Log-Normal, Triangular, Uniform and Beta distributions were assumed for individual activity durations in order to compare the effects of different probability distributions for each individual activity duration in a particular construction project.

The Monte Carlo simulations were carried out using the @Risk computer package [30,31]. In this particular package, probability distribution functions are used for adding uncertainty (in the form of probability distributions) to each of the activity duration. Distribution functions contain two elements; a function name and argument values which are enclosed in parentheses. The type of distribution which will be sampled is given by the name of the function and the parameters which specify the distribution are given by the arguments of the function.

In most construction projects, due to the lack of objective data, subjective data (estimates given by experienced estimators) such as minimum ( $a$ ), most likely ( $m$ ) and maximum ( $b$ ) are used. In the two models investigated,  $a$ ,  $m$ , and  $b$  are used for assessing the arguments (parameters) for different distributions.

#### **3.3.2.1 Normal and Log-Normal distributions**

The Normal and Log-Normal distributions in @Risk are expressed as:

- RiskNormal(mean, standard deviation)
- RiskLognorm(mean, standard deviation)

The parameters to be assessed in the above are the mean and standard deviation.

The three-point estimates are not suitable for assessing the parameters of mean and standard deviation of Normal and Log-Normal distributions. This is because the Normal distribution does not have finite boundaries and the Log-Normal distribution does not have an upper boundary. Due to this, the mean and the standard deviation are assessed by percentile confidence. In the two examples analysed, the confidence is set at 99%.

The parameters are assessed using the PERT procedure. Though the most likely values ( $m$ ) indicate that the individual activities are not symmetric, this conflict is ignored when these individual activities are assumed to have a Normal distribution.

The mean and the standard deviation therefore can be assessed by Equations 3.1 and 3.2.



A further analysis of how the mean and standard deviation should be set (assuming a symmetrical distribution for each activities) is continued in Section 3.3.3.

As the type of function is set by name in @Risk, therefore the mean and the standard deviation are the same for Normal and Log-Normal distributions in both cases.

### **3.3.2.2 Uniform and Triangular distributions**

The three-point estimates are easily applicable to estimate the parameters in both the Uniform and Triangular distributions.

The Uniform and Triangular distributions in @Risk package are entered as:

- RiskUniform(minimum, maximum)
- RiskTriang(minimum, most likely, maximum)

As such, the information (a, m and b) provided by the models can be used directly in the above.

### **3.3.2.3 Beta distribution**

The Beta distribution is expressed in @Risk as:

- RiskBeta(alpha1, alpha2)

In which alpha1 and alpha2 are shape parameters which must be assessed from available data.

In theory, the parameters of the Beta distribution are very difficult to estimate subjectively and as such the Beta distribution is not suitable for modelling with subjective data. Since the three point estimates are not sufficient for determining the Beta distribution which is a four-parameter probability distribution function, an approximate method using PERT procedures is applied to assess the parameters.

For the Beta distribution, the mean and variance are calculated as follows:

$$\mu = a + \frac{\alpha_1}{\alpha_1 + \alpha_2} (b - a) \quad (3.5)$$

$$\sigma^2 = \frac{\alpha_1 \cdot \alpha_2}{(\alpha_1 + \alpha_2)^2 (\alpha_1 + \alpha_2 + 1)} (b - a)^2 \quad (3.6)$$

In the equations above,  $\mu$  and  $\sigma$  are the mean and standard deviation,  $a$ ,  $m$ ,  $b$  are minimum, most likely, and maximum respectively and  $\alpha_1$  and  $\alpha_2$  are the shape parameters.

From these equations and in conjunction with PERT, (Equations 3.1 and 3.2), the shape parameters can be assessed by:

$$\alpha_1 = \alpha_2 \times \left( \frac{\mu - a}{b - \mu} \right) \quad (3.7)$$

$$\alpha_2 = \frac{36 \times \left( \frac{\mu - a}{b - \mu} \right) - \left( \frac{\mu - a}{b - \mu} + 1 \right)^2}{\left( \frac{\mu - a}{b - \mu} + 1 \right)^3} \quad (3.8)$$

It should be noted that the order of the  $\alpha_1$  and  $\alpha_2$  cannot be incorrect when they are placed in the @Risk program, otherwise the distribution generated will be skewed to the right and give more conservative results. This is because the Beta distribution is dependent on the shape parameters and  $B(\alpha_1, \alpha_2)$  is not equal to  $B(\alpha_2, \alpha_1)$ .

### 3.3.3 Altering the manner of setting the mean and standard deviation

This section investigates whether the results of the distributions investigated are truly dependent upon the shape of the distribution or merely on the manner in which the distributions' mean and standard deviation are set.

Further analysis in this section is aimed at assessing the extent to which a shift in the mean of the distribution is responsible for the sensitivity in overall project duration rather than the shape of the chosen distribution. This analysis is based on the following assumptions:

1. By retaining the minimum and maximum values in Table 3.1 and Table 3.2, the most likely values are set as  $(a+b)/2$  for all distributions.
2. For the Normal and Log-Normal distributions:

$$\mu = \frac{a+b}{2} \quad \text{and} \quad \sigma = \frac{b-a}{6}$$

where  $\mu$  is the mean,  $\sigma$  is the standard deviation,  $a$  is the minimum and  $b$  is the maximum value.

3. For the Uniform and Triangular distributions:

The parameters used are minimum and maximum for the Uniform distribution and minimum, most likely and maximum for the Triangular distribution.

4. For the Beta distribution:

The PERT procedures are again used to assess the parameters. As the most likely values are all set as  $(a+b)/2$ , using the Equations 3.1, 3.2, 3.5 and 3.6, the shape parameters can be assessed. Thus:  $\alpha_1 = \alpha_2 = 4$ .

### 3.4 Example project results

#### 3.4.1 Example 3.1 – a house construction project

Example 3.1 is a simple project concerned with the construction of a house. It is chosen from a published work by Risk Decisions Ltd. <sup>[80]</sup>. The project was chosen as an example due to its simplicity and because it contains all of the information needed for analysis, such as the dependency of the activities and the estimates of each activity. Figure 3.1 shows the arrow network for the activities involved. Their dependencies and estimates of the optimistic time ( $a$ ), the most likely time ( $m$ ) and the pessimistic time ( $b$ ) are shown in Table 3.1.

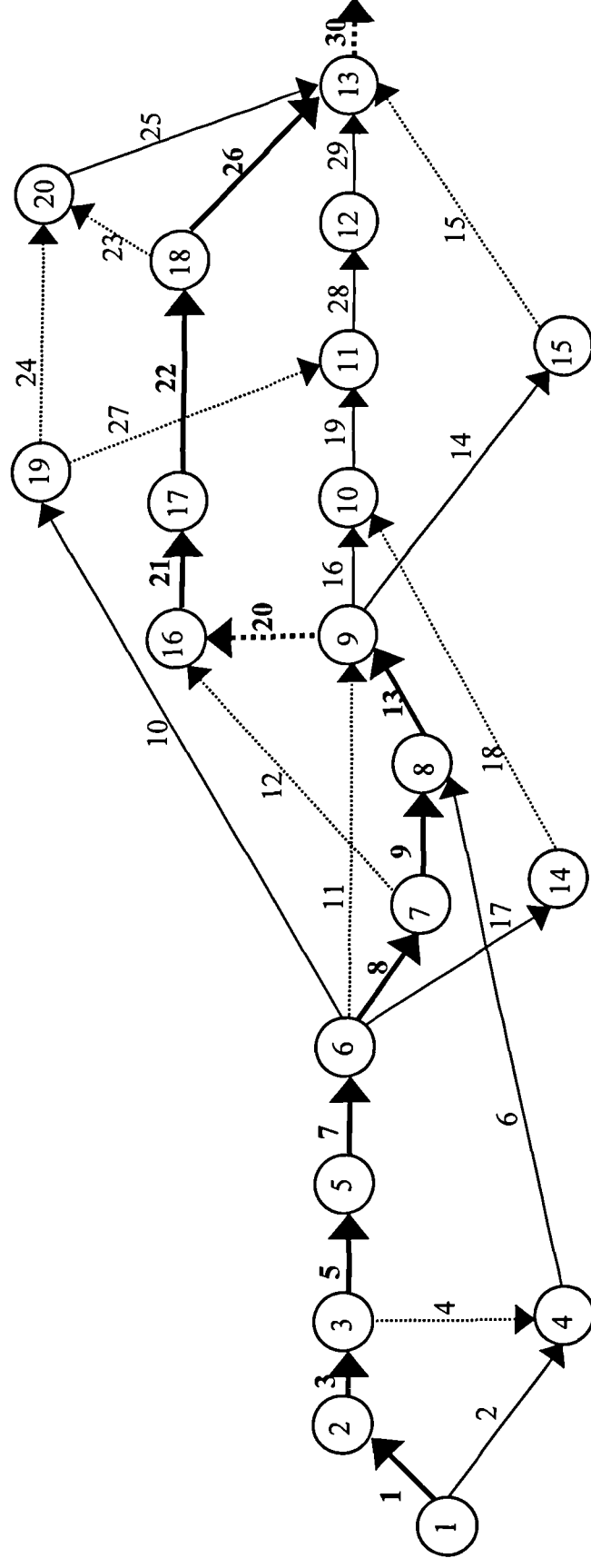


Figure 3.1 The network of Example 3.1 (bold activities are critical)

**Table 3.1 The activities involved in Example 3.1**

Activity		Description	Duration			Predecessors
code	No.		a	m	b	
<b>1-2</b>	<b>1</b>	<b>Clear site</b>	<b>3</b>	<b>4</b>	<b>5</b>	-
1-4	2	Bring utilities to site	10	15	23	-
<b>2-3</b>	<b>3</b>	<b>Excavate</b>	<b>3</b>	<b>5</b>	<b>9</b>	<b>1</b>
3-4	4	Dummy	0	0	0	3
<b>3-5</b>	<b>5</b>	<b>Pour foundation</b>	<b>7</b>	<b>8</b>	<b>10</b>	<b>3</b>
4-8	6	Outside plumbing	18	26	40	2, 4
<b>5-6</b>	<b>7</b>	<b>Frame house</b>	<b>13</b>	<b>14</b>	<b>16</b>	<b>5</b>
<b>6-7</b>	<b>8</b>	<b>Electric wiring</b>	<b>6</b>	<b>8</b>	<b>10</b>	<b>7</b>
<b>7-8</b>	<b>9</b>	<b>Lay floor</b>	<b>1</b>	<b>2</b>	<b>10</b>	<b>8</b>
6-19	10	Lay roof	1	2	15	7
6-9	11	Dummy	0	0	0	7
7-16	12	Dummy	0	0	0	8
<b>8-9</b>	<b>13</b>	<b>Inside plumbing</b>	<b>8</b>	<b>10</b>	<b>11</b>	<b>6, 9</b>
9-15	14	Shingling	4	8	10	11, 13
15-13	15	Dummy	0	0	0	14
9-10	16	Outside sheathing insul.	1	2	3	11, 13
6-14	17	Install windows	2	6	10	7
14-10	18	Dummy	0	0	0	17
10-11	19	Brick work	6	7	8	16, 18
<b>9-16</b>	<b>20</b>	<b>Dummy</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>11,13</b>
<b>16-17</b>	<b>21</b>	<b>Insulate walls + ceiling</b>	<b>8</b>	<b>10</b>	<b>12</b>	<b>12, 20</b>
<b>17-18</b>	<b>22</b>	<b>Cover walls + ceiling</b>	<b>4</b>	<b>6</b>	<b>10</b>	<b>21</b>
18-20	23	Dummy	0	0	0	22
19-20	24	Dummy	0	0	0	10
20-13	25	Insulate roof	2	3	20	23, 24
<b>18-13</b>	<b>26</b>	<b>Finish interior</b>	<b>13</b>	<b>15</b>	<b>18</b>	<b>22</b>
19-11	27	Dummy	0	0	0	10
11-12	28	Finish exterior	10	12	15	19, 27
12-13	29	Landscape	3	5	7	28
<b>13-21</b>	<b>30</b>	<b>End</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>15, 25, 26, 29</b>

Note: Activities in bold are those on the critical path.

### 3.4.1.1 PERT results

The results of PERT are summarised in Table 3.2. The overall properties of the distributions of the project durations and the percentile values are reported in Table 3.15.

**Table 3.2 The project duration analysed by PERT of Example 3.1**

Activity No.	Activities in critical path	Mean duration	Variance	Standard deviation
1	Clear site	4.0	0.11	0.3
3	Excavate	5.3	1.00	1.0
5	Pour foundation	8.2	0.25	0.5
7	Frame house	14.2	0.25	0.5
8	Electric wiring	8.0	0.44	0.7
9	Lay floor	3.2	2.25	1.5
13	Inside plumbing	9.8	0.25	0.5
20	Dummy	0.0	0.00	0.0
21	Insulate walls + ceiling	10.0	0.44	0.7
22	Cover walls + ceiling	6.3	1.00	1.0
26	Finish interior	15.2	0.69	0.8
30	End	0.0	0.00	0.0
<b>Project duration <math>E(T) = \sum t_i</math></b>		<b>84.2</b>		
<b>Project variance <math>S^2 = \sum S_i^2</math></b>			<b>6.69</b>	
<b>Project standard deviation <math>S = \sqrt{S^2}</math></b>				<b>2.6</b>

### 3.4.1.2 Different number of iterations and distributions of MCS

The results from Monte Carlo simulations for Example 3.1 are reported in Appendix A. Figures without a suffix are histograms and Figures with the suffix (a) are ascending cumulative curves.

Figures A.1-A.20(a) show the effect of the number of iterations for several different distributions. These are: Normal (Figures A.1-A.4(a)), Log-Normal (Figures A.5-A.8(a)), Triangular (Figures A.9-A.12(a)), Uniform (Figures A.13-A.16(a)) and Beta (Figures A.17-A.20(a)) distributions.

The number of iterations used with each distribution was: 100, 1000, 5000 and 10000 for both examples.

#### **3.4.1.3 Different distributions with altered means and standard deviations**

Figures A.21-A.24 (Appendix A) show the effect of varying the mean and standard deviation for alternative distributions (Normal, Log-Normal, Triangular and Beta) using a fixed number of iterations (10000) for Example 3.1.

#### **3.4.1.4 Comparison of different iterations**

Figures 3.2-3.6 show the comparison of the different number of iterations in all five of the different distribution forms.

Tables 3.3-3.7 show the equivalent quantitative aspects of the results of the different iterations in all five of the different distribution forms. The upper section of each table denotes the overall properties of the distribution and the lower section denotes the percentile values of the distribution for differing numbers of iterations (100, 1000, 5000 and 10000).

#### **3.4.1.5 Comparison of different distributions forms**

Figure 3.7 shows graphical results for project duration in days after 10,000 simulations for each of the different distributions used.

Table 3.8 shows the quantitative results for project duration in days after 10,000 simulations at varying percentile values for each of the different distributions.

#### **3.4.1.6 Comparison of different distributions when the mean is altered**

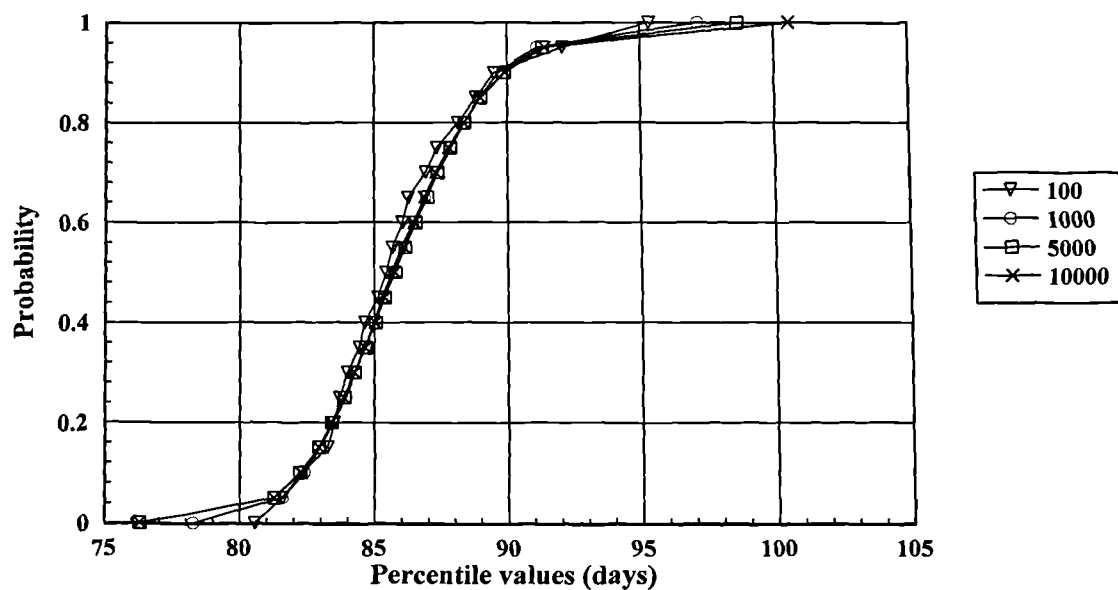
Figure 3.8 shows graphical results for project duration in days after 10,000 simulations at varying percentile values for the different distributions used with alternative means and standard deviations.

Table 3.9 shows the quantitative results for project duration in days after 10,000 simulations at varying percentile values for the different distributions when the mean is altered.

### 3.4.1.7 Comparison of PERT with MCS

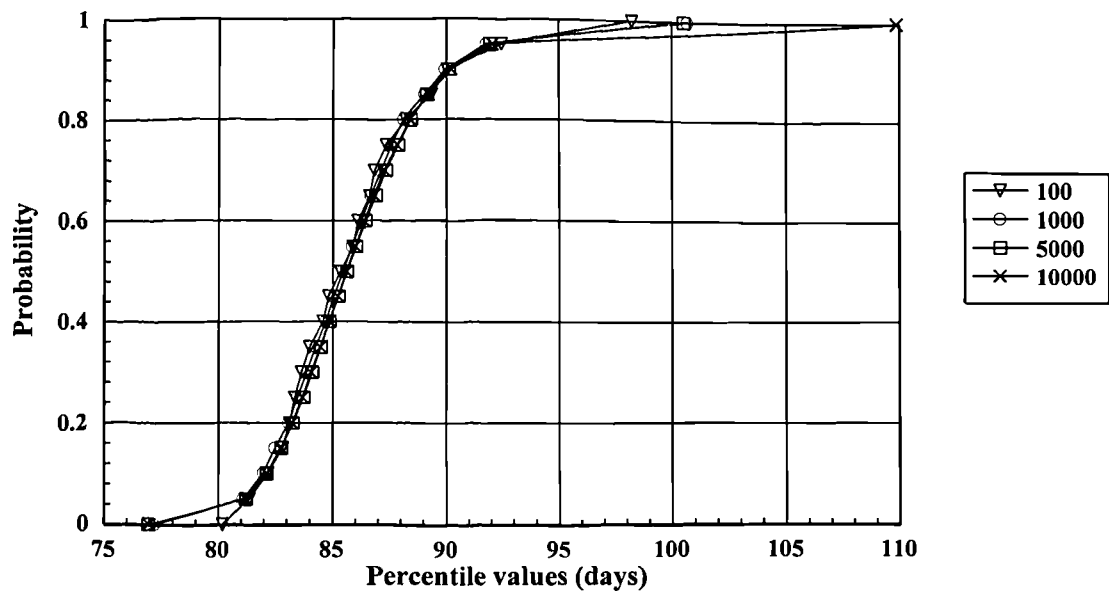
Figure 3.9 shows a comparison of PERT and Monte Carlo simulations with varying distributions and distributions with altered means and standard deviations.

Table 3.8 shows the estimated project duration range calculated using the PERT method and Monte Carlo simulation (at 10,000 simulations) using different distribution forms.

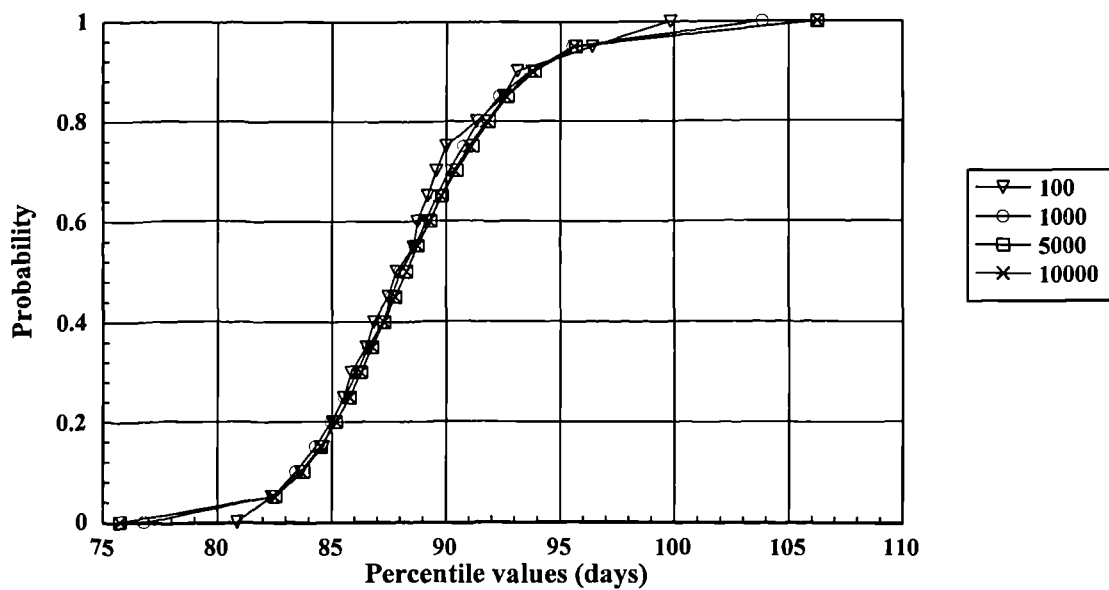


**Figure 3.2 Comparison of different iterations (Normal) Example 3.1**





**Figure 3.3 Comparison of different iterations (Log-Normal) Example 3.1**



**Figure 3.4 Comparison of different iterations (Triangular) Example 3.1**

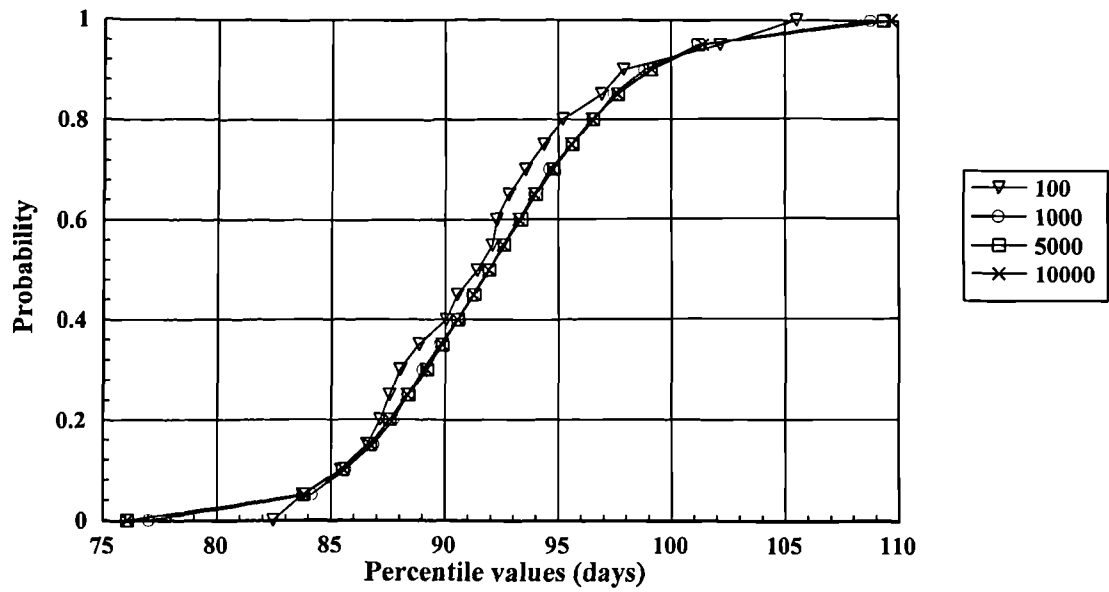


Figure 3.5 Comparison of different iterations (Uniform) Example 3.1

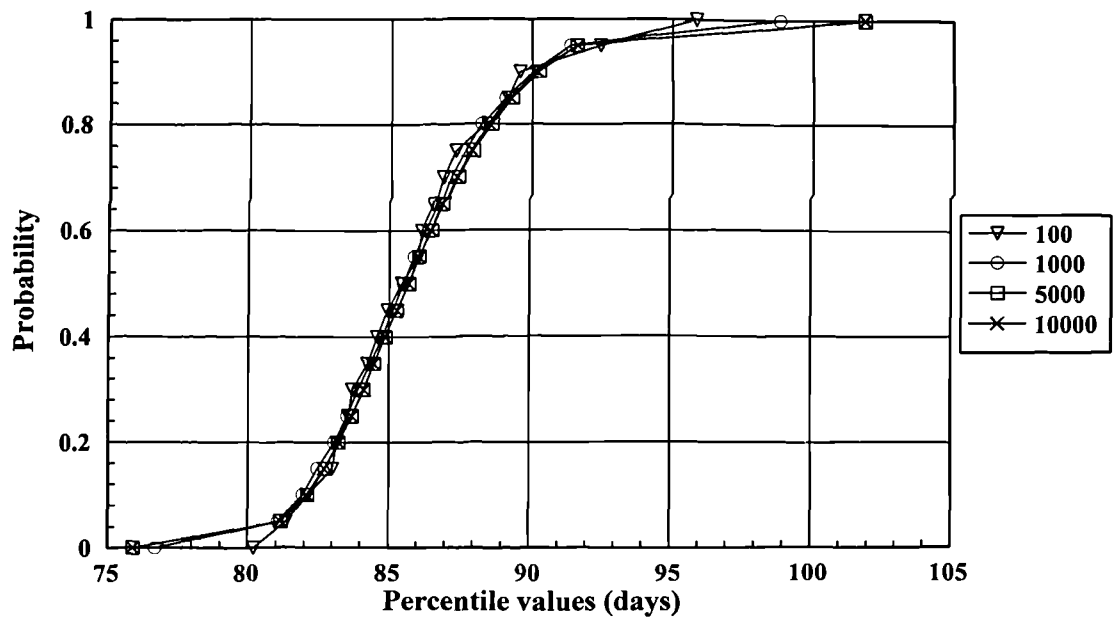
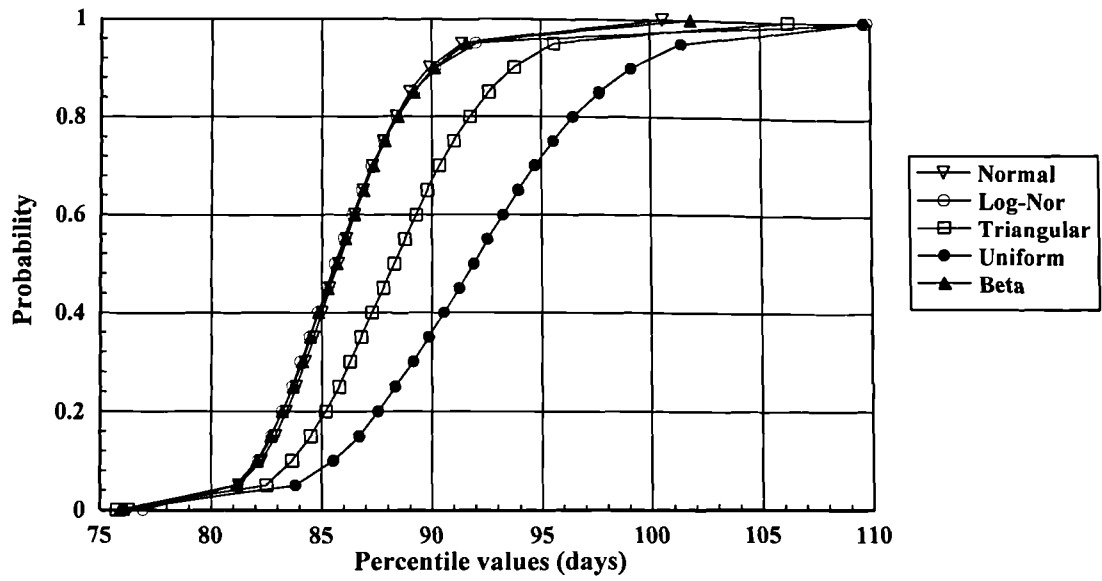
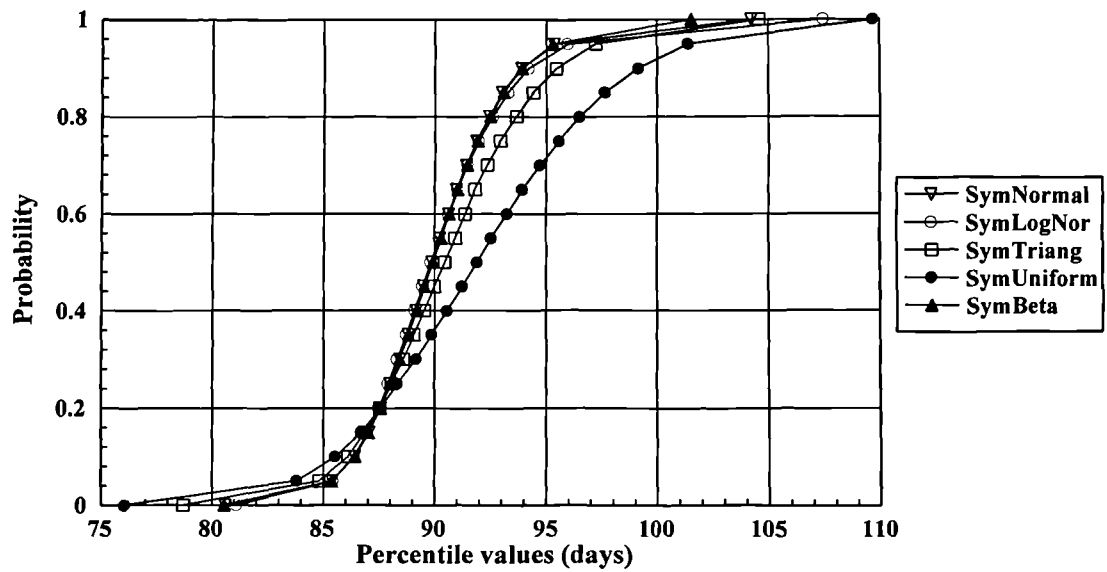


Figure 3.6 Comparison of different iterations (Beta) Example 3.1



**Figure 3.7 Comparison of different distribution at 10000 iterations, Example 3.1**



**Figure 3.8 Comparison of different distribution at 10000 iterations when the mean shifted, Example 3.1**

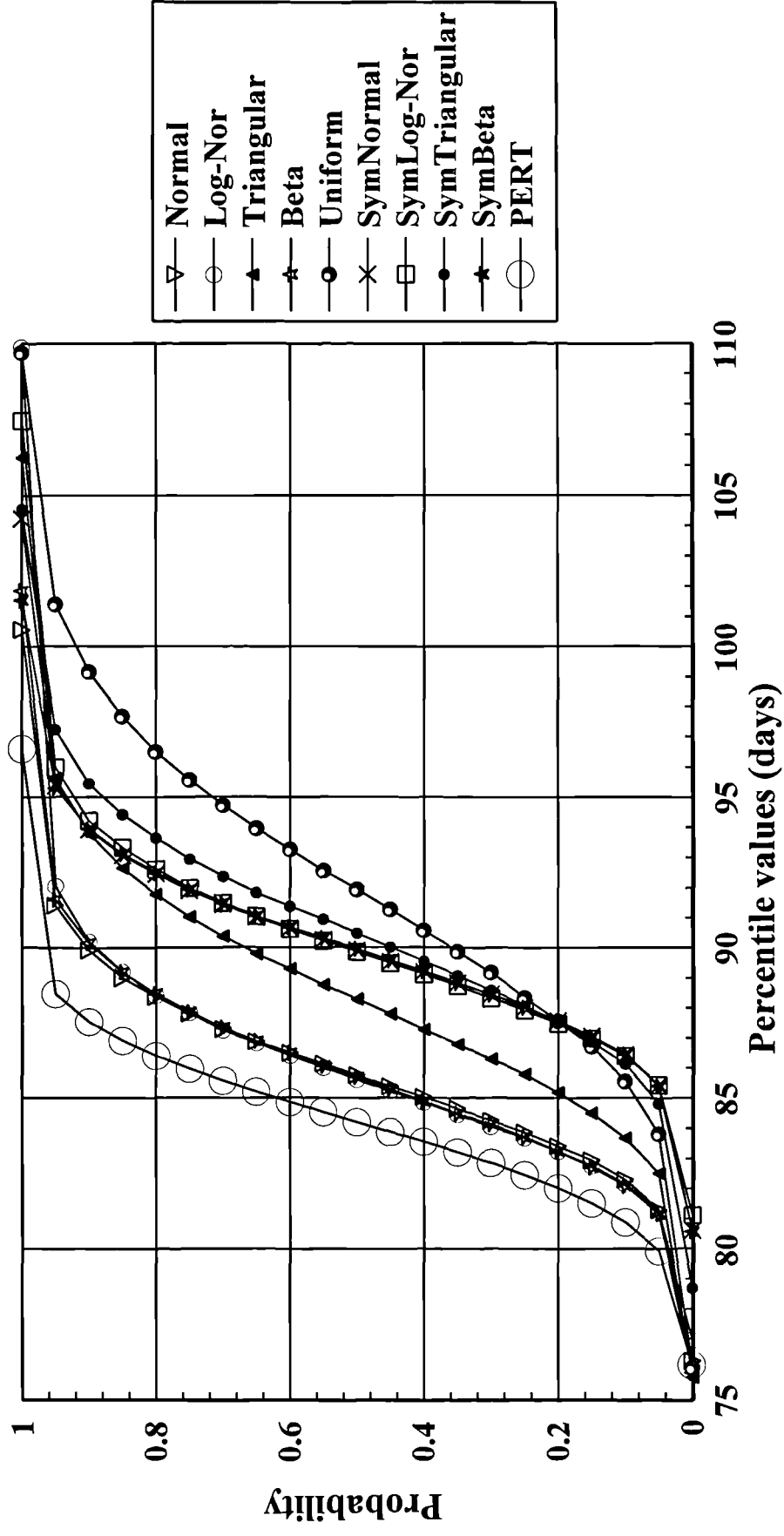


Figure 3.9 Comparison of different distributions at 10000 iterations when the mean before and after altered, Example 3.1

**Table 3.3 Comparison of different iterations (Normal) of Example 3.1**

<b>Iterations</b>	<b>100</b>	<b>1000</b>	<b>5000</b>	<b>10000</b>
<b>Minimum</b>	80.57	78.28	76.29	76.29
<b>Maximum</b>	95.34	97.17	98.64	100.56
<b>Mean</b>	85.90	85.98	85.98	85.96
<b>Std Deviation</b>	3.03	2.98	3.07	3.07
<b>Variance</b>	9.20	8.87	9.41	9.44
<b>Skewness</b>	0.94	0.51	0.42	0.45
<b>Kurtosis</b>	3.97	3.40	3.43	3.48
<b>Percentile Values</b>				
<b>5 %</b>	81.59	81.59	81.29	81.28
<b>10 %</b>	82.34	82.40	82.24	82.28
<b>15 %</b>	83.25	82.97	82.94	82.93
<b>20 %</b>	83.50	83.46	83.42	83.40
<b>25 %</b>	83.71	83.94	83.89	83.86
<b>30 %</b>	84.00	84.28	84.27	84.25
<b>35 %</b>	84.44	84.63	84.67	84.65
<b>40 %</b>	84.65	84.99	85.05	85.02
<b>45 %</b>	85.13	85.29	85.38	85.37
<b>50 %</b>	85.45	85.67	85.78	85.75
<b>55 %</b>	85.68	86.01	86.15	86.13
<b>60 %</b>	86.06	86.45	86.54	86.48
<b>65 %</b>	86.27	86.89	86.98	86.88
<b>70 %</b>	86.94	87.39	87.38	87.31
<b>75 %</b>	87.36	87.92	87.87	87.81
<b>80 %</b>	88.17	88.40	88.41	88.38
<b>85 %</b>	88.83	88.95	89.01	89.01
<b>90 %</b>	89.51	89.68	89.88	89.92
<b>95 %</b>	92.08	91.14	91.33	91.41

**Table 3.4 Comparison of different iterations (Log-Normal) of Example 3.1**

<b>Iterations</b>	<b>100</b>	<b>1000</b>	<b>5000</b>	<b>10000</b>
<b>Minimum</b>	80.19	77.19	76.96	76.96
<b>Maximum</b>	98.26	100.69	100.53	109.87
<b>Mean</b>	85.91	85.78	85.97	85.96
<b>Std Deviation</b>	3.45	3.23	3.26	3.30
<b>Variance</b>	11.87	10.46	10.64	10.91
<b>Skewness</b>	1.05	0.72	0.64	0.74
<b>Kurtosis</b>	4.34	4.01	3.71	4.18
<b>Percentile Values</b>				
<b>5 %</b>	81.36	81.10	81.22	81.20
<b>10 %</b>	82.18	81.99	82.12	82.13
<b>15 %</b>	82.82	82.49	82.77	82.75
<b>20 %</b>	83.13	83.08	83.26	83.22
<b>25 %</b>	83.35	83.46	83.72	83.68
<b>30 %</b>	83.64	83.87	84.10	84.05
<b>35 %</b>	83.99	84.25	84.49	84.46
<b>40 %</b>	84.58	84.71	84.86	84.84
<b>45 %</b>	84.82	85.05	85.26	85.22
<b>50 %</b>	85.34	85.45	85.63	85.60
<b>55 %</b>	85.89	85.85	86.04	86.00
<b>60 %</b>	86.11	86.27	86.46	86.40
<b>65 %</b>	86.63	86.68	86.91	86.82
<b>70 %</b>	86.84	87.14	87.35	87.27
<b>75 %</b>	87.37	87.59	87.90	87.84
<b>80 %</b>	88.24	88.13	88.44	88.42
<b>85 %</b>	89.35	89.06	89.16	89.22
<b>90 %</b>	90.05	89.96	90.13	90.21
<b>95 %</b>	92.46	91.79	91.95	92.05

**Table 3.5 Comparison of different iterations (Triangular) of Example 3.1**

<b>Iterations</b>	<b>100</b>	<b>1000</b>	<b>5000</b>	<b>10000</b>
<b>Minimum</b>	80.88	76.81	75.75	75.75
<b>Maximum</b>	99.84	103.87	106.25	106.25
<b>Mean</b>	88.36	88.38	88.58	88.54
<b>Std Deviation</b>	3.97	3.95	3.98	3.95
<b>Variance</b>	15.73	15.57	15.81	15.64
<b>Skewness</b>	0.72	0.45	0.39	0.38
<b>Kurtosis</b>	3.52	3.24	3.13	3.16
<b>Percentile Values</b>				
<b>5 %</b>	82.39	82.46	82.56	82.50
<b>10 %</b>	83.65	83.45	83.76	83.66
<b>15 %</b>	84.62	84.29	84.54	84.50
<b>20 %</b>	85.03	84.97	85.18	85.18
<b>25 %</b>	85.53	85.55	85.79	85.78
<b>30 %</b>	85.86	86.11	86.28	86.28
<b>35 %</b>	86.52	86.61	86.76	86.78
<b>40 %</b>	86.85	87.23	87.30	87.28
<b>45 %</b>	87.49	87.64	87.80	87.79
<b>50 %</b>	87.84	88.03	88.28	88.29
<b>55 %</b>	88.60	88.67	88.77	88.77
<b>60 %</b>	88.77	89.13	89.32	89.29
<b>65 %</b>	89.22	89.61	89.81	89.79
<b>70 %</b>	89.59	90.13	90.46	90.36
<b>75 %</b>	90.01	90.77	91.14	91.01
<b>80 %</b>	91.34	91.43	91.87	91.77
<b>85 %</b>	92.52	92.35	92.71	92.63
<b>90 %</b>	93.14	93.70	93.89	93.78
<b>95 %</b>	96.40	95.55	95.67	95.61

**Table 3.6 Comparison of different iterations (Uniform) of Example 3.1**

<b>Iterations</b>	100	1000	5000	10000
<b>Minimum</b>	82.46	77.02	76.07	76.07
<b>Maximum</b>	105.54	108.80	109.37	109.69
<b>Mean</b>	91.64	92.15	92.15	92.13
<b>Std Deviation</b>	5.07	5.17	5.30	5.30
<b>Variance</b>	25.68	26.69	28.12	28.04
<b>Skewness</b>	0.56	0.27	0.21	0.22
<b>Kurtosis</b>	2.98	2.93	2.93	2.91
<b>Percentile Values</b>				
<b>5 %</b>	83.76	84.16	83.79	83.81
<b>10 %</b>	85.44	85.65	85.53	85.54
<b>15 %</b>	86.59	86.86	86.72	86.71
<b>20 %</b>	87.15	87.74	87.58	87.57
<b>25 %</b>	87.57	88.37	88.41	88.34
<b>30 %</b>	88.02	89.02	89.20	89.17
<b>35 %</b>	88.86	89.78	89.88	89.87
<b>40 %</b>	90.06	90.55	90.58	90.56
<b>45 %</b>	90.55	91.21	91.28	91.26
<b>50 %</b>	91.42	91.89	91.95	91.92
<b>55 %</b>	92.08	92.66	92.60	92.55
<b>60 %</b>	92.29	93.28	93.37	93.27
<b>65 %</b>	92.83	93.97	94.03	93.95
<b>70 %</b>	93.56	94.62	94.78	94.71
<b>75 %</b>	94.37	95.61	95.63	95.57
<b>80 %</b>	95.21	96.61	96.56	96.50
<b>85 %</b>	96.94	97.48	97.66	97.66
<b>90 %</b>	97.92	98.85	99.09	99.15
<b>95 %</b>	102.18	101.35	101.21	101.40



**Table 3.7 Comparison of different iterations (Beta) of Example 3.1**

<b>Iterations</b>	100	1000	5000	10000
<b>Minimum</b>	80.22	76.72	75.91	75.91
<b>Maximum</b>	95.86	98.84	101.85	101.85
<b>Mean</b>	85.88	85.78	85.95	85.92
<b>Std Deviation</b>	3.22	3.16	3.19	3.17
<b>Variance</b>	10.37	10.01	10.18	10.08
<b>Skewness</b>	0.84	0.51	0.46	0.45
<b>Kurtosis</b>	3.71	3.39	3.29	3.32
<b>Percentile Values</b>				
<b>5 %</b>	81.36	81.08	81.17	81.16
<b>10 %</b>	82.13	81.94	82.10	82.08
<b>15 %</b>	83.00	82.47	82.75	82.70
<b>20 %</b>	83.16	83.07	83.22	83.23
<b>25 %</b>	83.57	83.54	83.70	83.70
<b>30 %</b>	83.72	83.92	84.11	84.13
<b>35 %</b>	84.26	84.35	84.47	84.48
<b>40 %</b>	84.60	84.78	84.88	84.87
<b>45 %</b>	84.94	85.13	85.27	85.28
<b>50 %</b>	85.45	85.51	85.71	85.69
<b>55 %</b>	86.05	85.89	86.07	86.06
<b>60 %</b>	86.17	86.31	86.52	86.48
<b>65 %</b>	86.58	86.76	86.92	86.89
<b>70 %</b>	86.87	87.15	87.40	87.35
<b>75 %</b>	87.32	87.67	87.93	87.86
<b>80 %</b>	88.41	88.23	88.57	88.48
<b>85 %</b>	89.13	89.05	89.28	89.20
<b>90 %</b>	89.57	90.03	90.23	90.16
<b>95 %</b>	92.43	91.36	91.59	91.60

**Table 3.8 Comparison of different distributions at 10000 iterations of  
Example 3.1**

<b>Iterations</b>	<b>Normal</b>	<b>Log-Nor.</b>	<b>Triangular</b>	<b>Uniform</b>	<b>Beta</b>	<b>PERT</b>
<b>Minimum</b>	76.29	76.96	75.75	76.07	75.91	76.40
<b>Maximum</b>	100.56	109.87	106.25	109.69	101.85	92.00
<b>Mean</b>	85.96	85.96	88.54	92.13	85.92	84.20
<b>Std Deviation</b>	3.07	3.30	3.95	5.30	3.17	2.60
<b>Variance</b>	9.44	10.91	15.64	28.04	10.08	6.69
<b>Skewness</b>	0.45	0.74	0.38	0.22	0.45	0.00
<b>Kurtosis</b>	3.48	4.18	3.16	2.91	3.32	
<b>Percentile Values</b>						
<b>5 %</b>	81.28	81.20	82.50	83.81	81.16	79.92
<b>10 %</b>	82.28	82.13	83.66	85.54	82.08	80.87
<b>15 %</b>	82.93	82.75	84.50	86.71	82.70	81.51
<b>20 %</b>	83.40	83.22	85.18	87.57	83.23	82.01
<b>25 %</b>	83.86	83.68	85.78	88.34	83.70	82.45
<b>30 %</b>	84.25	84.05	86.28	89.17	84.13	82.84
<b>35 %</b>	84.65	84.46	86.78	89.87	84.48	83.20
<b>40 %</b>	85.02	84.84	87.28	90.56	84.87	83.54
<b>45 %</b>	85.37	85.22	87.79	91.26	85.28	83.87
<b>50 %</b>	85.75	85.60	88.29	91.92	85.69	84.20
<b>55 %</b>	86.13	86.00	88.77	92.55	86.06	84.53
<b>60 %</b>	86.48	86.40	89.29	93.27	86.48	84.86
<b>65 %</b>	86.88	86.82	89.79	93.95	86.89	85.20
<b>70 %</b>	87.31	87.27	90.36	94.71	87.35	85.56
<b>75 %</b>	87.81	87.84	91.01	95.57	87.86	85.95
<b>80 %</b>	88.38	88.42	91.77	96.50	88.48	86.39
<b>85 %</b>	89.01	89.22	92.63	97.66	89.20	86.89
<b>90 %</b>	89.92	90.21	93.78	99.15	90.16	87.53
<b>95 %</b>	91.41	92.05	95.61	101.40	91.60	88.48

**Table 3.9 Comparison of different distributions at 10000 iterations when the mean is altered of Example 3.1**

<b>Iterations</b>	<b>Normal</b>	<b>Log-Nor.</b>	<b>Triangular</b>	<b>Uniform</b>	<b>Beta</b>
<b>Minimum</b>	80.60	81.11	78.70	76.07	80.56
<b>Maximum</b>	104.23	107.46	104.56	109.69	101.53
<b>Mean</b>	90.07	90.15	90.64	92.13	90.08
<b>Std Deviation</b>	3.01	3.22	3.71	5.30	2.98
<b>Variance</b>	9.06	10.36	13.77	28.04	8.88
<b>Skewness</b>	0.40	0.67	0.32	0.22	0.31
<b>Kurtosis</b>	3.47	3.97	3.17	2.91	3.20
<b>Percentile Values</b>					
<b>5 %</b>	85.41	85.43	84.81	83.81	85.39
<b>10 %</b>	86.41	86.39	86.11	85.54	86.44
<b>15 %</b>	87.07	86.99	86.86	86.71	87.05
<b>20 %</b>	87.57	87.48	87.53	87.57	87.58
<b>25 %</b>	88.01	87.93	88.07	88.34	88.02
<b>30 %</b>	88.44	88.32	88.59	89.17	88.44
<b>35 %</b>	88.82	88.73	89.07	89.87	88.83
<b>40 %</b>	89.19	89.11	89.54	90.56	89.21
<b>45 %</b>	89.52	89.47	90.00	91.26	89.56
<b>50 %</b>	89.89	89.84	90.46	91.92	89.94
<b>55 %</b>	90.25	90.22	90.94	92.55	90.31
<b>60 %</b>	90.63	90.61	91.37	93.27	90.67
<b>65 %</b>	90.99	91.03	91.81	93.95	91.04
<b>70 %</b>	91.42	91.47	92.36	94.71	91.47
<b>75 %</b>	91.89	91.98	92.95	95.57	91.93
<b>80 %</b>	92.45	92.61	93.65	96.50	92.49
<b>85 %</b>	93.05	93.32	94.42	97.66	93.10
<b>90 %</b>	93.90	94.21	95.44	99.15	93.93
<b>95 %</b>	95.37	95.97	97.23	101.40	95.30

### **3.4.2 Example 3.2 – a hypothetical project HABITAT**

Example 3.2 is a more complicated project example with the number of the activities increased to 50. The example is concerned with a hypothetical project HABITAT chosen from Ahuja, Dozzi and Abourizk <sup>[81]</sup>. Figure 3.10 shows the arrow network for the project. The dependencies and estimates of the optimistic time (a), the most likely time (m) and the pessimistic time (b) are shown in Table 3.10.

#### **3.4.2.1 PERT results**

The results of PERT are summarised in Table 3.11. The overall properties of the distributions of the project durations and the percentile values are reported in Table 3.17.

#### **3.4.2.2 Different number of iterations and distributions of MCS**

The results from Monte Carlo simulations for Example 3.2 are reported in Appendix B. Figures without a suffix are histograms and Figures with the suffix (a) are ascending cumulative curves.

Figures B.1-B.20(a) show the effect of the number of iterations for the five different distributions. They are: Normal (Figures B.1-B.4(a)), Log-Normal (Figures B.5-B.8(a)), Triangular (Figures B.9-B.12(a)), Uniform (Figures B.13-B.16(a)) and Beta (Figures B.17-B.20(a)) distributions.

The number of iterations used with each distribution was: 100, 1000, 5000 and 10000.

#### **3.4.2.3 Different distributions with altered means and standard deviations**

Figures B.21-B.24 (Appendix B) show the effect of varying the mean and standard deviation for alternative distributions (Normal, Log-Normal, Triangular and Beta) using a fixed number of iterations (10000).

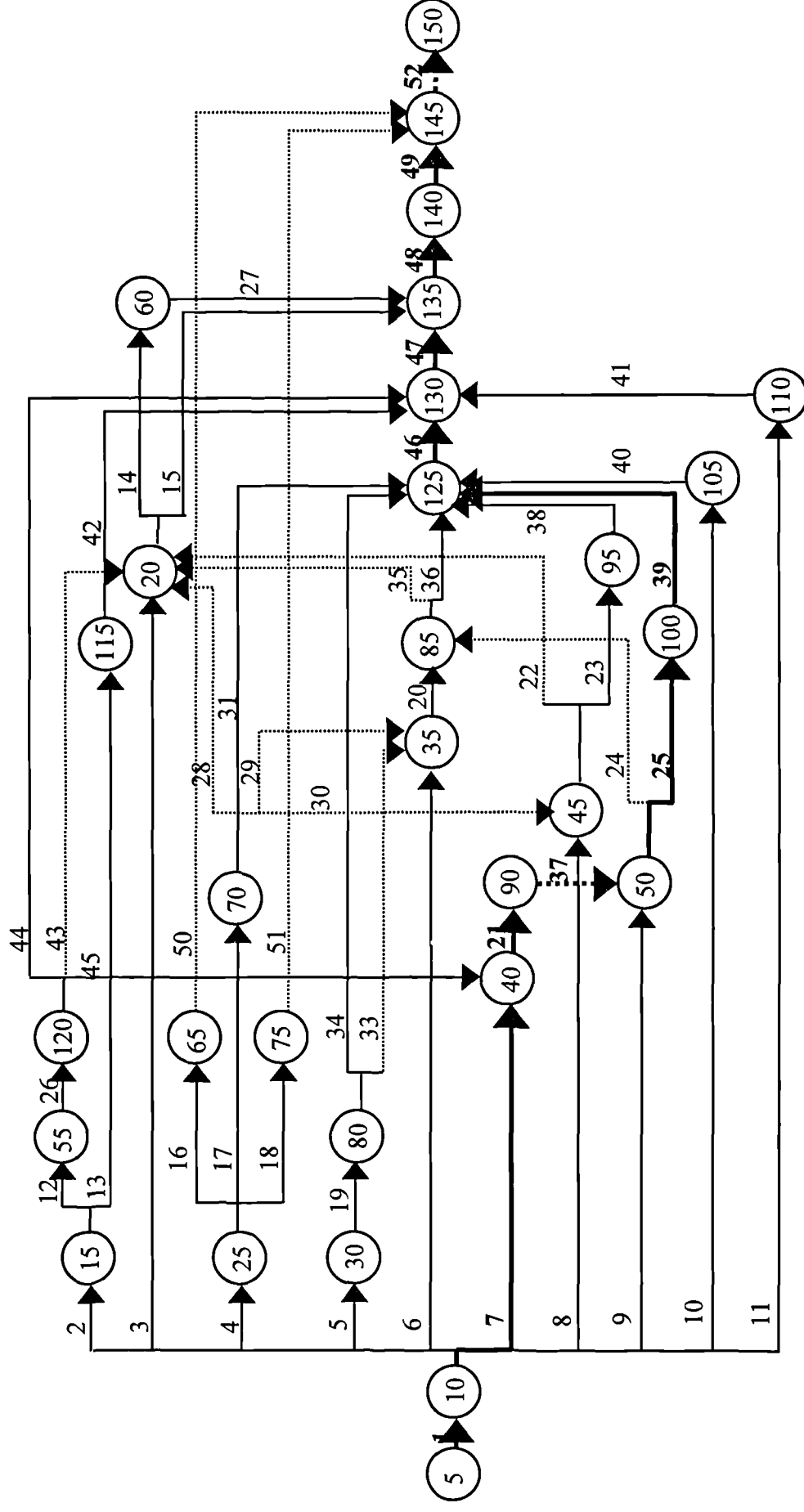


Figure 3.10 The network of Example 3.2 (bold activities are critical)

**Table 3.10 The activities involved in Example 3.2**

Activity		Description	Duration			Predecessors
code	No.		a	m	b	
<b>5-10</b>	<b>1</b>	<b>Obtain tank</b>	<b>9</b>	<b>10</b>	<b>11</b>	
<b>10-15</b>	<b>2</b>	<b>General area of site decided</b>	<b>14</b>	<b>15</b>	<b>21</b>	<b>1</b>
10-20	3	Umbilical initial design complete	59	60	80	1
10-25	4	Determine biological-environmental requirements	50	60	90	1
10-30	5	Determine communication system requirements	40	50	80	1
10-35	6	Determine electrical requirements	20	30	40	1
10-40	7	Determine required hull modifications	10	15	25	1
10-45	8	Determine plumbing requirements	10	15	30	1
10-50	9	Determine interior layout requirements	20	30	35	1
10-105	10	Testing procedure and apparatus designed	40	50	60	1
10-110	11	Submerging procedure designed	70	80	90	1
<b>15-55</b>	<b>12</b>	<b>Detailed data collection on area</b>	<b>14</b>	<b>15</b>	<b>16</b>	<b>2</b>
15-115	13	Shore station designed	14	15	16	2
<b>20-60</b>	<b>14</b>	<b>Umbilical final design complete</b>	<b>50</b>	<b>60</b>	<b>80</b>	<b>3,43,28,35,22</b>
20-135	15	Umbilical materials obtained	50	60	61	3,43,28,35,22
25-65	16	Environmental maintenance system designed	14	15	16	4
25-70	17	Sanitary and water facilities decided	14	15	16	4
25-75	18	Environmental monitoring and alarm system designed	29	30	31	4
30-80	19	System components designed	39	40	41	5
35-85	20	Electrical system designed	59	60	61	6,29,33
<b>40-90</b>	<b>21</b>	<b>Full modifications complete</b>	<b>89</b>	<b>90</b>	<b>91</b>	<b>7, 45</b>
45-20	22	Dummy	0	0	0	8, 30
45-95	23	System component design finished	9	10	11	8, 30
<b>50-85</b>	<b>24</b>	<b>Dummy</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>9, 37</b>
50-100	25	Interior layout designed	19	20	21	9, 37
<b>55-120</b>	<b>26</b>	<b>Final site selected</b>	<b>14</b>	<b>15</b>	<b>16</b>	<b>12</b>
<b>60-135</b>	<b>27</b>	<b>Umbilical constructed and tested</b>	<b>9</b>	<b>10</b>	<b>11</b>	<b>14</b>
70-20	28	Dummy	0	0	0	17
70-35	29	Dummy	0	0	0	17
70-45	30	Dummy	0	0	0	17
70-125	31	System components obtained and installed	29	30	31	17
80-30	32	Dummy	0	0	0	Invalid

80-35	33	Dummy	0	0	0	19
80-125	34	Components obtained and installed	29	30	31	19
<b>85-20</b>	<b>35</b>	<b>Dummy</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>20, 24</b>
85-125	36	Wiring fixtures, outlets, etc., installed	25	27	27	20, 24
<b>90-50</b>	<b>37</b>	<b>Dummy</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>21</b>
95-125	38	System installed	4	5	6	23
100-125	39	Interior construction finished	9	10	11	25
105-125	40	Testing apparatus constructed	19	20	21	10
110-130	41	Cradle and habitat transportation arrangements complete	29	30	31	11
115-130	42	Shore station constructed	29	30	31	13
120-20	43	Dummy	0	0	0	26
120-130	44	Cradle designed and constructed	39	40	41	26
<b>120-40</b>	<b>45</b>	<b>Cradle foundation designed and constructed</b>	<b>30</b>	<b>41</b>	<b>60</b>	<b>26</b>
125-130	46	Test and evaluation	19	20	21	31, 34, 36, 38, 39, 40
130-135	47	Cradle and habitat transported to site	4	5	6	42,41,44,46
<b>135-140</b>	<b>48</b>	<b>Habitat lowered and connections made</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>15,27,47</b>
<b>140-145</b>	<b>49</b>	<b>Habitat made operational</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>48</b>
65-145	50	Dummy	0	0	0	16
75-145	51	Dummy	0	0	0	18
<b>145-150</b>	<b>52</b>	<b>Dummy</b>	<b>0</b>	<b>0</b>	<b>0</b>	<b>49, 50, 51</b>

#### 3.4.2.4 Comparison of different iterations

Figures 3.11-3.15 show the comparison of the different number of iterations in all five of the different distribution forms.

Tables 3.12-3.16 show the equivalent quantitative aspects of the results of the different iterations in all five of the different distribution forms. The upper section of each table denotes the overall properties of the distribution and the lower section denotes the percentile values of the distribution for differing numbers of iterations (100, 1000, 5000 and 10000).

**Table 3.11 The project duration analysed by PERT of Example 3.2**

Activity No.	Activities in critical path	Mean duration	Variance	Standard deviation
1	Obtain tank	10.00	0.11	0.33
2	General area of site decided	15.83	1.36	1.17
12	Detailed data collection on area	15.00	0.11	0.33
14	Umbilical final design complete	61.67	25.00	5.00
21	Full modifications complete	90.00	0.11	0.33
24	Dummy	0.00	0.00	0.00
26	Final site selected	15.00	0.11	0.33
27	Umbilical constructed and tested	10.00	0.11	0.33
35	Dummy	0.00	0.00	0.00
37	Dummy	0.00	0.00	0.00
45	Cradle foundation designed and constructed	42.33	25.00	5.00
48	Habitat lowered and connections..	5.00	0.11	0.33
49	Habitat made operational	5.00	0.11	0.33
52	Dummy	0.00	0.00	0.00
<b>Project duration <math>E(T) = \sum t_i</math></b>		269.83		
<b>Project variance <math>S^2 = \sum S_i^2</math></b>			52.14	
<b>Project standard deviation <math>S = \sqrt{S^2}</math></b>				7.22

#### 3.4.2.5 Comparison of different distributions forms

Figures 3.16 shows graphical results for project duration in days after 10,000 simulations for each of the different distributions used.

Tables 3.17 shows the quantitative results for project duration in days after 10,000 simulations at varying percentile values for each of the different distributions.

#### 3.4.2.6 Comparison of different distributions when the mean is altered

Figures 3.17 shows graphical results for project duration in days after 10,000 simulations at varying percentile values for the different distributions used with alternative means and standard deviations.

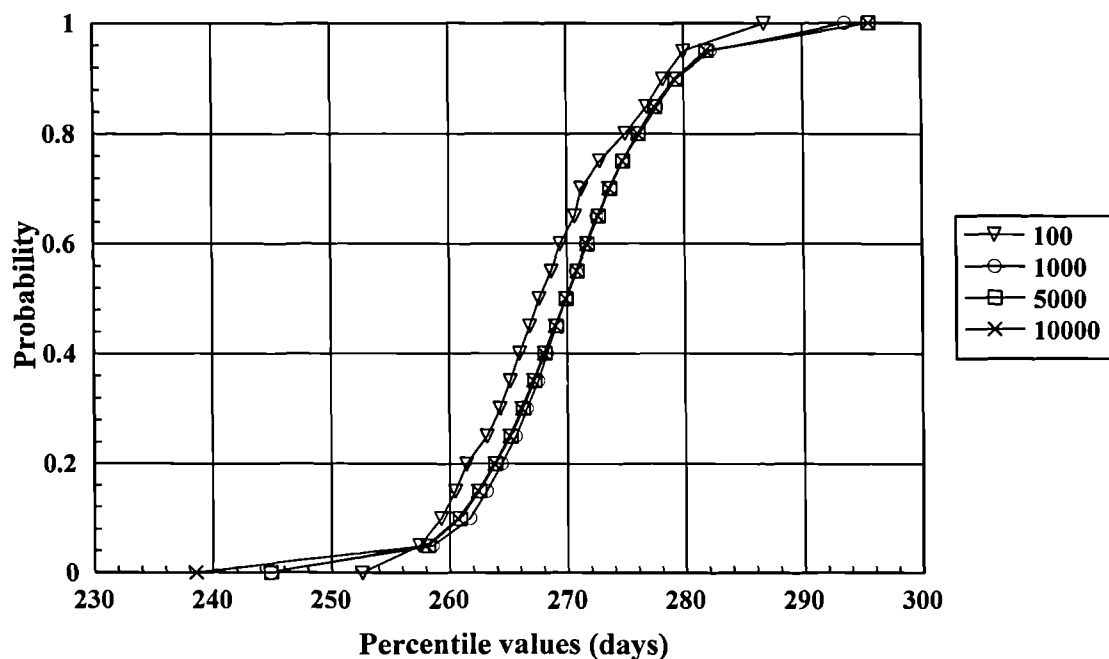


Tables 3.18 shows the quantitative results for project duration in days after 10,000 simulations at varying percentile values for the different distributions when the mean is altered.

### 3.4.2.7 Comparison of PERT with MCS

Figures 3.18 shows the comparison of PERT and Monte Carlo simulations with varying distributions and distributions with altered means and standard deviations.

Tables 3.17 shows the estimated project duration range calculated using the PERT method and Monte Carlo simulation (at 10,000 simulations) using different distribution forms.



**Figure 3.11 Comparison of different iterations (Normal) Example 3.2**

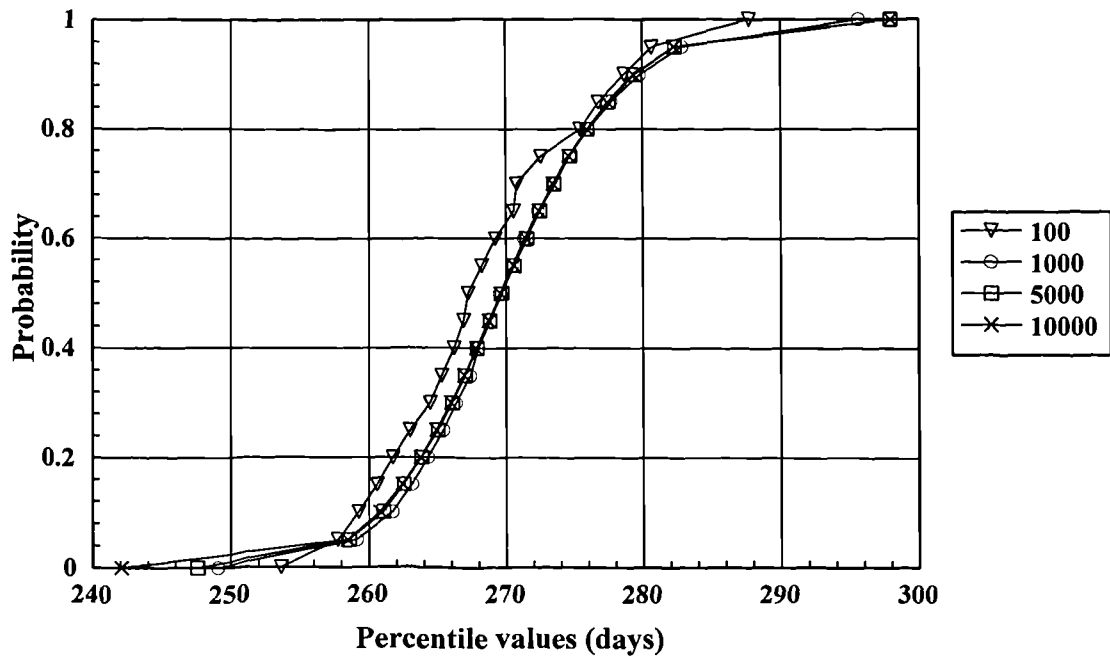


Figure 3.12 Comparison of different iterations (Log-Normal) Example 3.2

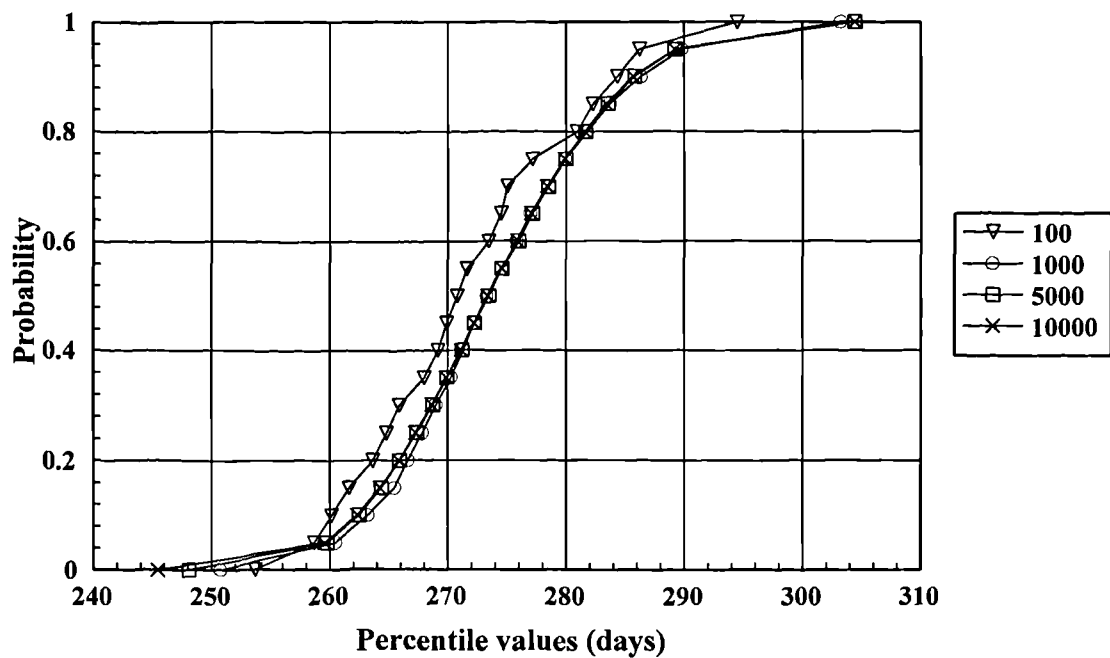
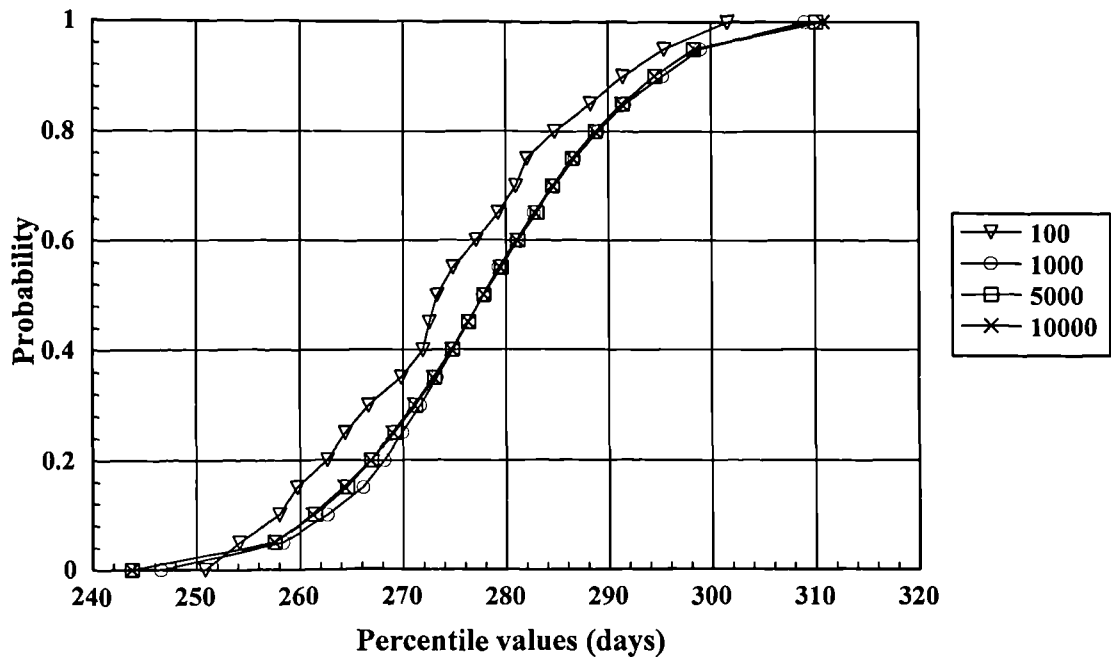
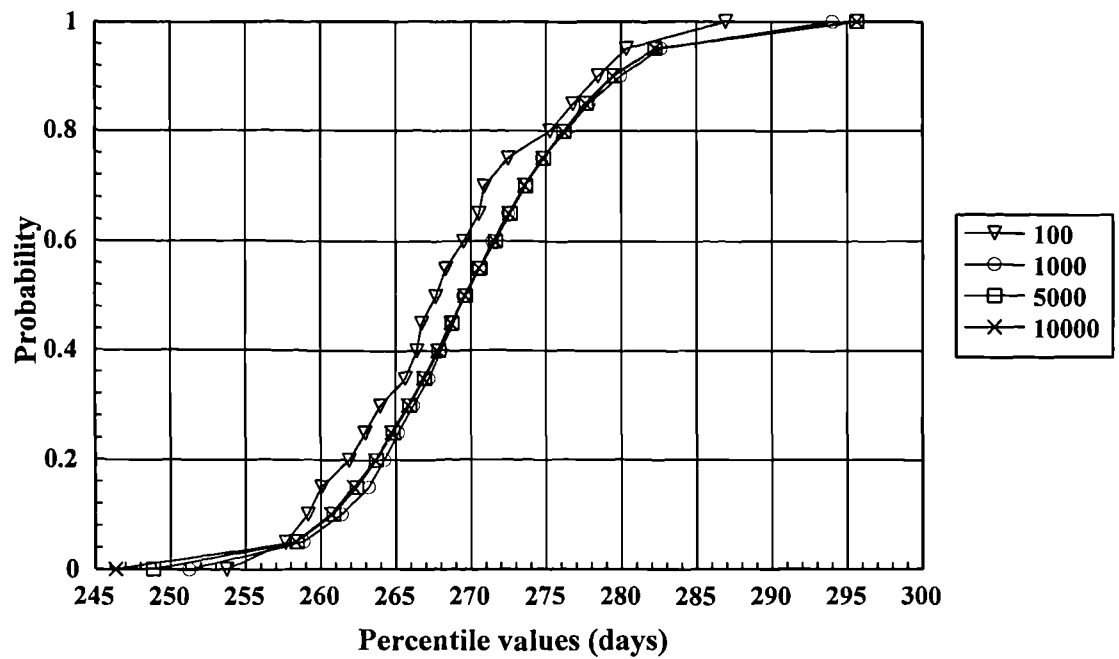


Figure 3.13 Comparison of different iterations (Triangular) Example 3.2



**Figure 3.14 Comparison of different iterations (Uniform) Example 3.2**



**Figure 3.15 Comparison of different iterations (Beta) Example 3.2**

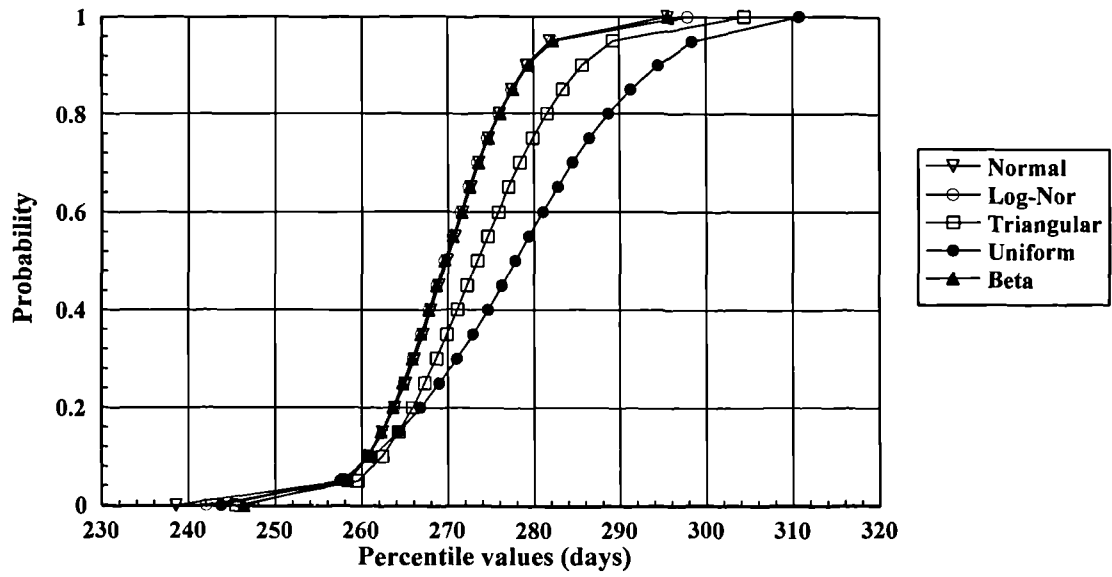


Figure 3.16 Comparison of different distributions at 10000 iterations, Example 3.2

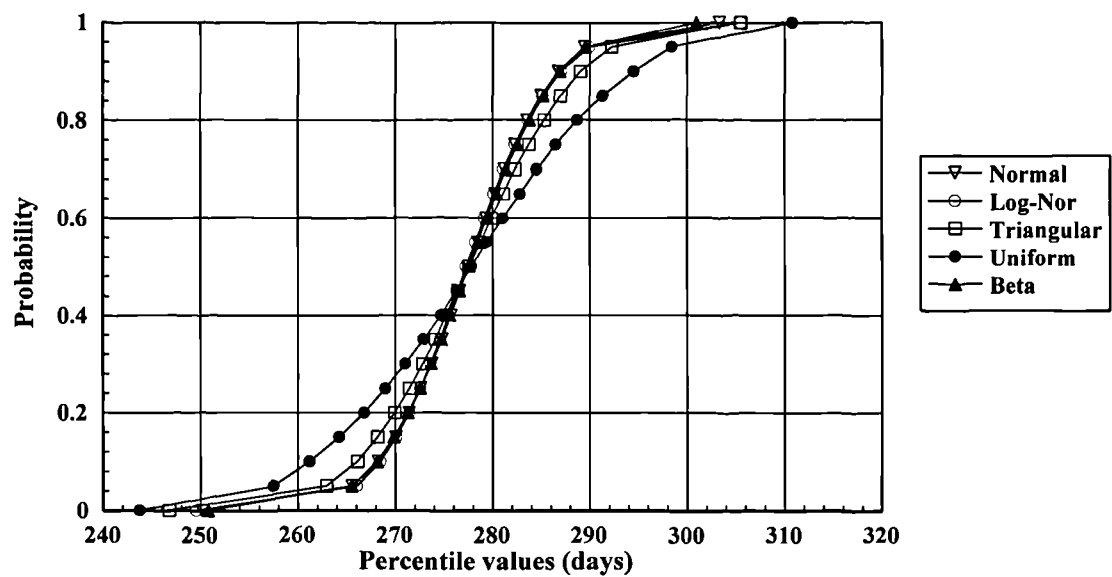


Figure 3.17 Comparison of different distributions at 10000 iterations when the mean shifted, Example 3.2

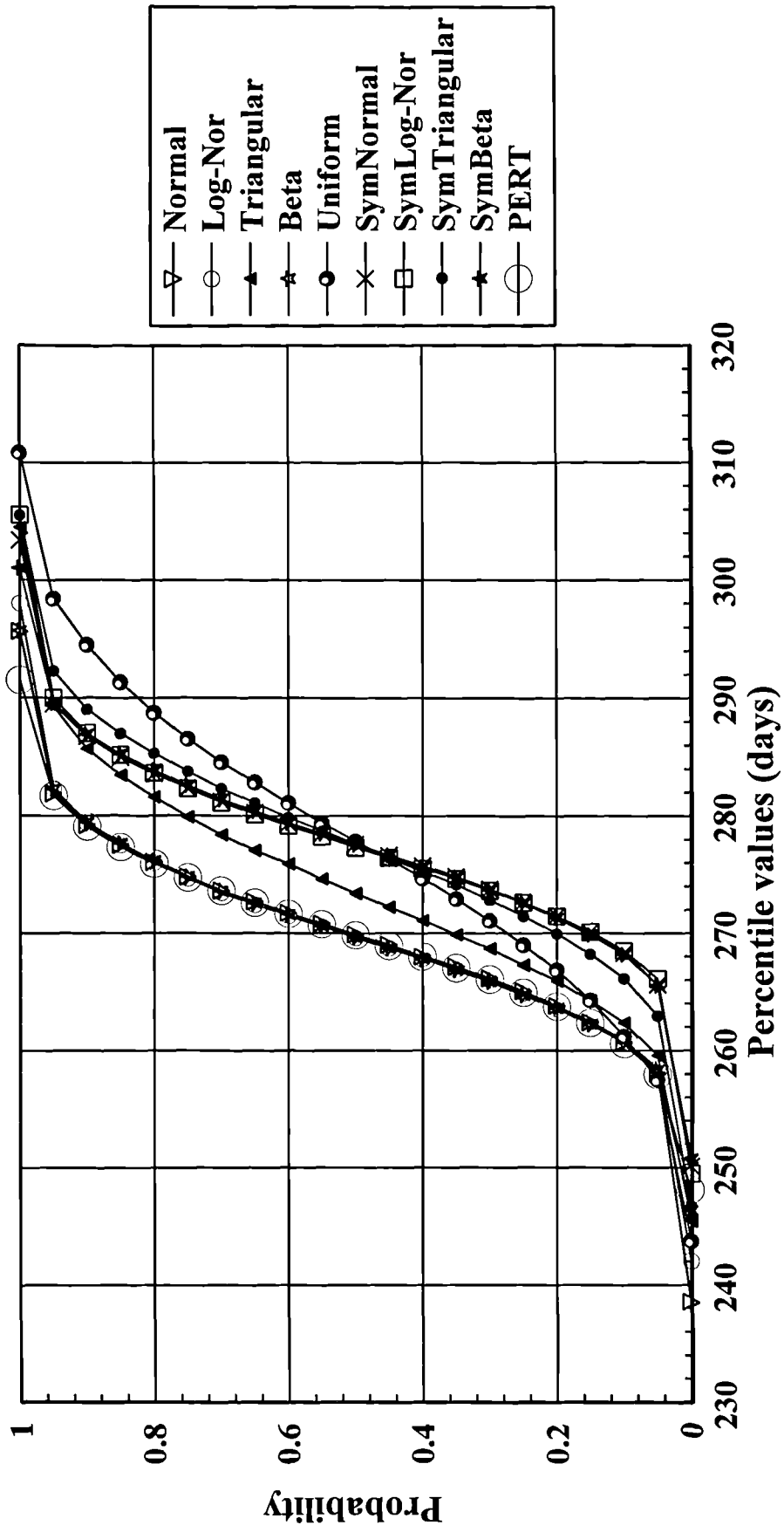


Figure 3.18 Comparison of different distributions at 10000 iterations when the mean before and after altered, Example 3.2

**Table 3.12 Comparison of different iterations (Normal) of Example 3.2**

<b>Iterations</b>	<b>100</b>	<b>1000</b>	<b>5000</b>	<b>10000</b>
<b>Minimum</b>	252.59	244.87	244.87	238.55
<b>Maximum</b>	286.78	293.62	295.61	295.61
<b>Mean</b>	268.27	270.26	269.99	269.91
<b>Std Deviation</b>	7.18	7.14	7.17	7.23
<b>Variance</b>	51.62	50.97	51.41	52.26
<b>Skewness</b>	0.11	0.09	0.01	0.02
<b>Kurtosis</b>	2.57	3.17	2.95	2.98
<b>Percentile Values</b>				
<b>5 %</b>	257.46	258.57	258.19	258.02
<b>10 %</b>	259.30	261.73	260.87	260.67
<b>15 %</b>	260.50	263.16	262.53	262.39
<b>20 %</b>	261.44	264.43	263.89	263.79
<b>25 %</b>	263.17	265.59	265.17	265.05
<b>30 %</b>	264.30	266.59	266.26	266.11
<b>35 %</b>	265.17	267.56	267.24	267.13
<b>40 %</b>	265.96	268.31	268.10	268.01
<b>45 %</b>	266.83	269.07	269.06	268.97
<b>50 %</b>	267.60	269.87	269.91	269.87
<b>55 %</b>	268.64	270.77	270.85	270.80
<b>60 %</b>	269.44	271.64	271.77	271.70
<b>65 %</b>	270.70	272.56	272.72	272.67
<b>70 %</b>	271.24	273.73	273.71	273.60
<b>75 %</b>	272.84	274.72	274.78	274.75
<b>80 %</b>	274.99	276.26	276.11	276.06
<b>85 %</b>	276.81	277.71	277.51	277.47
<b>90 %</b>	278.25	279.35	279.31	279.20
<b>95 %</b>	279.99	282.28	281.93	281.90

**Table 3.13 Comparison of different iterations (Log-Normal) of Example 3.2**

<b>Iterations</b>	<b>100</b>	<b>1000</b>	<b>5000</b>	<b>10000</b>
<b>Minimum</b>	253.64	249.08	247.59	242.08
<b>Maximum</b>	287.75	295.69	297.98	297.98
<b>Mean</b>	268.34	270.26	269.99	269.91
<b>Std Deviation</b>	7.18	7.20	7.20	7.25
<b>Variance</b>	51.60	51.89	51.80	52.60
<b>Skewness</b>	0.27	0.31	0.20	0.21
<b>Kurtosis</b>	2.62	3.22	3.01	3.06
<b>Percentile Values</b>				
<b>5 %</b>	257.76	259.12	258.56	258.47
<b>10 %</b>	259.28	261.74	261.05	260.81
<b>15 %</b>	260.62	263.15	262.55	262.44
<b>20 %</b>	261.76	264.30	263.83	263.77
<b>25 %</b>	263.01	265.43	265.02	264.92
<b>30 %</b>	264.47	266.36	266.08	265.97
<b>35 %</b>	265.31	267.40	267.01	266.95
<b>40 %</b>	266.22	267.99	267.88	267.82
<b>45 %</b>	266.92	268.86	268.83	268.74
<b>50 %</b>	267.23	269.56	269.70	269.61
<b>55 %</b>	268.23	270.49	270.60	270.57
<b>60 %</b>	269.25	271.36	271.53	271.48
<b>65 %</b>	270.59	272.38	272.49	272.45
<b>70 %</b>	270.82	273.65	273.54	273.46
<b>75 %</b>	272.58	274.67	274.70	274.63
<b>80 %</b>	275.46	276.07	275.99	275.98
<b>85 %</b>	276.80	277.75	277.53	277.48
<b>90 %</b>	278.70	279.82	279.47	279.37
<b>95 %</b>	280.69	282.89	282.38	282.32

**Table 3.14 Comparison of different iterations (Triangular) of Example 3.2**

<b>Iterations</b>	<b>100</b>	<b>1000</b>	<b>5000</b>	<b>10000</b>
<b>Minimum</b>	253.73	250.77	248.09	245.46
<b>Maximum</b>	294.52	303.25	304.45	304.45
<b>Mean</b>	271.65	274.17	273.86	273.77
<b>Std Deviation</b>	8.90	8.88	8.93	8.99
<b>Variance</b>	79.19	78.87	79.78	80.83
<b>Skewness</b>	0.22	0.27	0.18	0.19
<b>Kurtosis</b>	2.43	2.82	2.69	2.73
<b>Percentile Values</b>				
<b>5 %</b>	258.71	260.49	259.77	259.56
<b>10 %</b>	260.16	263.24	262.51	262.35
<b>15 %</b>	261.67	265.48	264.38	264.25
<b>20 %</b>	263.67	266.63	265.93	265.90
<b>25 %</b>	264.84	267.85	267.38	267.28
<b>30 %</b>	265.92	269.00	268.75	268.65
<b>35 %</b>	268.03	270.28	269.92	269.87
<b>40 %</b>	269.21	271.33	271.22	271.11
<b>45 %</b>	269.93	272.23	272.27	272.23
<b>50 %</b>	270.83	273.32	273.49	273.39
<b>55 %</b>	271.68	274.53	274.59	274.60
<b>60 %</b>	273.45	275.82	275.98	275.87
<b>65 %</b>	274.53	276.98	277.16	277.05
<b>70 %</b>	275.05	278.56	278.50	278.39
<b>75 %</b>	277.16	279.92	279.99	279.90
<b>80 %</b>	280.95	281.81	281.65	281.59
<b>85 %</b>	282.30	283.76	283.58	283.42
<b>90 %</b>	284.39	286.37	285.84	285.69
<b>95 %</b>	286.27	289.81	289.23	289.27



**Table 3.15 Comparison of different iterations (Uniform) of Example 3.2**

<b>Iterations</b>	<b>100</b>	<b>1000</b>	<b>5000</b>	<b>10000</b>
<b>Minimum</b>	250.87	246.61	243.76	243.76
<b>Maximum</b>	301.61	309.08	310.15	310.81
<b>Mean</b>	274.39	278.36	277.97	277.86
<b>Std Deviation</b>	12.23	12.02	12.20	12.29
<b>Variance</b>	149.53	144.51	148.90	151.08
<b>Skewness</b>	0.11	0.07	0.00	0.01
<b>Kurtosis</b>	2.32	2.53	2.44	2.46
<b>Percentile Values</b>				
<b>5 %</b>	254.17	258.39	257.57	257.52
<b>10 %</b>	258.02	262.68	261.46	261.22
<b>15 %</b>	259.73	266.14	264.56	264.27
<b>20 %</b>	262.66	268.15	266.89	266.82
<b>25 %</b>	264.43	269.96	269.23	268.99
<b>30 %</b>	266.68	271.67	271.24	271.06
<b>35 %</b>	269.84	273.32	273.05	272.94
<b>40 %</b>	271.94	274.86	274.79	274.69
<b>45 %</b>	272.55	276.38	276.38	276.30
<b>50 %</b>	273.31	277.79	277.85	277.83
<b>55 %</b>	274.85	279.33	279.56	279.40
<b>60 %</b>	277.12	281.03	281.18	281.07
<b>65 %</b>	279.25	282.77	283.05	282.84
<b>70 %</b>	280.99	284.74	284.56	284.55
<b>75 %</b>	282.07	286.76	286.51	286.52
<b>80 %</b>	284.81	289.06	288.76	288.75
<b>85 %</b>	288.31	291.65	291.42	291.34
<b>90 %</b>	291.48	295.32	294.58	294.53
<b>95 %</b>	295.47	299.00	298.31	298.43

**Table 3.16 Comparison of different iterations (Beta) of Example 3.2**

<b>Iterations</b>	<b>100</b>	<b>1000</b>	<b>5000</b>	<b>10000</b>
<b>Minimum</b>	253.75	251.26	248.86	246.36
<b>Maximum</b>	286.97	294.06	295.66	295.66
<b>Mean</b>	268.23	270.23	269.97	269.89
<b>Std Deviation</b>	7.18	7.16	7.20	7.24
<b>Variance</b>	51.57	51.32	51.78	52.49
<b>Skewness</b>	0.21	0.25	0.16	0.17
<b>Kurtosis</b>	2.46	2.89	2.74	2.78
<b>Percentile Values</b>				
<b>5 %</b>	257.69	258.86	258.43	258.35
<b>10 %</b>	259.16	261.41	260.85	260.67
<b>15 %</b>	260.06	263.22	262.39	262.21
<b>20 %</b>	261.90	264.23	263.68	263.61
<b>25 %</b>	262.97	265.19	264.85	264.73
<b>30 %</b>	264.01	266.20	265.95	265.85
<b>35 %</b>	265.66	267.22	266.95	266.86
<b>40 %</b>	266.43	268.04	267.87	267.79
<b>45 %</b>	266.75	268.86	268.76	268.69
<b>50 %</b>	267.69	269.54	269.66	269.60
<b>55 %</b>	268.35	270.58	270.61	270.57
<b>60 %</b>	269.54	271.46	271.66	271.60
<b>65 %</b>	270.55	272.50	272.64	272.55
<b>70 %</b>	270.91	273.64	273.66	273.57
<b>75 %</b>	272.50	274.75	274.85	274.78
<b>80 %</b>	275.27	276.35	276.20	276.15
<b>85 %</b>	276.81	277.86	277.69	277.62
<b>90 %</b>	278.48	279.97	279.55	279.44
<b>95 %</b>	280.37	282.67	282.26	282.26

**Table 3.17 Comparison of different distributions at 10000 iterations of  
Example 3.2**

<b>Iterations</b>	<b>Normal</b>	<b>Log-Nor.</b>	<b>Triangular</b>	<b>Uniform</b>	<b>Beta</b>	<b>PERT</b>
<b>Minimum</b>	238.55	242.08	245.46	243.76	246.36	248.17
<b>Maximum</b>	295.61	297.98	304.45	310.81	295.66	291.49
<b>Mean</b>	269.91	269.91	273.77	277.86	269.89	269.83
<b>Std Deviation</b>	7.23	7.25	8.99	12.29	7.24	7.22
<b>Variance</b>	52.26	52.60	80.83	151.08	52.49	52.14
<b>Skewness</b>	0.02	0.21	0.19	0.01	0.17	0.00
<b>Kurtosis</b>	2.98	3.06	2.73	2.46	2.78	
<b>Percentile Values</b>						
<b>5 %</b>	258.02	258.47	259.56	257.52	258.35	257.95
<b>10 %</b>	260.67	260.81	262.35	261.22	260.67	260.58
<b>15 %</b>	262.39	262.44	264.25	264.27	262.21	262.35
<b>20 %</b>	263.79	263.77	265.90	266.82	263.61	263.75
<b>25 %</b>	265.05	264.92	267.28	268.99	264.73	264.96
<b>30 %</b>	266.11	265.97	268.65	271.06	265.85	266.04
<b>35 %</b>	267.13	266.95	269.87	272.94	266.86	267.05
<b>40 %</b>	268.01	267.82	271.11	274.69	267.79	268.00
<b>45 %</b>	268.97	268.74	272.23	276.30	268.69	268.92
<b>50 %</b>	269.87	269.61	273.39	277.83	269.60	269.83
<b>55 %</b>	270.80	270.57	274.60	279.40	270.57	270.74
<b>60 %</b>	271.70	271.48	275.87	281.07	271.60	271.66
<b>65 %</b>	272.67	272.45	277.05	282.84	272.55	272.61
<b>70 %</b>	273.60	273.46	278.39	284.55	273.57	273.62
<b>75 %</b>	274.75	274.63	279.90	286.52	274.78	274.70
<b>80 %</b>	276.06	275.98	281.59	288.75	276.15	275.91
<b>85 %</b>	277.47	277.48	283.42	291.34	277.62	277.31
<b>90 %</b>	279.20	279.37	285.69	294.53	279.44	279.08
<b>95 %</b>	281.90	282.32	289.27	298.43	282.26	281.71

**Table 3.18 Comparison of different distributions at 10000 iterations when the mean is altered of Example 3.2**

<b>Iterations</b>	<b>Normal</b>	<b>Log-Nor.</b>	<b>Triangular</b>	<b>Uniform</b>	<b>Beta</b>
<b>Minimum</b>	250.15	249.56	246.78	243.76	250.79
<b>Maximum</b>	303.35	305.53	305.50	310.81	300.99
<b>Mean</b>	277.53	277.58	277.60	277.86	277.56
<b>Std Deviation</b>	7.22	7.26	8.84	12.29	7.24
<b>Variance</b>	52.18	52.66	78.09	151.08	52.38
<b>Skewness</b>	0.00	0.20	0.01	0.01	0.00
<b>Kurtosis</b>	2.99	3.06	2.72	2.46	2.75
<b>Percentile Values</b>					
<b>5 %</b>	265.57	266.09	262.95	257.52	265.54
<b>10 %</b>	268.27	268.48	266.13	261.22	268.17
<b>15 %</b>	270.07	270.12	268.20	264.27	269.87
<b>20 %</b>	271.44	271.45	269.95	266.82	271.30
<b>25 %</b>	272.62	272.60	271.46	268.99	272.56
<b>30 %</b>	273.75	273.64	272.84	271.06	273.68
<b>35 %</b>	274.80	274.61	274.13	272.94	274.74
<b>40 %</b>	275.74	275.50	275.22	274.69	275.62
<b>45 %</b>	276.62	276.42	276.44	276.30	276.61
<b>50 %</b>	277.57	277.30	277.57	277.83	277.55
<b>55 %</b>	278.50	278.25	278.75	279.40	278.51
<b>60 %</b>	279.34	279.17	279.87	281.07	279.41
<b>65 %</b>	280.27	280.14	281.10	282.84	280.42
<b>70 %</b>	281.23	281.14	282.30	284.55	281.40
<b>75 %</b>	282.32	282.30	283.75	286.52	282.58
<b>80 %</b>	283.57	283.66	285.32	288.75	283.85
<b>85 %</b>	284.95	285.16	287.00	291.34	285.24
<b>90 %</b>	286.75	287.03	289.02	294.53	286.92
<b>95 %</b>	289.45	289.98	292.26	298.43	289.55

### **3.5 Discussion**

#### **3.5.1 Sensitivity analysis of Monte Carlo simulations**

##### **3.5.1.1 Effect of different number of iterations**

From Figures 3.2 - 3.6 and 3.11 - 3.15, it can be observed that all of the cumulative curves at 100 iterations show a relatively variable path. From 1000 to 10000 iterations, as the number of iterations increase, the cumulative curves of the simulations tend toward more regular curves (a quantitative description of each set of iterations can be seen in Appendixes A and B). This shows that by increasing the number of iterations, a reduction of the sampling variability can be attained.

From these figures, it can also be seen that the percentile values from 5% to 95% are approximately the same at 1000, 5000 and 10000 iterations and only the minimum (less than 5%) and the maximum (greater than 95%) percentile values are different. This fact is irrespective of the type of probability distribution function specified for the activity durations. The project duration range (maximum minus minimum) is increased as the number of simulations increase. For distributions without a boundary, such as the Normal and Log-Normal (one side bounded) distributions, the project duration range will tend to infinity. On the other hand, distributions with a boundary, such as the Triangular, Uniform and Beta distributions, the project duration range will converge at certain points (the minimum and maximum values). For example, in Figure 3.4, the minimum percentile value decreases as the number of simulations increases until it converges at 75.75 days. In contrast, the maximum percentile value increases as the number of simulations increases until it converges at 106.25 days. These results mean that assuming a Triangular distribution for the individual activity duration, Example 3.1 is impossible to complete in 75.75 days or less and it can be completed in 106.25 days with 100% confidence.

Tables 3.3 - 3.7 and 3.12 - 3.16 give quantitative results and from these it can be seen that the results from 100 iterations are different from 1000 iterations in the same assumed distribution but this difference reduces as the number of iterations increase. It can also be observed that in all of the tables, the results of the overall properties

(except the minimum and the maximum values) and the percentile values at 1000, 5000 and 10000 iterations are so similar (the difference being less than the second decimal point in units of days) that they can be said to have converged.

These results show that for both of the examples analysed, 1000 iterations provide sufficiently accurate results. For a further increase in confidence, results at 10000 iterations are used for further comparison throughout. The accuracy of the MCS results can be seen in Table 3.19 and 3.20, where the standard error is calculated by:

$$\langle \mu \rangle_{1-\alpha} = \left( \bar{x} - k_{\alpha/2} \frac{\sigma}{\sqrt{n}}; \bar{x} + k_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right)$$

where  $(1 - \alpha)$  is the specified confidence level,  $k_{\alpha/2} = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$  is the value of the standard Normal variate with cumulative probability level  $\alpha/2$ ,  $\bar{x}$  is the sample mean,  $\sigma$  is the standard deviation and  $n$  is the sample size.

The standard error is the confidence intervals for the mean, these intervals contain the population mean and the smaller of the standard error, the more accurate of the sample mean.

**Table 3.19 The standard error (95 % intervals) for the mean of Example 3.1**

	100	1000	5000	10000
<b>Normal</b>	85.31 - 86.49	85.80 - 86.16	85.89 - 86.07	85.90 - 86.02
<b>Log-Normal</b>	85.23 - 86.59	85.58 - 85.98	85.88 - 86.06	85.90 - 86.02
<b>Triangular</b>	87.58 - 89.14	88.14 - 88.62	88.47 - 88.69	88.46 - 88.62
<b>Uniform</b>	90.65 - 92.63	91.83 - 92.47	92.00 - 92.30	92.03 - 92.23
<b>Beta</b>	85.25 - 86.51	85.58 - 85.98	85.86 - 86.04	85.86 - 85.98

**Table 3.20 The standard error (95 % intervals) for the mean of Example 3.2**

	100	1000	5000	10000
<b>Normal</b>	266.86-269.68	269.82-270.70	269.79-270.19	269.77-270.05
<b>Log-Normal</b>	266.93-269.75	269.81-270.71	269.79-270.19	269.77-270.05
<b>Triangular</b>	269.91-273.39	273.62-274.72	273.61-274.11	273.59-273.95
<b>Uniform</b>	271.99-276.79	277.61-279.11	277.63-278.31	277.62-278.10
<b>Beta</b>	266.82-269.64	269.79-270.67	269.77-270.17	269.75-270.03

### 3.5.1.2 Effect of different distributions

Figures 3.7 and 3.16 show the comparison of different distributions at 10000 iterations for the two examples investigated. It can be seen that between 5% and 95%, the percentile values of the Normal, Log-Normal and Beta distributions share virtually identical values but before the 5%<sup>th</sup> and after the 95%<sup>th</sup>, the percentile values of each of the different distributions can be seen to be more varied. The percentile values from the Uniform distribution are the most conservative and those of the Triangular distribution lie between the three distributions (Normal, Log-Normal and Beta) and the Uniform distribution.

Tables 3.8 and 3.17 show quantitative results for comparison. It can be observed that after 10000 simulations, the results of Normal, Log-Normal and Beta distributions show little difference. For example, in Table 3.8, the difference in the means of these three distributions is 0.04 days (0.05% of 85.96 days for the Normal distribution). The difference in the standard deviation is 0.23 days (7% of 3.07 days for the Normal distribution). The greatest difference of the percentile values from 5% to 95% is 0.64 days at the 95% percentile (0.7% of 91.41 days for the Normal distribution). This is because, when assessing the parameters for the different distributions for each individual activity, the mean and standard deviation are assumed to be the same for the three distributions. Example 3.2 (Table 3.17) confirms the same observation. Thus, it can be said that the results of the Normal, Log-Normal and Beta distributions are similar.

However, a small difference can still be seen and this shows the distinguishing characteristics for the different distributions. For example, the differences can easily be seen in the percentile values of less than 5% or greater than 95% percentile. The Beta distribution is a bounded distribution so that it converges quickly with a minimum of 75.91 days and a maximum of 101.85 days. The longest project duration is produced by the Log-Normal distribution with a maximum of 109.87 days. This is because the Log-Normal distribution is unbounded to the right so that a large number of simulations allows more probability of the extreme values being included in the

random number generation process. Characteristics of the Log-Normal distribution can also be seen in the skewness and kurtosis which show larger values.

The results for the Triangular distribution lie between the three distributions (Normal, Log-Normal and Beta) and the Uniform distribution. When compared to the Normal distribution quantitatively, in Example 3.1, the difference of the mean is 2.58 days (3% of the mean duration for the Normal distribution). The difference of the standard deviation is 0.88 days (29% of 3.07 days for the Normal distribution) and the greatest difference of the percentile values from 5% to 95% is 4.2 days (4.6% of 91.41 days for the Normal distribution) at the 95% percentile. In Example 3.2, the difference of the mean is 3.86 days (1.43% of 269.91 days for the Normal distribution). The difference of the standard deviation is 1.76 days (24.3% of 7.23 days for the Normal distribution) and the greatest difference of the percentile values from 5% to 95% is 7.37 days (2.6% of 281.9 days for the Normal distribution) at the 95% percentile. From this, it can be said that for both examples analysed, results from the Triangular distribution show only minor differences to those of the Normal distribution.

Figures 3.7 and 3.16 and Tables 3.8 and 3.17 also show that for both examples the Uniform distribution produces the most conservative results. The maximum project duration provided by Uniform distribution is 109.69 days (9% longer than the 100.56 days of the Normal distribution) for Example 3.1 and 310.81 days (5% longer than the 295.61 days of the Normal distribution) for Example 3.2.

Tables 3.21 and 3.22 are derived from Tables 3.8 and 3.17 and show the sensitivity of the project duration range (maximum-minimum) for different distributions. For example, the project duration range of the Uniform distribution is 34 days (shown in the 'Span 1' column of Table 3.19) and is the longest range in Example 3.1. Further comparisons to exclude extreme values are achieved by considering the ranges between the 5% and 95% percentile and between the 10% and 90% percentile. The same results are again observed (shown in the 'Span 2' and 'Span 3' columns of Table 3.21) and confirm that the Uniform distribution always produces conservative results. This is because the Uniform distribution allows extreme values to be included in the random number generation process with the same probability as any other values. Thus, it can



be said that the results of the Uniform distribution are pessimistic and the form of the distribution always overestimates the probability of the extremes of the variables' range.

It has also been observed that the overall properties of each distribution generated by the @Risk package show a characteristic probability distribution for the different distributions. This shows that the @Risk package executes in a way that correctly generates the random number according to the assigned distributions.

It can be concluded that the Normal, Log-Normal or Beta distribution functions for individual activities give very similar prediction for the total project duration. The Triangular distribution produces medium level duration which are proportionally similar to the Normal distribution results. The Uniform distribution always overestimates the probability of the extremes and therefore gives conservative results.

**Table 3.21 The sensitivity of the project duration range for different distributions of Example 3.1**

	<b>Span 1</b> Maximum - Minimum	<b>Span 2</b> 95% - 5%	<b>Span 3</b> 90% - 10%
<b>Normal</b>	24.27	10.13	7.64
<b>Log-Normal</b>	32.91	10.85	8.08
<b>Triangular</b>	30.50	13.11	10.12
<b>Uniform</b>	<b>33.62</b>	<b>17.59</b>	<b>13.61</b>
<b>Beta</b>	25.94	10.44	8.08
<b>PERT</b>	15.60	8.56	6.66
<b>When the mean is altered</b>			
<b>Sym. - Normal</b>	23.63	9.96	7.49
<b>Sym. - Log-Normal</b>	26.35	10.54	7.82
<b>Sym. - Triangular</b>	25.86	12.42	9.33
<b>Sym. - Uniform</b>	<b>33.62</b>	<b>17.59</b>	<b>13.61</b>
<b>Sym. - Beta</b>	20.97	9.91	7.49

**Table 3.22 The sensitivity of the project duration range for different distributions of Example 3.2**

	<b>Span 1</b> Maximum - Minimum	<b>Span 2</b> 95% - 5%	<b>Span 3</b> 90% - 10%
<b>Normal</b>	57.06	23.88	18.53
<b>Log-Normal</b>	55.90	23.85	18.56
<b>Triangular</b>	58.99	29.71	23.34
<b>Uniform</b>	<b>67.05</b>	<b>40.91</b>	<b>33.31</b>
<b>Beta</b>	49.30	23.91	18.77
<b>PERT</b>	43.32	23.76	18.5
<b>When the mean is altered</b>			
<b>Sym. - Normal</b>	53.20	23.88	18.48
<b>Sym. - Log-Normal</b>	55.97	23.89	18.55
<b>Sym. - Triangular</b>	58.72	29.31	22.89
<b>Sym. - Uniform</b>	<b>67.05</b>	<b>40.91</b>	<b>33.31</b>
<b>Sym. - Beta</b>	50.20	24.01	18.75

### 3.5.1.3 Effect of the manner of setting the mean and standard deviation

Figures 3.8 and 3.17 show a *comparison of different distributions when the mean and standard deviation are altered* (all of the distributions are assumed to be symmetric as discussed in Section 3.3.3). Tables 3.9 and 3.18 provide the quantitative results.

When comparing the altered Normal, Log-Normal and Beta distributions, the results show little difference. For example, in Table 3.9, the difference of the means of these three distributions is 0.08 days (0.08% of the mean duration for the Normal distribution). The difference of the standard deviation is 0.24 days (8% of 3.01 days for the Normal distribution). The greatest difference of the percentile values from 5% to 95% is 0.67 days (0.7% of 95.37 days for the Normal distribution) at the 95<sup>th</sup> percentile. This is because when the means and standard deviations are altered, the means and standard deviations are assumed to be the same for these three distributions.

However, a small difference can be seen and this shows the different characteristics of the different distributions. The percentile values of less than 5% or greater than 95%

percentile and also properties such as the Log-Normal distribution have a greater value of skewness and kurtosis.

When the altered Triangular distribution is compared to the altered Normal distribution, an intersection point (between the 20%-25% percentile) can be observed in both examples. Before this point, the Normal distribution provides more pessimistic results. After the point the situation is reversed. This shows that the results of the Normal distribution only have 25% probability of being larger than those of the Triangular distribution. This is characteristic to the two distributions as when they are assumed to be symmetric with a same extreme values of minimum and maximum, the Normal distribution has more probabilities closer to the mean and the Triangular distribution has more probabilities closer to the extreme values.

However, there are differences between the two distributions. For example, in Table 3.9, the difference of the means is 0.57 days (6% of the mean duration for the Normal distribution). The difference of the standard deviations is 0.7 days (23% of 3.01 days for the Normal distribution) and the greatest difference of the percentile values from 5% to 95% is 1.86 days (2% of 95.37 days for the Normal distribution) at the 95<sup>th</sup> percentile. Example 3.2 (Table 3.18) shows similar results. From this, it can be said that for both examples analysed, results from the altered Triangular distribution show only minor differences to those of the altered Normal distribution.

The Uniform distribution is the same in Figures 3.7 and 3.8 (before and after the altering). It is also the same in Figures 3.16 and 3.17. This is because when altering the means and standard deviations for the different distributions, the minimum and maximum values are retained and the Uniform distribution only uses these two parameters and thus the means and standard deviations of the distribution have not been changed.

When the Uniform distribution is compared to the altered Normal distribution, a intersection point at approximately the 25% percentile can also be observed. Below this point, the altered Normal distribution produces more pessimistic results and after this point the situation is reversed. It was noted that the differences of the results

between the altered Normal distribution and the Uniform distribution are larger than those between the altered Normal distribution and the altered Triangular distribution.

Figures 3.8 and 3.17 show the same characteristic probability distribution for the different distributions as those of Figures 3.7 and 3.16 (before alteration). Thus, it can be said that the results of the five distributions are truly dependent upon the shape of the distribution.

Figures 3.9 and 3.18 show the comparison of different distributions at 10000 iterations before and after the means and standard deviations are altered for the two examples. It can be seen that the results after alteration of the different distributions are all more pessimistic than those prior to the alteration. This is because most of the three point estimates of each activities provided by the two examples are skewed to the right (the most likely values closer to the minimum rather than to the maximum), thus the results are more optimistic (prior to altering). When the distributions are assumed to be symmetrical (the means and standard deviations are shifted), the probabilities of being closer to the maximum or the minimum are equal and therefore the results will be more pessimistic.

The comparisons show that the simulation results are not only dependent upon the shape of the distributions but are also dependent upon the manner of setting the mean and standard deviation. However, it can be seen that the effect of the choice of the distribution is greater than the effect of the manner of setting the mean and standard deviation in the two examples investigated.

### **3.5.2 Comparison of PERT and MCS methods**

Figures 3.9 and 3.18 show the comparison of the PERT and different distributions of Monte Carlo simulation methods. Tables 3.8 and 3.17 give quantitative results respectively as the project durations predicted by the PERT are assumed following a Normal distribution. It can be seen that the PERT gives the most optimistic results when compared to those of the Monte Carlo simulation methods in both cases. This is because in the PERT, the estimation of project duration is based on the uncertainty in

the activities that lie on the critical path. Some paths have shorter expected durations but may have a much higher level of uncertainty. The Monte Carlo simulation method considers all of the paths having equal probability of being included in the random generation process and thus produces more pessimistic results.

In the two examples investigated, PERT produces the most optimistic results, which is in agreement with the common views in PERT studies <sup>[44-48]</sup>. However, from a quantitative point of view, the two examples studied show an insignificant difference when the PERT results are compared to MCS. The difference can be ignored especially when the way of the data collection (such as subjective estimations by experience estimators) is taken into account. In present study, PERT has the advantages of being easy to understand and can be calculated manually, thus saving computational time.

On the other hand, the Monte Carlo simulation method can accommodate the different distribution forms for the duration of individual activities and with large samples it can provide more conservative results. It is easy to understand but it takes a considerable time to be executed. In the two example projects studied, the simulations (10000 iterations) take approximately 4 - 5 hours to complete on a PC, IBM compatible 586 with 16 Megabytes of RAM and a 133 MHz processing chip.

It should be mentioned that one advantage that the MCS has when compared to the PERT is that it can determine critical indices for a given network. Once uncertainty is added to a project model, the critical path becomes less definite. A task that is critical for one iteration may not be critical for the next. The critical index of a task is a measurement of how often a task is critical during a simulation, or how often a task falls on the critical path. The critical index is measured in percentages, or the percent of time a task is critical during the simulation. The critical index gives managers the ability to rate the importance of tasks.

For example, with PERT a project manager may have the following information:

Task 1	Critical
Task 2	Critical

Task 3	Non-critical
Task 4	Non-critical

Thus allowing him to conclude that Task 1 and 2 deserve the most attention, while Task 3 and 4 are less important. But, using Monte Carlo simulation, the following information may be available:

Task 1	Critical Index = 0.55, or 55%
Task 2	Critical Index = 0.85, or 85%
Task 3	Critical Index = 0.5, or 50%
Task 4	Critical Index = 0.2, or 20%

With simulation, it can be concluded that Task 2 is the most important, while Tasks 1 and 3 are of medium importance and Task 4 is least important. The critical index demonstrates that Task 3 is more important than the project manager first realised, which may lead to a change in planning.

It is unfortunate that in the early version (1.12) of the @Risk package, when the two examples were analysed, the Critical Index function did not function well and the results can not be stated in this thesis. It is certain that in the later version (4.2) the simulation output will have a value of 1 for each iteration that the task is critical and a value of 0 when the task is not critical and the critical indices can be viewed easily on the @Risk statistics report.

### **3.6 Summary of findings**

The total project duration of two example projects has been studied by two existing methods, the PERT and MCS methods. From the results of two case studies, the following conclusions can be drawn.

#### **3.6.1 Sensitivity analysis of Monte Carlo simulations**

1. With an increasing number of iterations, a reduction of the sampling variability can be attained.

2. After a certain number of iterations, the percentile values from 5% to 95% approximately tend to be similar and increasing the number of iterations only affects the minimum (less than 5%) and the maximum (greater than 95%) percentile values. This fact is irrespective of the type of probability distribution function specified for the activity durations.
3. The project duration range (maximum minus minimum) is increased as the number of simulations increase. For distributions without a boundary, such as the Normal and Log-Normal (one side bounded) distributions, the project duration range will tend to infinity. On the other hand, distributions with a boundary, such as the Triangular, Uniform and Beta distributions, project duration ranges will converge at certain points (the minimum and maximum values).
4. For the two projects studied, with 30-52 activities, 1000 iterations was sufficient to provide accurate results.
5. After certain iterations, for example, 10000 iterations in the example projects, it can be seen that between 5% and 95%, the percentile values of the Normal, Log-Normal and Beta distributions share virtually identical values but before the 5% and after the 95%, the percentile values of each of the different distributions can be seen to be more varied.
6. For the parameters assessed in the present research, the Normal, Log-Normal or Beta distribution functions for individual activities give very similar predictions for the total project duration. The Triangular distribution produces medium level durations which are proportionally similar to the Normal distribution results. The Uniform distribution always overestimates the probability of extremes and therefore gives conservative results
7. When the mean is altered, that is all of the distributions are considered symmetric, Normal, Log-Normal and Beta distribution functions again give very similar prediction for the total project duration. Before a turn point (20%-25% percentile values), these three distributions (Normal, Log-Normal and Beta) provide more

pessimistic results whereas a Uniform distribution provides the most optimistic results. After that point, the three distributions provide more optimistic results whereas a Uniform distribution provides the most pessimistic results. The Triangular distribution produces medium level durations which are proportionally similar to the Normal distribution results.

8. The results of the five distributions are dependent upon the shape of the distribution. After altering the mean, the results of different distributions are all more pessimistic than those of before the alteration.
9. The simulation results are not only dependent upon the shape of the distributions but are also dependent upon the manner of setting the mean and standard deviation. However, it can be seen that the effect of the choice of the distribution is greater than the effect of the manner of setting the mean and standard deviation in the two examples investigated.
10. The overall properties of each distribution generated by the @Risk package show a characteristic probability distribution for the different distributions. The @Risk package executes in a way that correctly generates the random number according to the assigned distributions.

### **3.6.2 Comparison of PERT and MCS methods**

1. PERT gives the most optimistic results when they are compared to those of the MCS in both of the example projects.
2. PERT is easy to understand, can be calculated manually and saves computational time.
3. Monte Carlo simulation method can accommodate the different distribution forms for the duration of individual activities and with large samples it can provide more conservative results.



4. MCS is also very easy to understand but requires considerable computational time.
5. The MCS can determine critical indices for a given network whereas the PERT cannot.

# CHAPTER 4 THE MODIFIED STOCHASTIC ASSIGNMENT MODEL (MSAM)

## 4.1 Introduction

As stated in Chapter 2, the network analysis methods usually used in construction utilise classic analytical methods or involve some form of Monte Carlo simulation <sup>[82]</sup>. These methods are still commonly used in construction network analysis even though most of them are up to 40 years old. The best known method is the CPM <sup>[37]</sup>. The CPM is based on the assumption that the durations of the activities in a project network are deterministic; accordingly, the project duration estimated with CPM are naturally also deterministic <sup>[83]</sup>.

In most projects, the durations of the various activities are not always predictable. The activity durations should therefore be modelled as random variables, and evaluation of the project time considered as a problem of probabilistic network analysis.

The PERT method <sup>[38,39,43]</sup> is the most commonly used method which considers the activity durations as random variables. However, the required project duration is usually determined solely on the basis of the mean critical path (the network path with the longest expected time). This invariably underestimates the required completion time for a given network <sup>[47,48,51,54,84,85]</sup>.

Unlike PERT, which only considers the mean critical path, PNET <sup>[48]</sup> takes all of the paths into consideration and can therefore be expected to generate more accurate results <sup>[50,51,86]</sup>. The main disadvantage of PNET is that it requires path enumeration and in a large network, the number of paths can be very large.

With the advent of increased computer processor power over the previous two decades, Monte Carlo simulation methods <sup>[52]</sup> have become increasingly popular. However, for large networks this requires considerable expense in terms of computational time that may not be necessary <sup>[48,82]</sup>.

The above summarised limited review of related methods indicate that there is a considerable need for probabilistic analysis of construction networks. A practical method for evaluating activity networks under uncertainty is needed to provide the probabilistic information required for making proper decisions.

One of the objectives of this investigation is to propose a new analytical method, the Modified Stochastic Assignment Model (MSAM) for the estimation of the project completion time under uncertainty. The proposed method is inspired by a previous method used solely in traffic networks, the Stochastic Assignment Model (SAM) <sup>[87]</sup>. The MSAM method employs a computational procedure proposed by Clark <sup>[88]</sup>. This produces estimates of the first four moments for the maximum of two Normally distributed random variables (representing activity durations). By repeated application of Clark's result through a network of activities, it is possible to estimate the mean and standard deviation of the project duration time. There is no literature describing computational experience with an efficient, detailed algorithm which incorporates Clark's results in construction network analysis. In contrast, within transportation research, the SAM has been used to model drivers' route choice. As drivers are assumed to select the shortest route, Clark's results are applied in an opposite way to find the expected minimum perceived path costs. A computer program has been written to implement the SAM algorithm by Maher <sup>[87]</sup> and Maher and Hughes <sup>[89]</sup>. In the present research, the proposed new method is modified from the SAM. This chapter shows that the new method is an analytical counterpart of the Monte Carlo simulation method, which can be applied to construction networks and provides meaningful probabilistic information.

In the remainder of the chapter, a detailed description of the original SAM is given in which the basic assumptions and mathematical formulation of SAM are described and a summary of the algorithm is given. The proposed new method, the MSAM, is presented through explaining the differences and similarities between transportation and construction networks. This shows how and why the SAM can be applied to construction network analysis. The MSAM algorithm is also described. Five example projects are used to demonstrate the validity of the MSAM and to illustrate its

application in construction project evaluations. The accuracy of the MSAM method is assessed by comparing to the Monte Carlo simulation method. A comparison of the MSAM with other analytical methods commonly used in construction network analysis, such as PERT and PNET, has also been presented. Finally, a summary of conclusions are given.

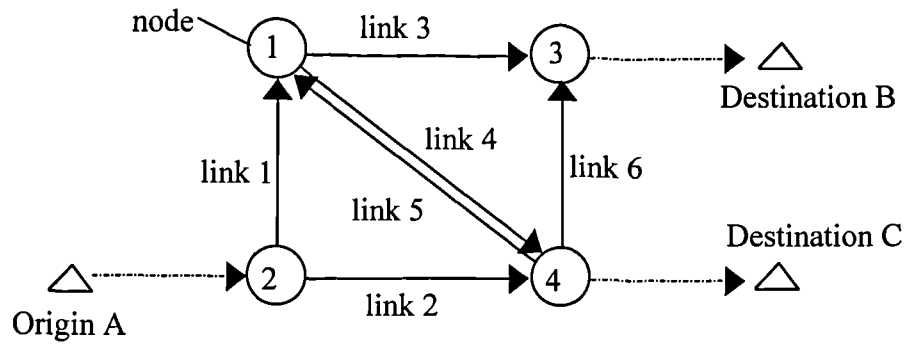
## 4.2 The original SAM

### 4.2.1 Description and terminology

The Stochastic Assignment Model (SAM) described in Maher <sup>[87]</sup> and Maher and Hughes <sup>[89]</sup>, was originally developed for solving the stochastic traffic assignment problem in transportation studies. It was the first to implement a probit assignment model (that is assuming a Normal distribution for the link costs) without recourse to full path enumeration in a transportation network. This was done by using a scanning method, in conjunction with Clark's approximation <sup>[88]</sup>, to calculate the choice probabilities at each node or junction.

To better understand the SAM, some terms used in transportation study are explained below.

A network is referred to as a *pure network* if only its topology and connectivity are considered. If a network is characterised by its topology and flow properties (such as origin-destination demands, capacity constraints, path choice and link cost functions) it is then referred to as a *flow network* (see Du and Nicholson <sup>[90]</sup>). A *transportation network* is a flow network representing the movement of people, vehicles or goods. Any transportation network can be represented as a *graph* in the mathematical sense, consisting of a set of *links* and a set of *nodes*. The links represent the movements between the nodes, which in turn represent points in space (and possibly also in time). The link may also refer to a specific mode of transport (for example, a movement by car, bus, train bicycle or on foot), in which case a *path* in the transportation network specifies both the route and the mode(s) of transport. An example transportation network can be seen in Figure 4.1.



**Figure 4.1 An example of transportation network**

Travel behaviour with the network is governed by *costs*. Of particular interest is the *path cost*. At equilibrium, drivers choose the paths that they perceive at the time to be the least cost. Where the network is appropriately specified, this may represent both route and mode choice. The *trip cost* is then equal to the cost of the path(s) chosen.

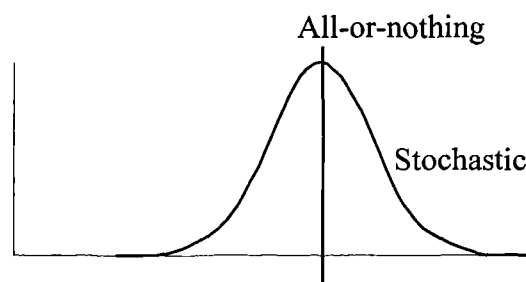
The level of demand is determined by the trip costs. Underlying both path and trip costs are the *link costs*, as a path cost is the sum of the costs of the links constituting the path, and the trip cost is the cost of the path with the minimum perceived cost. The relationship between link cost and link flow is called the *link cost function*.

*Traffic assignment* is the problem of predicting drivers' route choices, and hence the congestion which will occur in a road network. The simplest of all assignment methods is the "all-or-nothing" model, in which the drivers' routes are found by means of a shortest path algorithm such as that of Dijkstra<sup>[91]</sup> using constant link costs. There is therefore no multi-routing. Traditionally there have been two techniques for modelling the multi-routing which occurs in practice. *Equilibrium methods* and *stochastic methods*. Equilibrium methods assume that drivers are identical, perfectly knowledgeable and rational, and hence arrange themselves so that they minimise their travel costs. The methods are deterministic using functional relationships between link cost and link flow. Stochastic methods model the random variations in drivers' perception of costs, due to their different levels of knowledge and priorities. Link costs are therefore random variables. Within the class of stochastic methods, most models either assume a Normal distribution for the link costs (*probit model*), or use the logistic function to split traffic between a set of available routes (*logit model*).

#### 4.2.2 Assumptions

Stochastic assignment models are an extension of the all-or nothing model, where all link costs are fixed, and drivers select the same minimum cost path between a particular O-D (origin-destination) pair. All drivers are assumed to be the same in their perception and knowledge of the network. However, in stochastic models, the assumption is that drivers perceive cost differently from each other, either through differing levels of knowledge of the network, or through differing priorities. For instance some may be “time-minimisers” and others “distance-minimisers” and are assumed to minimise their perceived cost. The consequence is that drivers choose different “minimum cost” routes, and several routes will potentially be used between any O-D pair.

The cost is no longer assumed to be a single value, instead the perceived cost is a random variable taken from a probability distribution (Figure 4.2). The perceived costs on different links are independent of each other. The independence assumption ensures that in certain cases (the Normal and Gumbel distributions) the link cost variables  $X_a$  can be summed to give the path cost variables  $Z_k$  of the same distribution. The assignment problem then becomes one of calculating the choice probabilities between the different  $Z_k$ . It is possible to imagine scenarios where this assumption does not hold, for instance where drivers who prefer (or dislike) motorways will consistently perceive all motorway links as having a lower (or higher) cost. However, the assumption makes the problem much more tractable, and most authors have used it, on the basis that it will have a small effect on actual link volumes, in comparison to the difference between a stochastic and a deterministic method.



**Figure 4.2 All-or-nothing versus stochastic models**

The choice of distribution for the link costs, the main distinguishing factor between different approaches to stochastic assignment, determines the ease of calculation of the choice probabilities.

SAM assumes that the cost of travel along link  $k$  is  $X_k$ , a random variable, independent of any other link cost and whose distribution will be taken to be Normal (a probit model). It follows then that the cost of travel along any prescribed path will also be Normal with a mean and variance which are the sums of the means and variances of the links which make up the path.

SAM operates by starting at an origin node and steadily moving a frontier through the network until all destinations have been reached. There are two types of operation. The first is the merging of two (or more links) where they meet at a node (enter the node), and the second is the scanning from a node to the ends of links which exit from that node. In the following description, the set of links which enter node  $n$  (the “before” links) is referred to as  $B_n$  and the set of links which exit from node  $n$  (the “after” links) as  $A_n$ .

At each merge the split of traffic between the two routes is calculated and recorded for later use. When the forward pass of the frontier is complete, a backward pass uses the stored splits at each merge and uses these to load the traffic from that origin to all destinations on to the network. Each origin is then considered in turn, and the traffic loads built up incrementally. The main advantage of the method over other stochastic methods is that proper account is taken of the correlations between routes at the merge point.

The cost of travel from the origin to the end of link  $j$  is  $Y_j$  and the cost from the origin to node  $n$  is the random variable  $W_n$ . If the set of all links which end at node  $n$  is  $B_n$ , then:

$$W_n = \min_{j \in B_n} Y_j \quad (4.1)$$

It is only possible to deal completely with a node (in the sense of constructing the distribution of its  $W$ ) when all contributing  $Y_s$  are known. When all such  $Y_s$  are known and  $W$  has been constructed, the node is “complete”. At the same time are found the proportions of traffic  $p_j$  which arrive at node  $n$  by link  $j$ . These proportions are defined by:

$$p_j = \text{Pr ob} \left( Y_j = \min_{k \in B_n} (Y_k) \right) \quad j \in B_n \quad (4.2)$$

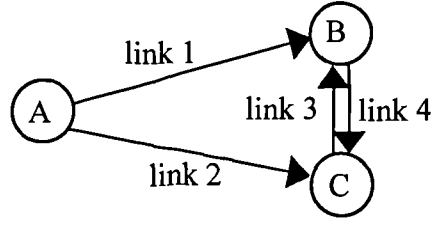
In scanning from node  $n$  which is complete and for which the set of all links which start there is  $A_n$ :

$$Y_j = W_n + X_j \quad j \in A_n \quad (4.3)$$

It is clear that as soon as node  $n$  is complete the distribution of  $Y_j$  for all links in  $A_n$  can be found. In principle the specification of the distribution of the  $X_s$  will lead, through the repeated application of the merging and scanning processes of Equation 4.1 and 4.3, to the final result by the end of the forward pass through the network. A backward pass is then made to load the traffic on to the network, using the splits determined at each merging point by Equation 4.2.

Unfortunately, it can happen when treating a real network that there comes a point in the process at which no node is complete. For example, the simple network in Figure 4.3, in which A is the origin. Initially scanning is done from A, to the ends of links AB and AC. At that point neither B or C is complete and no further progress can be made, without some alternative course of action. This occurs when there is “looping” in a network. This will be referred to as “deadlock” and is discussed in (Maher <sup>[87]</sup> and Maher and Hughes <sup>[89]</sup> ). Since it does not exist in a construction network (no looping is allowed in construction networks), the discussion is not included here.





**Figure 4.3 Three-nodes network**

The second point concerns the manner by which the distributions (of the  $Y_s$  and  $W_s$ ) are determined. Mathematically it would be most convenient to use some known distributional form, so that only the values of a few parameters such as mean, variance and covariance need be evaluated. It soon becomes apparent though, when studying Equation 4.1, that no such distributional form exists. To make progress therefore some approximate method is required.

#### 4.2.3 The Clark approximation

Clark <sup>[88]</sup> proposed an analytic result for the greatest of a finite set of Normal random variables. This very useful result deals with the maximum of two Normal variables  $X_1$  and  $X_2$ , which have expected values  $\mu_1, \mu_2$  and variances  $\sigma_1^2, \sigma_2^2$ , and such that the correlation between  $X_1$  and  $X_2$  is  $\rho_{12}$ . The resulting variable is also, approximately, Normal with the following distribution:

$$\text{Max}(X_1, X_2) \rightarrow N(\nu_1, \nu_2 - \nu_1^2) \quad (4.4)$$

where  $\nu_1, \nu_2 - \nu_1^2$  are the expected value and variance of the random variable  $\text{Max}(X_1, X_2)$ , Clark has proved that:

$$\nu_1 = \mu_1 \Phi(\gamma) + \mu_2 \Phi(-\gamma) + a \phi(\gamma) \quad (4.5)$$

$$\nu_2 = (\mu_1^2 + \sigma_1^2) \Phi(\gamma) + (\mu_2^2 + \sigma_2^2) \Phi(-\gamma) + (\mu_1 + \mu_2) a \phi(\gamma) \quad (4.6)$$

and

$$a^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12} \quad (4.7)$$

$$\gamma = (\mu_1 - \mu_2)/a \quad (4.8)$$

where  $\Phi$  and  $\phi$  are respectively the probability distribution function and density function for the Normal distribution:

$$\Phi(x) = \int_{-\infty}^x \phi(t) dt \quad (4.9)$$

and

$$\phi(x) = (2\pi)^{-1/2} \exp(-x^2 / 2) \quad (4.10)$$

The probability of  $X_1$  being the larger is  $p_1$  where:

$$p_1 = \Phi(\gamma) \quad (4.11)$$

and hence:

$$p_2 = \Phi(-\gamma) \quad (4.12)$$

If  $r$  denotes the coefficient of linear correlation, we write  $r(X_1, X_2) = \rho_{12}$ ,  $r(X_1, X_3) = \rho_{13}$ ,  $r(X_2, X_3) = \rho_{23}$ . If the correlations  $\rho_{13}$  and  $\rho_{23}$  of  $X_1$  and  $X_2$  with a third Normal variable  $X_3$  are known, then the correlation of the new variable  $\text{Max}(X_1, X_2)$  with  $X_3$  can be found:

$$r[X_3, \text{Max}(X_1, X_2)] = [\sigma_1 \rho_{13} \Phi(\gamma) + \sigma_2 \rho_{23} \Phi(-\gamma)] / (\nu_2 - \nu_1^2)^{1/2} \quad (4.13)$$

Equation 4.13 is used in estimating moments of the greatest of more than two Normally distributed variables. By repeated use of this result, the maximum of any number of Normal random variable can be calculated recursively:

$$\text{Max}(X_1, X_2, \dots, X_n) = \text{Max}(\text{Max}(\dots \text{Max}(X_{n-2}, \text{Max}(X_{n-1}, X_n)) \dots)) \quad (4.14)$$

SAM uses Clark's results in an opposite way, to find the minimum of two random variables, a simple modification is needed, using the fact that:

$$\text{Min}(X_1, X_2) = -\text{Max}(-X_1, -X_2) \quad (4.15)$$

so if:

$$Y = -\text{Max}(-X_1, -X_2) \quad (4.16)$$

then the Clark formula becomes:

$$\begin{aligned} v_i &= E(Y) \\ &= -[\mu_1 \Phi(-\gamma) - \mu_2 \Phi(\gamma) + a\phi(-\gamma)] \\ &= \mu_1 \Phi(-\gamma) + \mu_2 \Phi(\gamma) - a\phi(\gamma) \end{aligned} \quad (4.17)$$

$$v_2 = (\mu_1^2 + \sigma_1^2) \Phi(-\gamma) + (\mu_2^2 + \sigma_2^2) \Phi(\gamma) - (\mu_1 + \mu_2) a \phi(\gamma) \quad (4.18)$$

where

$$a^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12} \quad (4.19)$$

$$\gamma = (\mu_1 - \mu_2) / a \quad (4.20)$$

The probability that it is  $Y_i$  which is the smaller is  $p_i$  where:

$$p_i = \Phi(-\gamma) \quad (4.21)$$

and hence:

$$p_j = \Phi(\gamma) \quad (4.22)$$

Finally the covariance between  $W$  and some other variable  $Y_k$  is:

$$\text{Cov}(W, Y_k) = p_i v_{ik} + p_j v_{jk} \quad (4.23)$$

where  $v_{ik}$  is the covariance between  $Y_i$  and  $Y_k$ . The accuracy of these results is remarkably good, and certainly of sufficient quality to make the application to traffic assignment feasible. This result gives a possible method for calculating the route flows in a network, if the route cost variables, with their covariances, are known.

#### 4.2.4 The SAM algorithm

The SAM algorithm is now described in some detail; as stated above, the link costs are assumed to be Normally distributed and independent with specified means and variances. For simplicity, the variability parameter  $\beta$  (the ratio of variance to mean) is assumed to be constant for all links in the network, although this parameter could be allowed to take different values for different links, with no alterations to the basic method. In any case, it is often convenient keep the same value of  $\beta$  for the whole network which ensures that a series of links will sum together to give a distribution which has the same value of  $\beta$ :

$$\sigma^2 = \sum_a \sigma_a^2 = \sum_a \beta \mu_a = \beta \sum_a \mu_a \quad (4.24)$$

In this way, the modeller can be assured that the ratio of variance to mean will not depend on the level of detail of the network.

The algorithm loads each origin in turn, and comprises a forward pass, during which choice probabilities at junctions are calculated, and a backward pass, in which the probabilities are used to load the traffic. During the forward pass, the algorithm progresses through the network outwards from the current origin, by a series of ‘scanning’ and ‘merging’ operations.

The first step is to scan outwards from the current origin  $r$  to the ends of all links which leave the origin. This gives the travel time distributions from the origin to the ends of those links (these random variables will be referred to as  $Y_a$ , for any link  $a$  in the network, dropping the reference to origin  $r$  for ease of notation). For the initial scan from the origin, the means and the variances of the  $Y_a$  are simply the means and variances of the link costs.

Next comes the ‘merge’ step: find a node  $n$ , such that the distributions of the  $Y_a$  variables have been determined for all links  $a$  in the set  $B_n$ . There will always be such a node in the cases where the network contains no loops. (since looping does not exist

in a construction network, the discussion is restricted to such cases) The merge calculation uses the Clark approximation (Equations 4.17 - 4.23) to find the minimum of the  $Y_a$  at this node, determining the distribution of the minimum perceived travel time  $W_n$  from the origin to this node. The random variable  $W_n$  is defined as:

$$W_n = \min_{a \in B_n} Y_a \quad (4.25)$$

The calculations give the mean and variance of  $W_n$ , as well as the probabilities  $p_{an}$  that, of all the traffic entering node  $n$ , it is the link  $a$  which has the minimum travel time. That is:

$$p_{an} = \text{prob}\left(Y_a = \min_{k \in B_n}(Y_k)\right) \quad a \in B_n \quad (4.26)$$

$p_{an}$  are defined such that  $\sum_{a \in B_n} p_{an} = 1 \forall n$ . These probabilities will later be used to split the traffic between the entering links. The node  $n$  is marked as ‘complete’.

After merging at node  $n$ , scan from there. This involves calculating the  $Y_a$  for the  $a$  in  $A_n$ . The means of these  $Y_a$  are simply the mean of  $W_n$  plus the mean of the link cost distribution; the variances are calculated similarly. The covariances of the  $Y_a$  for  $a$  in  $A_n$  are also calculated, with all the other  $Y_a$  known so far. The covariance between the  $Y_a$  for  $a$  in  $A_n$  are all equal to the variance of  $W_n$ . Other covariances are calculated by the following formula:

$$v_{ab} = \sum_{c \in B_n} p_{an} v_{cb} \quad \forall a \in A_n, \quad b \in L, \quad b \notin A_n \quad (4.27)$$

in which  $v_{ab}$  is the covariance between activities  $a$  and  $b$ , where  $a$  belongs to one of the activities after event  $n$ . Activity  $b$  belongs to  $L$ , in which  $L$  denotes all the known activities excluding the set of activities  $B_n$ . After the scanning step, return to the merge step, to find a node which is not yet complete, but has known  $Y_a$ , for all the  $a$  in  $B_n$ . Repeat the merging and scanning steps for this new node and subsequent ones until all nodes in the network are complete. This constitutes the end of the forward pass for the

current origin  $r$ ; all the  $p_{an}$  have been calculated, and will be used in a backward pass to load the traffic.

In the backward pass the nodes are processed in reverse order to that of the forward pass; for each node  $n$ , the links  $a$  in  $B_n$  are allocated a flow  $x_a$ , using the  $p_{an}$  from the forward pass:

$$x_a = p_{an} \left( q_{rn} + \sum_{a' \in A_n} x_{a'} \right) \quad \forall a \in B_n \quad (4.28)$$

where  $q_{rn}$  is the demand from  $r$  to  $n$ . This ensures that the demand from  $r$  to all destinations is loaded.

Having carried out the forward and backward pass for origin  $r$ , the loading for  $r$  is complete. The loading is carried out for all origins, with the link flows being summed over the origins. This constitutes the end of a complete SAM loading. In summary then, the steps for loading the demand, origin by origin, are as follows:

1. Scan outwards from the current origin to the ends of all links which leave the origin.
2. Find a node  $n$ , such that the distributions of the  $Y_a$  variables have been determined for all links  $a$  which enter node  $n$ . “merge” at this node, determining the distribution of the minimum perceived travel time  $W_n$  from the origin to this node.
3. Scan out from node  $n$  (which was completed in step 2), estimating the distributions of the  $Y_a$  for all  $a \in A_n$ .
4. If there are any nodes which are not complete, return to step 2.

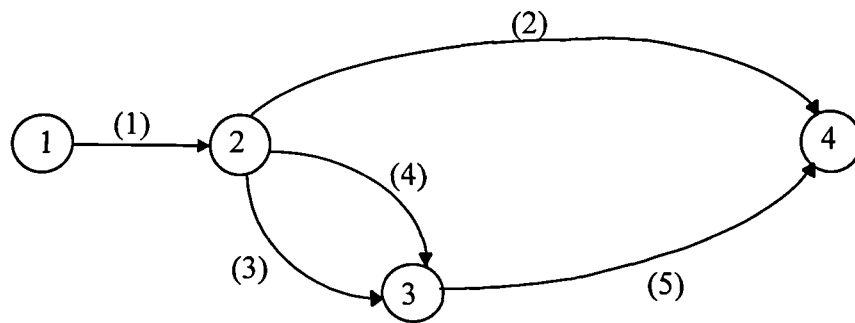
Steps 1-4 comprise the forward pass for the current origin, in which all the required probabilities  $p_{an}$  are calculated.

Then carry out the backward pass to load the traffic:

5. Using the splitting probabilities  $p_{an}$  from step 2, load the traffic, starting at the last node at which merging took place, and work back through each node  $n$  towards the origin, in the reverse order to which merging took place.
6. Repeat steps 1-5 for each node in turn.

#### 4.2.5 SAM applied to a simple transportation network

In this subsection, a simple network is used to show the application of SAM. The example is chosen from Philip Hughes's PhD thesis <sup>[92]</sup>. The network in Figure 4.4 has five links, with the link number shown in brackets.



**Figure 4.4 A transportation network with correlated routes**

To illustrate the algorithm's workings, consider first the set of costs:

$$c_1 = 5$$

$$c_2 = 10$$

$$c_3 = 3$$

$$c_4 = 3.5$$

$$c_5 = 6$$

The parameter  $\beta$  is set equal to 0.5 for this example, and assume that 100 units of demand travel from node 1 to 4. For this simple case, the method is followed through in some detail; when SAM is applied to the network, the steps of the algorithm will be as follows:

1. Scan outwards from the origin, node 1, to the ends of all links which leave the origin (link 1 only). Then  $E(Y_1) = 5$ .
2. Find a node  $n$ , such that the distributions of the  $Y_a$  variables have been determined for all links  $a$  which enter node  $n$ . This node is node 2, as  $Y_1$  was found in step 1. The “merge” at this node is trivial, the distribution of the minimum perceived travel time  $W_2$  is just the distribution of  $Y_1$ , with mean 5 and variance 2.5.
3. Scan outwards from node 2, to the ends of all links which leave node 2 (links 2,3, and 4). Then  $E(Y_2) = 15$ ,  $E(Y_3) = 8$ ,  $E(Y_4) = 8.5$ , found by adding  $E(W_2)$  (5 units) to the link costs  $c_1$ ,  $c_2$ , and  $c_3$  respectively. The covariances of these distributions with each other, and with  $Y_1$ , are all equal to 2.5, the variance of  $Y_1$ .
4. Find a node  $n$ , such that the distributions of the  $Y_a$  variables have been determined for all links  $a$  which enter node  $n$ . This node is node 3, as  $Y_3$  and  $Y_4$  are now known. Merge at this node, determined the distribution of  $W_3$ , which will be the minimum of  $Y_3$  and  $Y_4$ . The mean of  $W_3$  is 7.5033, and the variance is 3.6026, as given by  $\nu_1$  and  $\nu_2 - \nu_1^2$  in the Equations 4.17 and 4.18 above. The probability for link 3 is 0.6093, and for link 4 it is 0.3907.
5. Scan out from node 3, which has just been completed, calculating the distribution  $Y_5$ ; the mean is 13.5033 ( $E(W_3) + c_5$ ) and the variance is 4.1026 (found by calculating  $Var(W_3) + \beta c_5 = 3.6026 + 3 = 6.6026$ ). The covariances of  $Y_5$  with the other known  $Y_a$  distributions are also found now. For each link  $a$ , these are calculated by multiplying the split for each link  $b$  in  $B_3$  by the covariance  $Cov(Y_a, Y_b)$ , and summing:



$$\begin{aligned}
Cov(Y_5, Y_1) &= 0.6093 * Cov(Y_3, Y_1) + 0.3907 * Cov(Y_4, Y_1) = 0.6093 * 2.5 + 0.3907 * 2.5 = 2.5 \\
Cov(Y_5, Y_2) &= 0.6093 * Cov(Y_3, Y_2) + 0.3907 * Cov(Y_4, Y_2) = 0.6093 * 2.5 + 0.3907 * 2.5 = 2.5 \\
Cov(Y_5, Y_3) &= 0.6093 * Cov(Y_3, Y_3) + 0.3907 * Cov(Y_4, Y_3) = 0.6093 * 4 + 0.3907 * 2.5 = 3.4139 \\
Cov(Y_5, Y_4) &= 0.6093 * Cov(Y_3, Y_4) + 0.3907 * Cov(Y_4, Y_4) = 0.6093 * 2.5 + 0.3907 * 4.25 = 3.1838
\end{aligned}$$

6. Merge at node 4;  $Y_2$  and  $Y_5$  are now known, so this is possible. The mean of  $W_4$  is 12.9025 and the variance is 5.6204, and the split between links 1 and 4 is 0.3097 and 0.6903. All nodes are now complete, so the forward pass is finished.

For completeness the full covariance matrix  $V$  is given, where the diagonal elements are the variances of the  $Y_a$  distributions:

$$V = \begin{pmatrix} 2.5 & 2.5 & 2.5 & 2.5 & 2.5 \\ 2.5 & 7.5 & 2.5 & 2.5 & 2.5 \\ 2.5 & 2.5 & 4.0 & 2.5 & 3.4139 \\ 2.5 & 2.5 & 2.5 & 4.25 & 3.1838 \\ 2.5 & 2.5 & 3.4139 & 3.1838 & 6.6026 \end{pmatrix}$$

Then carry out the backward pass to load the traffic using the splitting probabilities  $p_{an}$  from step 2. The merge steps were carried out at nodes 2 then 3, so node 3 is now processed first:

Node 3:

$$x_l = 0.3097 \times 100 = 30.97$$

$$x_r = 0.6903 \times 100 = 69.03$$

Node 2:

$$x_2 = 0.6093 \times 69.03 = 42.0581$$

$$x_3 = 0.3907 \times 69.03 = 26.9413$$

#### **4.2.6 Concluding remarks**

The basis of SAM has been described, which was originally developed for solving traffic assignment problems. It is the first to implement a probit assignment model, without recourse to full path enumeration in a transportation network. This is done by using a scanning method, in conjunction with Clark's approximation, to calculate the choice probabilities at each junction. It is an analytical counterpart of the Monte Carlo simulation method and takes proper account of the correlations between routes. A computer program has been written, in FORTRAN 77, to implement the SAM algorithm (Maher <sup>[87]</sup> and Maher and Hughes <sup>[89]</sup>). Tests have been carried out and the results agree closely with those from Monte Carlo simulation.

### **4.3 The proposed new method- Modified SAM (MSAM)**

#### **4.3.1 Similarities of transportation and construction networks**

As already stated in Chapter 2 and the previous subsection 4.2.1, transport and construction networks are quite similar. A construction network is a flow network as well, it represents the activity durations and their dependencies in a construction project. Both transport and construction networks considered as a network approach being made up from arrows and nodes. Arrows represent links in transport networks and activities in construction networks. Nodes represent points in space (and possibly also in time) in transport networks and events in construction networks. In both cases there are various paths from the starting point to the finishing point. In traffic networks, more than one destination can sometimes be seen, but the network is always analysed each O-D pair (Origin - Destination) in turn.

However, there are differences between the two types of networks. For instance, "bidirected" or "parallel" cases can not be found in construction networks. "Bidirected" means there are two orientations between the same pair of nodes (Figure 4.1, links 4 and 5 between nodes 1 and 4). Two links are said to be "parallel" if they connect the same pair of nodes in the same direction. The logic of connectivity of the "parallel" case is represented by a dummy activity (broken or dotted arrows) in construction networks.

Construction networks have a very special structure when compared to transport networks. In particular, such representations contain arrows directed (which only have one orientation) and form circuitless networks. In terms of a network, a circuit is defined as a finite chain with the first and last nodes being coincident (looping). It can also be stated that looping is not allowed in construction networks.

Further detailed descriptions of network concepts and their definitions are referred to by Bell and Iida <sup>[93]</sup> for transport networks; Battersby <sup>[42]</sup>, Lockyer and Gordon <sup>[33]</sup> for construction networks; and Phillips and Garcia-Diaz <sup>[94]</sup> for both types of network.

Network analysis problems are also quite similar in the construction and transportation industries. The problem in the case of transportation is to minimise the transportation costs (or time), in contrast, in the case of the construction, the problem is to determine the project completion time being the longest path in network. Both problems are solved under a network approach.

As explained in the previous section, SAM assumes that *the driver's perceptions of* travel times along the links in a traffic network are random variables, independent of any other link and whose distribution will be Normal. The cost of travel along any prescribed path will also be Normal with a mean and variance which are the sums of the means and variances of the links which make up the path.

SAM operates by starting at an origin node and steadily moving a frontier through the network until all destinations have been reached. There are two types of operation. The merging operation is used to determine the distribution of the minimum cost to that node and also to find the proportions of traffic  $p_j$  which arrive at node  $n$  by link  $j$  (taking account of the correlations between routes). This is achieved by applying Clark's formula in an opposite sense to find the least of the link costs. The scanning operation is used to determine the distribution of the costs to the ends of links which exit from that node. The repeated operations result in the distribution of the minimum cost to any destination node  $n$ . The backward pass then uses the stored splits at each merge and to load the traffic.

It can be seen that under the same assumptions, that is if the duration of activities in a construction network are random variables, independent of any other activity and whose distribution will be Normal, the duration along any prescribed path will also be Normal with a mean and variance which are the sums of the means and variances of the activities which make up the path.

Using the same procedures as SAM, the repeated merging and scanning processes, the maximum time to any destination node (end event) can be found. It should be noticed that Clark's results are applied here in their original sense in contrast to that in the SAM. The backward pass is not needed in construction network analysis since there is no "loading" (SAM is a stochastic method assuming link costs are variables hence the backward pass is needed to load the traffic). However, it can be used to determine the probability of an activity being critical.

It is clear that with only minor modifications, the SAM method is applicable to construction network analysis for project planning.

#### **4.3.2 The MSAM algorithm**

The MSAM algorithm is now described in detail. As stated above, the same assumptions of the SAM still hold. The assumption of the randomness of individual activity duration in a construction network is well accepted due to the many uncontrollable factors, such as weather and resource availability, encountered in practice <sup>[48,95]</sup>. The Normality assumption for individual activity duration has often been made in the literature <sup>[48,59,71,96]</sup>. For a general network, if there are a large number of activities in a path, the path duration will tend toward a Normal variate (by virtue of the Central Limit Theorem <sup>[39]</sup>). For most medium to large scale construction networks there are enough activities for the C. L. T. to be valid. However, for small scale networks, the accuracy of MSAM results need closer examination but the method could still provide a useful approximate estimate of project duration.

The algorithm treats each event in turn, carries out a forward pass, during which choice probabilities at junctions are calculated. The algorithm progresses through the network outwards from the start event, by a series of ‘scanning’ and ‘merging’ operations. At the end of the forward pass, the expected value of the maximum path length (the project duration) is known.

The first step is to scan outwards from the start event  $r$  to the ends of all activities which leave the event. This gives the duration distributions from the event to the ends of those activities (these random variables will be referred to as  $Y_a$ , for any activity  $a$  in the network, dropping the reference to event  $r$  for ease of notation). For the initial scan from the start event, the means and the variances of the  $Y_a$  are simply the means and variances of the activity durations.

Next comes the ‘merge’ step: find an event  $n$ , such that the distributions of the  $Y_a$  variables have been determined for all activities  $a$  in the set  $B_n$  (denoting the set of activities “Before” event  $n$ ). The merge calculation uses the Clark approximation (Equations 4.4-4.14) to find the parameters of the distribution of the maximum activity duration  $W_n$  from the start event to this event. The random variable  $W_n$  is defined as:

$$W_n = \max_{a \in B_n} Y_a \quad (4.29)$$

The calculations give the mean and variance of  $W_n$ , as well as the probabilities  $p_{an}$  that, of all the activities entering event  $n$ , it is activity  $a$  which was the latest to be completed. That is:

$$p_{an} = \text{prob}\left(Y_a = \max_{k \in B_n} (Y_k)\right) \quad a \in B_n \quad (4.30)$$

$p_{an}$  are defined such that  $\sum_{a \in B_n} p_{an} = 1 \forall n$ . The event  $n$  is then marked as ‘complete’.

After merging at event  $n$ , scan from there. This involves calculating the  $Y_a$  for the  $a$  in  $A_n$  (denoting the set of activities “After” event  $n$ ). The means of these  $Y_a$  are simply the mean of  $W_n$  plus the mean of the duration distribution; the variances are calculated

similarly. The covariances of the  $Y_a$  for  $a$  in  $A_n$  are also calculated, with all the other  $Y_a$  known so far. The covariance between the  $Y_a$  for  $a$  in  $A_n$  are all equal to the variance of  $W_n$ . Other covariances are calculated by Equation 4.31:

$$v_{ab} = \sum_{c \in B_n} p_{an} v_{cb} \quad \forall a \in A_n, \quad b \in L, \quad b \notin A_n \quad (4.31)$$

in which  $v_{ab}$  is the covariance between activities  $a$  and  $b$ , where  $a$  belongs to one of the activities after event  $n$ . Activity  $b$  belongs to  $L$ , in which  $L$  denotes all the known activities excluding the set of activities  $B_n$ . After the scanning step, return to the merge step, to find an event which is not yet complete, but has known  $Y_a$ , for all the  $a$  in  $B_n$ . Repeat the merging and scanning steps for this new event and subsequent ones until all events in the network are complete. In construction networks, which contain no loops, it is always possible to find such an event.

In summary, the steps of the MSAM are as follows:

1. Scan outwards from the start event to the ends of all activities which leave the event.
2. Find a event  $n$ , such that the distributions of the  $Y_a$  variables have been determined for all activity  $a$  which enter event  $n$ . “merge” at this event, determining the distribution of the maximum duration  $W_n$  from the start event to this event.
3. Scan out from event  $n$  (which was completed in step 2), estimating the distributions of the  $Y_a$  for all  $a \in A_n$ .
4. If there are any events which are not complete, return to step 2.

Steps 1-4 comprise the forward pass for the network, in which all the required probabilities  $p_{an}$  are calculated.

### 4.3.3 The MSAM as applied to a construction network

A case study project is used to illustrate the algorithm of the MSAM. The network is shown in Figure 4.5 and the set of activity durations are shown in Table 4.1:

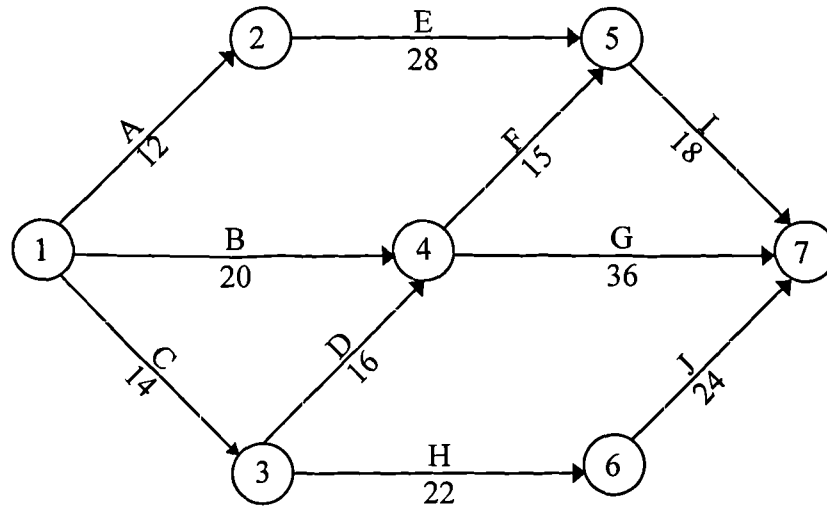


Figure 4.5 Arrow network of Example 4.1

Table 4.1 The mean and variance of each activity of Example 4.1

Activity	Expected value	Variance	Standard Deviation
A	12.00	1.78	1.33
B	20.00	9.00	3.00
C	14.00	4.00	2.00
D	16.00	16.00	4.00
E	28.00	40.11	6.33
F	15.00	4.00	2.00
G	36.00	16.00	4.00
H	22.00	7.11	2.67
I	18.00	2.78	1.67
J	24.00	11.11	3.33

When the MSAM is applied to the network the steps of the algorithm will be as follows:

1. Scan outwards from the start event, node 1, to the ends of all activities which leave the node. They are activities A, B, and C. Then:

$$Y_A \quad \mu_A = 12, \sigma_A^2 = 1.78$$

$$Y_B \quad \mu_B = 20, \sigma_B^2 = 9$$

$$Y_C \quad \mu_C = 14, \sigma_C^2 = 4$$

The covariances of these distributions with each other are all equal to zero because of the assumption of independence.

$$\text{Cov}(Y_A, Y_B) = 0$$

$$\text{Cov}(Y_A, Y_C) = 0$$

$$\text{Cov}(Y_B, Y_C) = 0$$

2. Find an event  $n$ , such that the distributions of the  $Y_a$  variables have been determined for all activities  $a$  which enter event  $n$ . Such an event is node 2, as  $Y_A$  was found in step 1. The “merge” at this node is trivial, the distribution of the maximum duration  $W_2$  is just the distribution of  $Y_A$ , with mean 12 and variance 1.78.

3. Scan outwards from node 2, to the ends of all activities which leave event 2 (Activity E only). Then:

$$Y_E = W_2 + E, \text{ Thus: } \mu_E = 12 + 28 = 40, \sigma_E^2 = 1.78 + 40.11 = 41.89$$

The covariances of these distributions are:

$$\text{Cov}(Y_E, Y_A) = \text{Cov}(Y_A, Y_A) = 1.78$$

$$\text{Cov}(Y_E, Y_B) = \text{Cov}(Y_A, Y_B) = 0$$

$$\text{Cov}(Y_E, Y_C) = \text{Cov}(Y_A, Y_C) = 0$$

4. Find a event  $n$ , such that the distributions of the  $Y_a$  variables have been determined for all activities  $a$  which enter event  $n$ . This event is node 3, as  $Y_C$  was found in step



1. Merge at this node, the distribution of the maximum duration  $W_3$  is just the distribution of  $Y_C$ , with mean 14 and variance 4.

5. Scan outwards from event 3, to the ends of all activities which leave event 3 (Activity D and H). Then:

$$Y_D = W_3 + D, \text{ Thus: } \mu_D = 14 + 16 = 30, \sigma_D^2 = 4 + 16 = 20$$

$$Y_H = W_3 + H, \text{ Thus: } \mu_H = 14 + 22 = 36, \sigma_H^2 = 4 + 7.11 = 11.11$$

The covariances of these distributions are:

$$\text{Cov}(Y_D, Y_H) = \text{Cov}(W_3) = 4$$

$$\text{Cov}(Y_D, Y_A) = \text{Cov}(Y_C, Y_A) = 0$$

$$\text{Cov}(Y_D, Y_B) = \text{Cov}(Y_C, Y_B) = 0$$

$$\text{Cov}(Y_D, Y_C) = \text{Cov}(Y_C, Y_C) = 4$$

$$\text{Cov}(Y_D, Y_E) = \text{Cov}(Y_C, Y_E) = 0$$

$$\text{Cov}(Y_H, Y_A) = \text{Cov}(Y_C, Y_A) = 0$$

$$\text{Cov}(Y_H, Y_B) = \text{Cov}(Y_C, Y_B) = 0$$

$$\text{Cov}(Y_H, Y_C) = \text{Cov}(Y_C, Y_C) = 4$$

$$\text{Cov}(Y_H, Y_E) = \text{Cov}(Y_C, Y_E) = 0$$

6. Merge at node 4;  $Y_B$  and  $Y_D$  are now known, so this is possible.  $W_4$  is the maximum duration of the  $Y_B$  and  $Y_D$ :

$$\nu_1 = \mu_B \Phi(\gamma) + \mu_D \Phi(-\gamma) + a\phi(\gamma)$$

$$\nu_2 = (\mu_B^2 + \sigma_B^2)\Phi(\gamma) + (\mu_D^2 + \sigma_D^2)\Phi(-\gamma) + (\mu_B + \mu_D)a\phi(\gamma)$$

$$a^2 = \sigma_B^2 + \sigma_D^2 - 2\sigma_B\sigma_D\rho_{BD}$$

$$\gamma = \frac{\mu_B - \mu_D}{a}$$

Thus:

$$a^2 = 9 + 20 - 2 * \sqrt{9} * \sqrt{20} * \frac{Cov(Y_B, Y_D)}{\sqrt{9} * \sqrt{20}} = 9 + 20 - 2 * 0 = 29$$

$$\gamma = \frac{20 - 30}{\sqrt{29}} = -1.85695$$

$$v_1 = 20 * \Phi(-1.85695) + 30 * \Phi(1.85695) + \sqrt{29} * \phi(-1.85695) = 30.06652$$

$$v_2 = (20^2 + 9) * \Phi(-1.85695) + (30^2 + 20) * \Phi(1.85695) + (20 + 30) * \sqrt{29} * \phi(-1.85695) = 922.9779$$

$$v_2 - v_1^2 = 922.9779 - 30.06652^2 = 18.9821$$

$$W_4 \quad \mu_{W_4} = 30.06652, \sigma_{W_4}^2 = 18.9821$$

The proportion are:

$$p_B = \Phi\left(\frac{\mu_B - \mu_D}{a}\right) = \Phi(\gamma) = 0.031659$$

$$p_D = \Phi(-\gamma) = 0.968341$$

7. Scan out from node 4, which has just been completed, to the ends of all activities which leave node 4 (Activity F and G). Then:

$$Y_F = W_4 + F, \text{ Thus: } \mu_F = 30.06652 + 15 = 45.06652, \sigma_F^2 = 18.9821 + 4 = 22.9821$$

$$Y_G = W_4 + G, \text{ Thus: } \mu_G = 30.06652 + 36 = 66.06652, \sigma_G^2 = 18.9821 + 16 = 34.9821$$

The covariances:

$$Cov(Y_F, Y_G) = Cov(W_4) = 18.9821$$

$$Cov(Y_F, Y_A) = p_B Cov(Y_B, Y_A) + p_D Cov(Y_D, Y_A) = 0$$

$$Cov(Y_F, Y_B) = p_B Cov(Y_B, Y_B) + p_D Cov(Y_D, Y_B) = 0.031659 * 9 + 0 = 0.284929$$

$$Cov(Y_F, Y_C) = p_B Cov(Y_B, Y_C) + p_D Cov(Y_D, Y_C) = 0 + 0.968341 * 4 = 3.873365$$

$$Cov(Y_F, Y_D) = p_B Cov(Y_B, Y_D) + p_D Cov(Y_D, Y_D) = 0 + 0.968341 * 20 = 19.36682$$

$$Cov(Y_F, Y_E) = p_B Cov(Y_B, Y_E) + p_D Cov(Y_D, Y_E) = 0$$

$$Cov(Y_F, Y_H) = p_B Cov(Y_B, Y_H) + p_D Cov(Y_D, Y_H) = 0 + 0.968341 * 4 = 3.873365$$

$$\begin{aligned}
Cov(Y_G, Y_A) &= p_B Cov(Y_B, Y_A) + p_D Cov(Y_D, Y_A) = 0 \\
Cov(Y_G, Y_B) &= p_B Cov(Y_B, Y_B) + p_D Cov(Y_D, Y_B) = 0.031659 * 9 + 0 = 0.284929 \\
Cov(Y_G, Y_C) &= p_B Cov(Y_B, Y_C) + p_D Cov(Y_D, Y_C) = 0 + 0.968341 * 4 = 3.873365 \\
Cov(Y_G, Y_D) &= p_B Cov(Y_B, Y_D) + p_D Cov(Y_D, Y_D) = 0 + 0.968341 * 20 = 19.36682 \\
Cov(Y_G, Y_E) &= p_B Cov(Y_B, Y_E) + p_D Cov(Y_D, Y_E) = 0 \\
Cov(Y_G, Y_H) &= p_B Cov(Y_B, Y_H) + p_D Cov(Y_D, Y_H) = 0 + 0.968341 * 4 = 3.873365
\end{aligned}$$

8. Merge at node 5;  $Y_E$  and  $Y_F$  are now known, so this is possible.  $W_5$  is the maximum duration of the  $Y_E$  and  $Y_F$ :

$$\begin{aligned}
\nu_1 &= \mu_E \Phi(\gamma) + \mu_F \Phi(-\gamma) + a \phi(\gamma) \\
\nu_2 &= (\mu_E^2 + \sigma_E^2) \Phi(\gamma) + (\mu_F^2 + \sigma_F^2) \Phi(-\gamma) + (\mu_E + \mu_F) a \phi(\gamma) \\
a^2 &= \sigma_E^2 + \sigma_F^2 - 2\sigma_E \sigma_F \rho_{EF} \\
\gamma &= \frac{\mu_E - \mu_F}{a}
\end{aligned}$$

Thus:

$$a^2 = 41.89 + 22.9821 - 2 * \sqrt{41.89} * \sqrt{22.9821} * \frac{Cov(Y_E, Y_F)}{\sqrt{41.89} * \sqrt{22.9821}} = 64.8721$$

$$\gamma = \frac{40 - 45.06652}{\sqrt{64.8721}} = -0.62904$$

$$\nu_1 = 40 * \Phi(-0.62904) + 45.06652 * \Phi(0.62904) + \sqrt{64.8721} * \phi(-0.62904) = 46.36203$$

$$\begin{aligned}
\nu_2 &= (40^2 + 41.89) * \Phi(\gamma) + (45.06652^2 + 22.9821) * \Phi(-\gamma) + (40 + 45.06652) * \\
&\quad \sqrt{64.8721} * \phi(\gamma) = 2169.182
\end{aligned}$$

$$\nu_2 - \nu_1^2 = 2169.182 - 46.36203^2 = 19.74418$$

$$W_5 \quad \mu_{W_5} = 46.36203, \quad \sigma_{W_5}^2 = 19.74418$$

The proportion are:

$$p_E = \Phi\left(\frac{\mu_E - \mu_F}{a}\right) = \Phi(\gamma) = 0.26466$$

$$p_F = \Phi(-\gamma) = 0.73534$$

9. Scan out from node 5, which has just been completed, to the ends of all activities which leave node 5 (Activity I only). Then:

$Y_I = W_5 + I$ , Thus:

$$\mu_I = 46.36203 + 18 = 64.36203, \sigma_I^2 = 2.78 + 19.74418 = 22.52418$$

The covariances:

$$Cov(Y_I, Y_A) = p_E Cov(Y_E, Y_A) + p_F Cov(Y_F, Y_A) = 0.26466 * 1.78 + 0 = 0.471095$$

$$Cov(Y_I, Y_B) = p_E Cov(Y_E, Y_B) + p_F Cov(Y_F, Y_B) = 0 + 0.73534 * 0.284929 = 0.20952$$

$$Cov(Y_I, Y_C) = p_E Cov(Y_E, Y_C) + p_F Cov(Y_F, Y_C) = 0 + 0.73534 * 3.873365 = 2.84824$$

$$Cov(Y_I, Y_D) = p_E Cov(Y_E, Y_D) + p_F Cov(Y_F, Y_D) = 0 + 0.73534 * 19.36682 = 14.2412$$

$$Cov(Y_I, Y_E) = p_E Cov(Y_E, Y_E) + p_F Cov(Y_F, Y_E) = 0.26466 * 41.89 + 0 = 11.08661$$

$$Cov(Y_I, Y_F) = p_E Cov(Y_E, Y_F) + p_F Cov(Y_F, Y_F) = 0 + 0.73534 * 22.9821 = 16.89965$$

$$Cov(Y_I, Y_G) = p_E Cov(Y_E, Y_G) + p_F Cov(Y_F, Y_G) = 0 + 0.73534 * 18.9821 = 13.95829$$

$$Cov(Y_I, Y_H) = p_E Cov(Y_E, Y_H) + p_F Cov(Y_F, Y_H) = 0 + 0.73534 * 3.873365 = 2.84824$$

10. Find a node  $n$ , such that the distributions of the  $Y_a$  variables have been determined for all activities  $a$  which enter node  $n$ . This node is node 6, as  $Y_H$  was found in step 5. Merge at this node, the distribution of the maximum duration  $W_6$  is just the distribution of  $Y_H$ , with mean 36 and variance 11.11.

11. Scan outwards from node 6, to the ends of all activities which leave node 6 (Activity J only). Then:

$$Y_J = W_6 + H, \text{ Thus: } \mu_J = 36 + 24 = 60, \sigma_J^2 = 11.11 + 11.11 = 22.22$$

The covariances:

$$Cov(Y_J, Y_A) = Cov(Y_H, Y_A) = 0$$

$$Cov(Y_J, Y_B) = Cov(Y_H, Y_B) = 0$$

$$Cov(Y_J, Y_C) = Cov(Y_H, Y_C) = 4$$

$$Cov(Y_J, Y_D) = Cov(Y_H, Y_D) = 4$$

$$\text{Cov}(Y_J, Y_E) = \text{Cov}(Y_H, Y_E) = 0$$

$$\text{Cov}(Y_J, Y_F) = \text{Cov}(Y_H, Y_F) = 3.873365$$

$$\text{Cov}(Y_J, Y_G) = \text{Cov}(Y_H, Y_G) = 3.873365$$

$$\text{Cov}(Y_J, Y_H) = \text{Cov}(Y_H, Y_H) = 11.11$$

$$\text{Cov}(Y_J, Y_I) = \text{Cov}(Y_H, Y_I) = 2.84824$$

12. Merge at node 7, as  $Y_I$ ,  $Y_G$ , and  $Y_J$  are all known. First find the maximum of  $Y_I$  and

$Y_G$ :

$$v_I = \mu_I \Phi(\gamma) + \mu_G \Phi(-\gamma) + a \phi(\gamma)$$

$$v_2 = (\mu_I^2 + \sigma_I^2) \Phi(\gamma) + (\mu_G^2 + \sigma_G^2) \Phi(-\gamma) + (\mu_I + \mu_G) a \phi(\gamma)$$

$$a^2 = \sigma_I^2 + \sigma_G^2 - 2\sigma_I \sigma_G \rho_{IG}$$

$$\gamma = \frac{\mu_I - \mu_G}{a}$$

Thus:

$$\begin{aligned} a^2 &= 22.52418 + 34.9821 - 2 * \sqrt{22.52418} * \sqrt{34.9821} * \frac{\text{Cov}(Y_I, Y_G)}{\sqrt{22.52418} * \sqrt{34.9821}} \\ &= 22.52418 + 34.9821 - 2 * 13.95829 \\ &= 29.58969 \end{aligned}$$

$$\gamma = \frac{64.36203 - 66.06652}{\sqrt{29.58969}} = -0.313346$$

$$\begin{aligned} v_I &= 64.36203 * \Phi(-0.313346) + 66.06652 * \Phi(0.313346) + \sqrt{29.58969} * \phi(-0.313346) \\ &= 67.49005 \end{aligned}$$

$$\begin{aligned} v_2 &= (64.36203^2 + 22.52418) * \Phi(\gamma) + (66.06652^2 + 34.9821) * \Phi(-\gamma) + \\ &\quad (64.36203 + 66.06652) * \sqrt{29.58969} * \phi(\gamma) \\ &= 4580.74 \end{aligned}$$

$$v_2 - v_I^2 = 4580.74 - 67.49005^2 = 25.83253$$

$$\mu_{IG} = 67.49005 \quad \sigma_{IG}^2 = 25.83253$$

The correlation:

$$\rho_{IJ} = \frac{Cov(Y_I, Y_J)}{\sigma_I \sigma_J} = \frac{2.84824}{\sqrt{22.52418} * \sqrt{22.22}} = 0.127315$$

$$\rho_{GJ} = \frac{Cov(Y_G, Y_J)}{\sigma_G \sigma_J} = \frac{3.873365}{\sqrt{34.9821} * \sqrt{22.22}} = 0.138929$$

The correlation between  $Y_J$  and the maximum of  $Y_I$  and  $Y_G$ ,

$$\begin{aligned} \rho_{J,IG} &= \frac{\sigma_I \rho_{IJ} \Phi(\gamma) + \sigma_G \rho_{GJ} \Phi(-\gamma)}{\sqrt{v_2 - v_I^2}} \\ &= \frac{\sqrt{22.52418} * 0.127315 * \Phi(-0.313346) + \sqrt{34.9821} * 0.138929 * \Phi(0.313346)}{\sqrt{25.83253}} \\ &= 0.14554 \end{aligned}$$

Find the  $Max(Y_J, Max(Y_I, Y_G))$ :

$$v_I = \mu_J \Phi(\gamma) + \mu_{IG} \Phi(-\gamma) + a \phi(\gamma)$$

$$v_2 = (\mu_J^2 + \sigma_J^2) \Phi(\gamma) + (\mu_{IG}^2 + \sigma_{IG}^2) \Phi(-\gamma) + (\mu_J + \mu_{IG}) a \phi(\gamma)$$

$$a^2 = \sigma_J^2 + \sigma_G^2 - 2\sigma_J \sigma_{IG} \rho_{J,IG}$$

$$\gamma = \frac{\mu_J - \mu_{IG}}{a}$$

Thus:

$$a^2 = 22.22 + 25.83253 - 2 * \sqrt{22.22} * \sqrt{25.83253} * 0.14554 = 41.07876$$

$$\gamma = \frac{60 - 67.49005}{\sqrt{41.07876}} = -1.16863$$

$$\begin{aligned} v_I &= 60 * \Phi(-1.16863) + 67.49005 * \Phi(1.16863) + \sqrt{41.07876} * \phi(-1.16863) \\ &= 67.87338 \end{aligned}$$

$$\begin{aligned} v_2 &= (60^2 + 22.22) * \Phi(\gamma) + (67.49005^2 + 25.83253) * \Phi(-\gamma) + (60 + 67.49005) \\ &\quad * \sqrt{41.07876} * \phi(\gamma) = 4629.172 \end{aligned}$$

$$v_2 - v_I^2 = 4629.172 - 67.87338^2 = 22.37629$$

$$W_7 \quad \mu_{w_7} = 67.87338, \quad \sigma_{w_7}^2 = 22.37629$$

Therefore, the maximum of project duration is with mean 67.87338 and variance 22.37629.

13. For completeness the full covariance matrix  $V$  is given, where the diagonal elements are of course the variances of the  $Y_a$  distributions:

$$V = \begin{pmatrix} 1.78 & 0 & 0 & 0 & 1.78 & 0 & 0 & 0 & 0.471095 & 0 \\ 0 & 9 & 0 & 0 & 0 & 0.284929 & 0.284929 & 0 & 0.20952 & 0 \\ 0 & 0 & 4 & 4 & 0 & 3.873365 & 3.873365 & 4 & 2.84824 & 4 \\ 0 & 0 & 4 & 20 & 0 & 19.36682 & 19.36682 & 4 & 14.2412 & 4 \\ 1.78 & 0 & 0 & 0 & 41.89 & 0 & 0 & 0 & 11.08661 & 0 \\ 0 & 0.284929 & 3.873365 & 19.36682 & 0 & 22.9821 & 18.9821 & 3.873365 & 16.89965 & 3.873365 \\ 0 & 0.284929 & 3.873365 & 19.36682 & 0 & 18.9821 & 34.9821 & 3.873365 & 13.95829 & 3.873365 \\ 0 & 0 & 4 & 4 & 0 & 3.873365 & 3.873365 & 11.11 & 2.84824 & 11.11 \\ 0.471095 & 0.20952 & 2.84824 & 14.2412 & 11.08661 & 16.89965 & 13.95829 & 2.84824 & 22.52418 & 2.84824 \\ 0 & 0 & 4 & 4 & 0 & 3.873365 & 3.873365 & 11.11 & 2.84824 & 22.22 \end{pmatrix}$$

#### 4.3.4 Validations and applications

##### 4.3.4.1 Introduction

The proposed new analytical method, the MSAM, was developed in the previous sections. In addition, a computer program has been modified from SAM, in FORTRAN 77, to implement the MSAM algorithm. When applying MSAM in practice, network analysis begins in a similar manner as any PERT networks: the durations of the individual activity are estimated by experienced estimators, or historical data is used to determine the mean and standard deviation of the activity durations. The dependency of activities are then clearly stated (network is constructed). The MSAM program can then be applied to estimate the total completion time of the project duration. The program requires the event (node) numbers, mean and standard deviation for each activity, number of predecessors, and the dependency of activities. There is no limit on the number of activities in the network. The calculation normally can be done in few seconds in a PC (IBM compatible 586 computer with 16 Megabytes of R.A.M. and a 133 Mhz processing chip).

This section describes validation and applications of the analytical method. In most of the examples presented in this section it is difficult to separate the validation studies from the applications. Therefore, it will be helpful to the reader if the results from the analytical method are viewed as applications and those from Monte Carlo simulations are viewed as validations.

The only means of validating the analytical method is with Monte Carlo simulation. This is because the results from Monte Carlo simulation can be said as “exact” if the sample size is infinite <sup>[51]</sup>. Five examples were studied. Initially the first three examples were intended for assessing the accuracy of the MSAM. The first example, the case study project, was used to demonstrate the use of the MSAM program, such as the input data needed and the output data which would be obtained. It can also be compared to the manual calculation, which is in section 4.3.3. Other two examples (Examples 4.4 and 4.5) were chosen from a published paper <sup>[48]</sup> and intended to use for comparing with different analytical methods such as PERT and PNET. However, all the examples were studied by both MSAM program and Monte Carlo simulation.

The simulations were performed by using the @Risk Monte Carlo simulation package <sup>[30,31]</sup>. The results of Monte Carlo simulation are set at 10,000 iterations. They are treated as “exact” because the sample size is large enough so that the standard error is insignificant. The 99% confidence interval in 10,000 samples has calculated for all projects using  $\frac{\sigma}{\sqrt{n}} k_{.005}$ . These intervals are small, (such as the intervals of Example 4.4 and 4.5 are 0.14 and 0.25 days respectively), so the simulation results can be said to be approximately “exact”.

#### **4.3.4.2 Example 4.1 - the case study project**

The case study project illustrated step by step in section 4.3.3 was to show the MSAM algorithm. The program is needed to input the event (node) numbers, mean and variance for each activity (or the three estimate times of each activity: minimum, most likely and maximum), number of predecessors, and the dependency of activities. For example, in this case, the inputs are as listed in Table 4.2:



**Table 4.2 The input data required by MSAM for Example 4.1**

Node number	Mean	Standard Deviation	Number of predecessors	Predecessors		
101	0.00	0.00	0			
102	12.00	1.33	1	101		
103	20.00	3.00	1	101		
104	14.00	2.00	1	101		
105	16.00	4.00	1	104		
106	28.00	6.33	1	102		
107	15.00	2.00	2	103	105	
108	36.00	4.00	2	103	105	
109	22.00	2.67	1	104		
110	18.00	1.67	2	106	107	
111	24.00	3.33	1	109		
112	0.00	0.00	3	108	110	111

Then the result is calculated by MSAM within seconds given the output as in Table 4.3:

**Table 4.3 The results from the MSAM for Example 4.1**

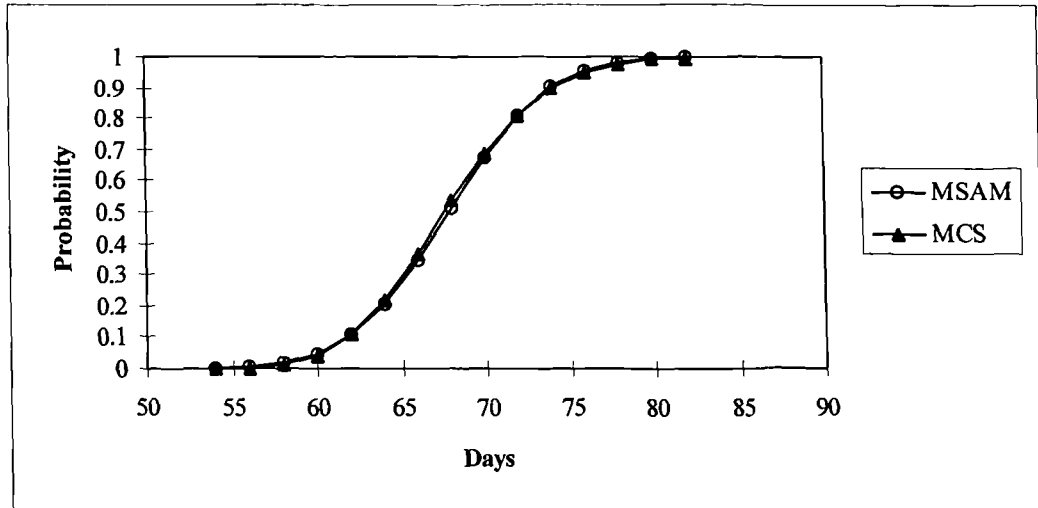
Mean:	67.8802762953
Variance:	22.3480833290
Standard Deviation:	4.72737594539
Run Time:	0.2197802

These results suggest that the total project duration of Example 4.1 analysed by MSAM is approximated to a Normal distribution with a mean 67.88 days and the standard deviation of 4.73 days.

The input data for the MSAM program and output results of Example 4.1 are also shown in Appendix C.1.

The results of the Monte Carlo simulation are listed in Appendix D.1.

From the results by MSAM and Monte Carlo simulations, a chart can be easily produced to show the comparison of two method. This is shown in Figure 4.6:



**Figure 4.6 Comparison of MSAM with MCS of Example 4.1**

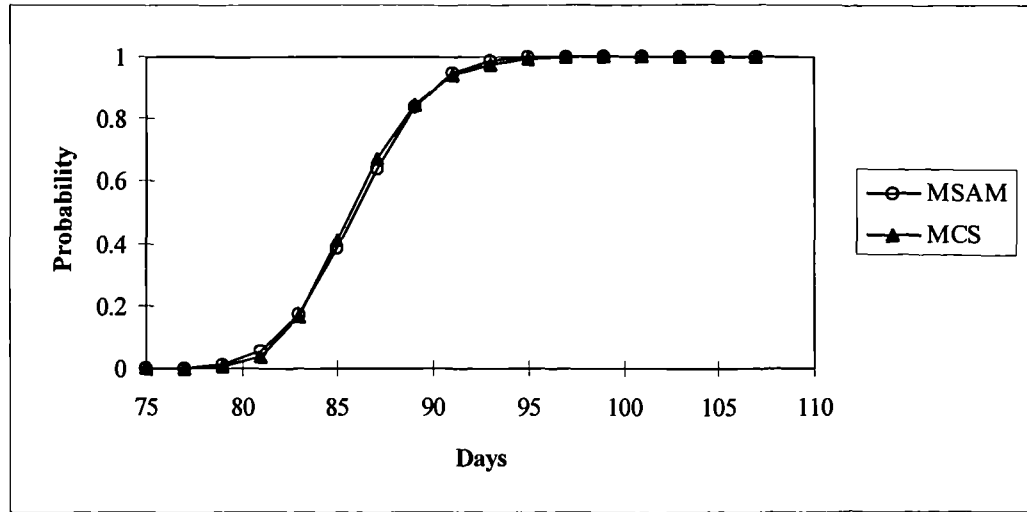
#### 4.3.4.3 Example 4.2 - a house construction project

Example 4.2 uses a example studied in Chapter 3, Example 3.1 <sup>[80]</sup>. This example has been studied using PERT and MCS methods in Chapter 3. In this Chapter, the example is studied by using MSAM method, the results can then compare to MCS method. The arrow network of the activities involved and their dependencies, the estimates of the optimistic time ( $a$ ), the most likely time ( $m$ ) and the pessimistic time ( $b$ ) are already shown in Figure 3.1 and Table 3.1.

The input and output data from the MSAM program are shown in Appendix C.2. Those results suggest that the total project duration of Example 4.2 analysed by MSAM is approximated to a normal distribution with a mean 85.89 days and the standard deviation of 3.11 days.

The results of the Monte Carlo simulation are listed in appendix D.2.

The result of the comparison of the MSAM with Monte Carlo simulation is shown in Figure 4.8:



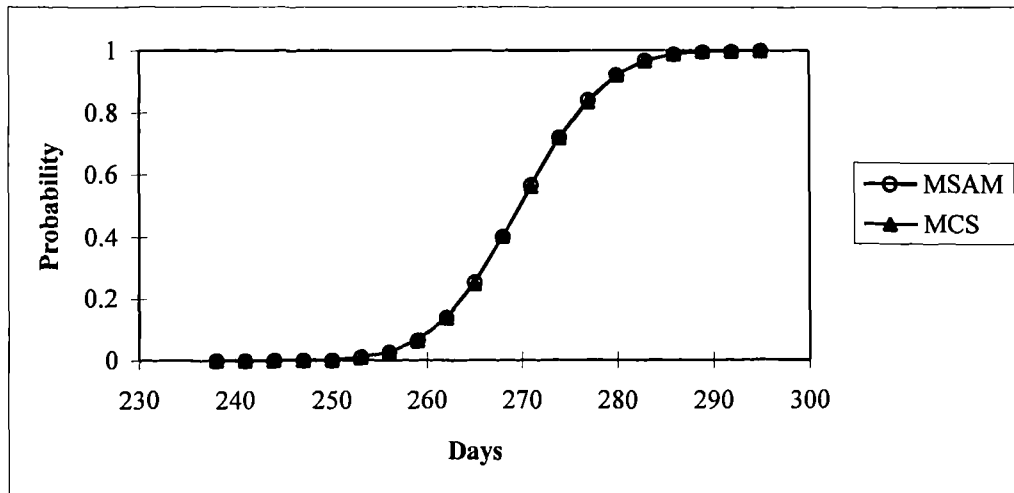
**Figure 4.7 Comparison of MSAM with MCS of Example 4.2**

#### 4.3.4.4 Example 4.3 - a hypothetical project HABITAT

Example 4.3 again uses a example studied in Chapter 3, Example 3.2 <sup>[81]</sup>. It has been studied by PERT and MCS methods in Chapter 3. In this Chapter, the example is studied by using MSAM method, the results can then compare to MCS method. The arrow network of the activities involved and their dependencies, the estimates of the optimistic time ( $a$ ), the most likely time ( $m$ ) and the pessimistic time ( $b$ ) are already shown in Figure 3.10 and Table 3.10.

The input and output data from the MSAM program are shown in Appendix C.3. Those results suggest that the total project duration of Example 4.3 analysed by MSAM is approximated to a Normal distribution with a mean 269.8 days and the standard deviation of 7.21 days.

The results of the Monte Carlo simulation are listed in appendix D.3. The result of comparison of the MSAM with Monte Carlo simulation is shown in Figure 4.10:



**Figure 4.8 Comparison of MSAM with MCS of Example 4.3**

#### **4.3.4.5 Example 4.4 - road pavement project**

To compare the MSAM method with others (PERT, PNET and MCS), two examples (this one and the next) were chosen from a published paper by Ang, Abdelnour and Chaker <sup>[48]</sup>. In the paper, the authors used the examples to show the PNET method compare to the PERT method and the Monte Carlo simulation method. The examples are used here so that the results from MSAM can be compared to those from PERT, PNET and Monte Carlo simulation methods.

Example 4.4 original taken from Brooks, Leahy and Shaffer <sup>[97]</sup> and involves the paving of 2.2 miles of roadway pavement and the construction of appurtenant drainage structures, excavation to grade, placement of macadam shoulders, erection of guardrails, and landscaping.

The various activities of the project are described in Table 4.6, where the respective mean durations and corresponding standard deviations are also listed. The project network is shown in Figure 4.11. All of the nine paths of the network are listed in Table 4.7, arranged in order of decreasing mean durations. The standard deviations of these paths are also given in Table 4.7, calculated in accordance with the fact that the corresponding variance is the sum of the individual variance of each activity comprising that path because of the assumption of independence.

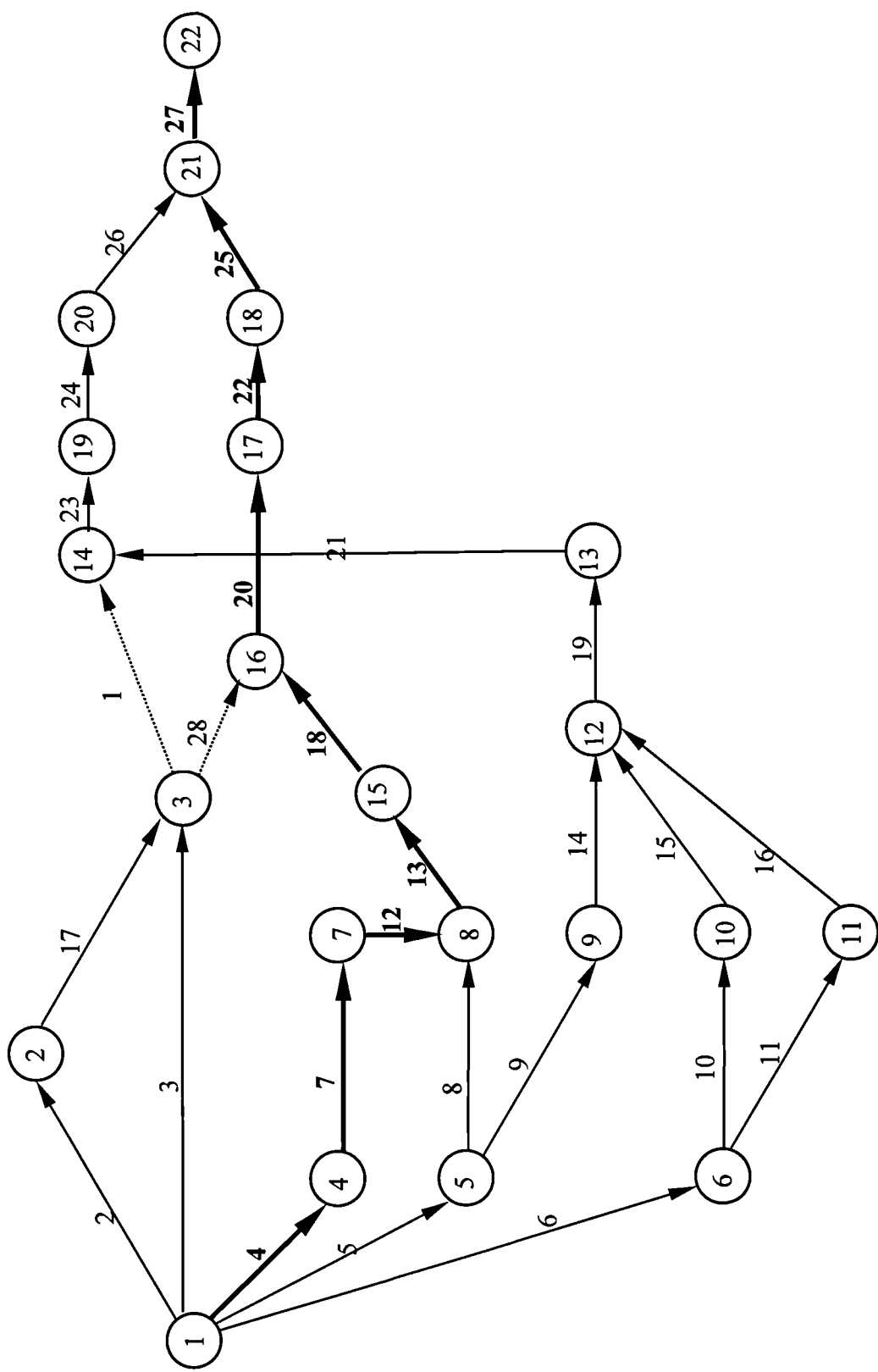


Figure 4.9 Arrow network of Example 4.4

**Table 4.4 Activities and estimated durations of Example 4.4**

Activity		Description of activities	Duration (days)		Predecessors
code	No.		Mean	S.D.	
3-14	1	Dummy	0	0.0	3, 17
1-2	2	Set-up batch plant	2	0.5	-
1-3	3	Order and deliver paving mesh	5	1.0	-
<b>1-4</b>	<b>4</b>	<b>Deliver rebars for double barrel culvert</b>	<b>6</b>	<b>1.5</b>	-
1-5	5	Move in equipment	3	0.5	-
1-6	6	Deliver rebars for small box culvert	7	4.0	-
<b>4-7</b>	<b>7</b>	<b>Build double barrel culvert</b>	<b>10</b>	<b>2.0</b>	<b>4</b>
5-8	8	Clear and grub from station 42 - station 100	3	1.0	5
5-9	9	Clear and grub from station 100 - station 158	7	1.5	5
6-10	10	Build box culvert at station 127	5	2.0	6
6-11	11	Build box culvert at station 138	3	1.5	6
<b>7-8</b>	<b>12</b>	<b>Cure double barrel culvert</b>	<b>9</b>	<b>2.0</b>	<b>7</b>
<b>8-15</b>	<b>13</b>	<b>Move dirt between station 42 - station 100</b>	<b>5</b>	<b>1.5</b>	<b>8, 12</b>
9-12	14	Start moving dirt between station 100 - station 158	3	0.5	9
10-12	15	Cure box culvert at station 127	9	4.5	10
11-12	16	Cure box culvert at station 138	6	2.0	11
2-3	17	Order and stockpile paving material	2	0.5	2
<b>15-16</b>	<b>18</b>	<b>Place subbase from station 42 - station 100</b>	<b>7</b>	<b>1.73</b>	<b>13</b>
12-13	19	Finish moving dirt between station 100 - station 158	5	2.0	14,15,16
<b>16-17</b>	<b>20</b>	<b>Pave from station 42 - station 100</b>	<b>10</b>	<b>2.0</b>	<b>18, 28</b>

13-14	21	Place subbase from station 100 - station 158	7	3.31	19
17-18	22	<b>Cure pavement from station 42 - station 100</b>	<b>6</b>	<b>1.5</b>	<b>20</b>
14-19	23	Pave from station 100 - station 158	10	4.5	1, 21
19-20	24	Cure pavement from station 100 - station 158	6	1.5	23
18-21	25	<b>Place shoulders from station 42 - station 100</b>	<b>3</b>	<b>1.0</b>	<b>22</b>
20-21	26	Place shoulders from station 100 - station 158	3	1.0	24
21-22	27	<b>Place guardrail and landscape</b>	<b>5</b>	<b>1.5</b>	<b>25, 26</b>
3-16	28	Dummy	0	0.0	3, 17

**Table 4.5 Ordered paths and duration statistics of Example 4.4**

Path	Activities in path	Mean duration (days)	Standard deviation (days)
1	4, 7, 12, 13, 18, 20, 22, 25, 27	61	5.0
2	6, 10, 15, 19, 21, 23, 24, 26, 27	57	9.0
3	6, 11, 16, 19, 21, 23, 24, 26, 27	52	7.94
4	5, 9, 14, 19, 21, 23, 24, 26, 27	49	6.54
5	5, 8, 13, 18, 20, 22, 25, 27	42	4.0
6	3, 28, 20, 22, 25, 27	29	3.24
7	3, 1, 23, 24, 26, 27	29	5.19
8	2, 17, 28, 20, 22, 25, 27	28	3.16
9	2, 17, 1, 23, 24, 26, 27	28	5.12

The project duration was analysed by using the MSAM method. The input data for the MSAM program and the results output from the MSAM method are shown in Appendix C.4. Those results suggest that the total project duration of Example 4.4

analysed by MSAM is approximated to a Normal distribution with a mean 63.39 days and the standard deviation of 5.29 days.

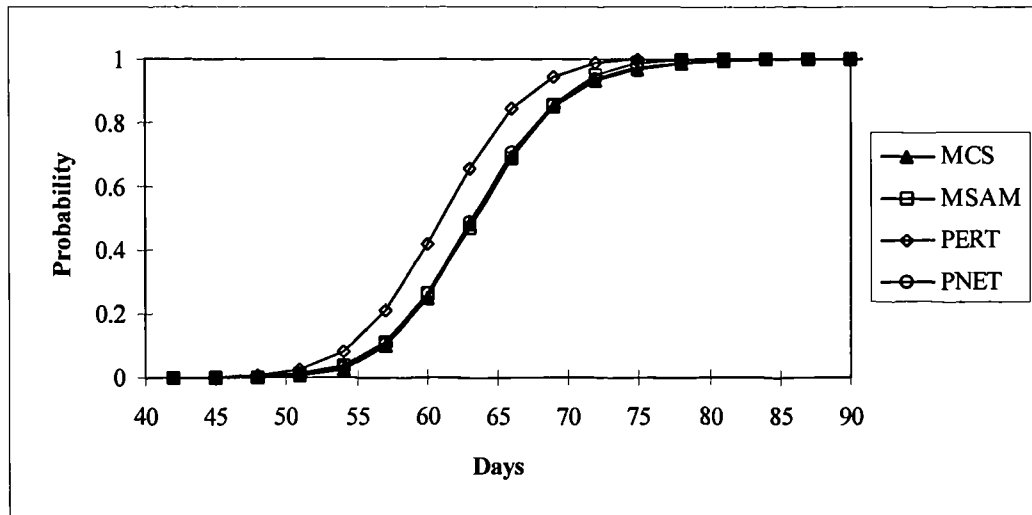
Using the PNET method, Ang et al <sup>[48]</sup> had identified the “representative” paths among the first five major paths and had given the completion-time probability of the project (assuming normal distributions for  $T_1$  and  $T_2$ ) as:

$$p(t) \approx P(T_1 \leq t)P(T_2 \leq t) = \Phi\left(\frac{t-61}{5}\right)\Phi\left(\frac{t-57}{9}\right)$$

in which  $T_1$  and  $T_2$  are the completion times for path 1 and path 2.  $\Phi(-)$  is the cumulative probability of the standard normal distribution.

The results of the Monte Carlo simulation are listed in Appendix D.4.

The completion-time probabilities determined with the PNET method, MSAM method, and Monte Carlo simulation are shown in Figure 4.12 for comparison. In this figure, result of the PERT method,  $P(T_1 \leq t)$ , is also shown.



**Figure 4.10 Comparison of MSAM with other methods of Example 4.4**



#### **4.3.4.6 Example 4.5 - industrial building project**

Example 4.5 is an industrial building project original taken from Brand, Meyer and Shaffer <sup>[98]</sup>. This project involves the construction of a single story industrial building with an adjoining parking lot. The building is comprised of reinforced concrete piers, frost walls, structural steel columns, and a precast roof deck. The corresponding project network is shown in Figure 4.13. The various activities are described in Table 4.8, with the means and standard deviations of the respective durations also listed. In Table 4.9 are listed the first 10 paths arranged in decreasing mean path durations; also shown are the corresponding standard deviations.

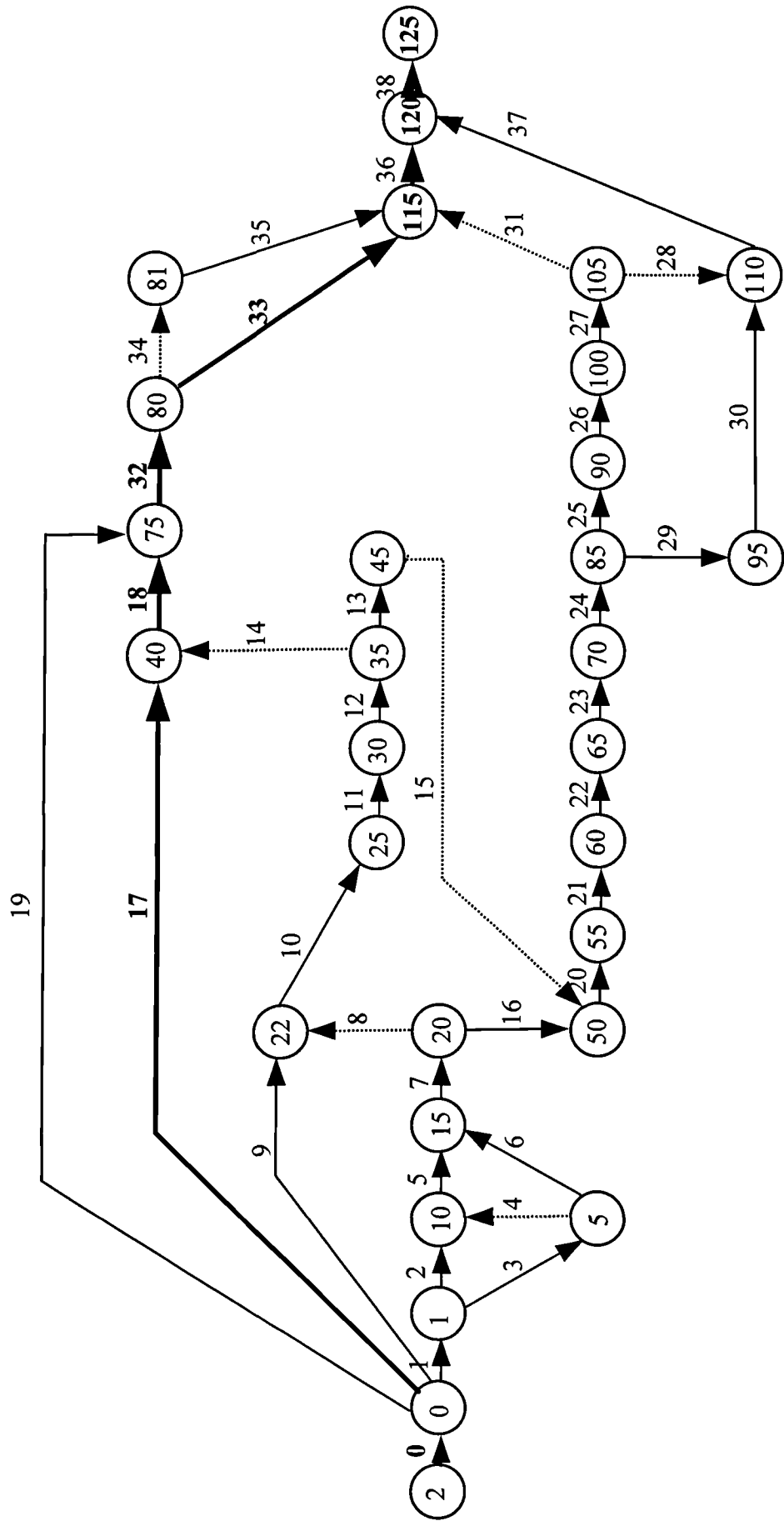


Figure 4.11 Arrow network of Example 4.5

**Table 4.6 Activities and estimated durations of Example 4.5**

Activity		Description of activities	Duration(days)		Predecessors
code	No.		Mean	S.D.	
<b>2-0</b>	<b>0</b>	<b>Dummy</b>	<b>0</b>	<b>0</b>	<b>-</b>
0-1	1	Mobilisation	32	3.2	0
1-10	2	Move in	2	0.5	1
1-5	3	Initial layout	2	0.5	1
5-10	4	Dummy	0	0	3
10-15	5	Site rough grading	2	0.5	2, 4
5-15	6	Layout of piers	1	0.5	3
15-20	7	Excavate piers	2	1.0	5, 6
20-22	8	Dummy	0	0	7
0-22	9	Order and deliver rebars	40	12	0
22-25	10	Form and rebars piers	2	0.5	8, 9
25-30	11	Pour piers	2	0.5	10
30-35	12	Cure piers	4	0.8	11
35-45	13	Strip piers	1	0.1	12
35-40	14	Dummy	0	0	12
45-50	15	Dummy	0	0	13
20-50	16	Excavate frost walls	1	0.5	7
<b>0-40</b>	<b>17</b>	<b>Order and deliver structural steel columns</b>	<b>60</b>	<b>12</b>	<b>0</b>
<b>40-75</b>	<b>18</b>	<b>Erect structural steel columns</b>	<b>5</b>	<b>1</b>	<b>14, 17</b>
0-75	19	Order and deliver precast roof deck	30	6	0
50-55	20	Form and mesh frost walls	3	0.9	15, 16
55-60	21	Pour frost walls	1	0.3	20
60-65	22	Cure frost walls	4	0.4	21
65-70	23	Strip frost walls	1	0.1	22

70-85	24	Backfill	2	0.5	23
85-90	25	Grade and compact gravel for floor	2	0.2	24
90-100	26	Rebar floor and set screeds	2	0.5	25
100-105	27	Pour and finish floor	2	0.5	26
105-110	28	Dummy	0	0	27
85-95	29	Excavate and grade parking	2	0.2	24
95-110	30	Stone base for parking	1	0.2	29
105-115	31	Dummy	0	0	27
<b>75-80</b>	<b>32</b>	<b>Set roof deck</b>	<b>5</b>	<b>1.5</b>	<b>18, 19</b>
<b>80-115</b>	<b>33</b>	<b>Hang siding and waterproof roof</b>	<b>6</b>	<b>1.2</b>	<b>32</b>
80-81	34	Dummy	0	0	32
81-115	35	Hang doors	4	1.2	34
<b>115-120</b>	<b>36</b>	<b>Clean up</b>	<b>2</b>	<b>0.5</b>	<b>31,33,35</b>
110-120	37	Bituminous surface in parking	3	0.3	28, 30
<b>120-125</b>	<b>38</b>	<b>Dummy</b>	<b>0</b>	<b>0</b>	<b>36, 37</b>

**Table 4.7 Ordered paths and duration statistics of Example 4.5**

Path	Activities in path	Mean duration (days)	Standard deviation (days)
1	0, 17, 18, 32, 33, 36	78	12.20
2	0, 17, 18, 32, 34, 35, 36	76	12.20
3	0, 9, ... <sup>a</sup> , 13, 15, 20, ... <sup>a</sup> , 28, 37	69	12.12
4	0, 9, ... <sup>a</sup> , 13, 15, 20, ... <sup>a</sup> , 27, 31, 36	68	12.14
5	0, 1, 2, 5, 7, 8, 10, ... <sup>a</sup> , 13, 15, 20, ... <sup>a</sup> , 28, 37	67	3.85
6	0, 1, 3, ... <sup>a</sup> , 5, 7, 8, 10, ... <sup>a</sup> , 13, 15, 20, ... <sup>a</sup> , 28, 37	67	3.85
7	0, 1, 3, 6, ... <sup>a</sup> , 8, 10, ... <sup>a</sup> , 13, 15, 20, ... <sup>a</sup> , 28, 37	66	3.85
8	0, 9, ... <sup>a</sup> , 13, 15, 20, ... <sup>a</sup> , 24, 29, 30, 37	66	12.09
9	0, 1, 2, 5, 7, 8, 10, ... <sup>a</sup> , 13, 15, 20, ... <sup>a</sup> , 27, 31, 36	66	3.71
10	0, 1, 3, ... <sup>a</sup> , 5, 7, 8, 10, ... <sup>a</sup> , 13, 15, 20, ... <sup>a</sup> , 27, 31, 36	66	3.87
	<sup>a</sup> Includes all intervening activities.		

In the use of the MSAM method, the input data and results output from the MSAM are shown in Appendix C.5. Those results suggest that the total project duration of the example 5 analysed by MSAM is approximated to a Normal distribution with a mean 83.3 days and the standard deviation of 8.16 days.

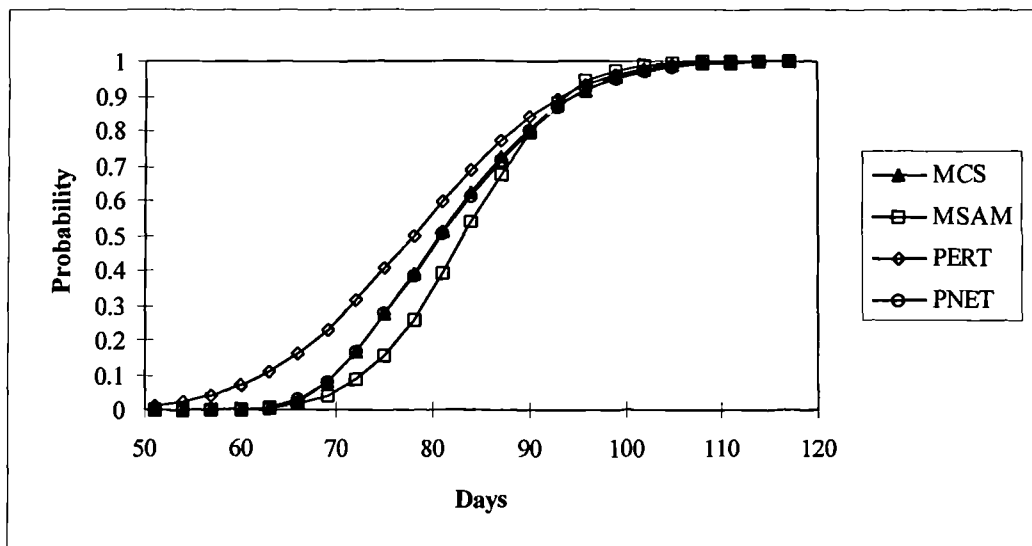
Using the PNET method, Ang et al <sup>[48]</sup> identified the “representative” paths among the first ten paths and gave the completion-time probability of the project (assuming normal distributions for all paths) as:

$$p(t) \approx P(T_1 \leq t)P(T_3 \leq t)P(T_5 \leq t) = \Phi\left(\frac{t-78}{12.20}\right)\Phi\left(\frac{t-69}{12.12}\right)\Phi\left(\frac{t-67}{3.85}\right)$$

where  $\Phi(-)$  is the cumulative probability of the standard normal distribution.

The results of the Monte Carlo simulation are listed in Appendix D.5.

The completion-time probabilities determined with the PERT method, PNET method, MSAM method, and Monte Carlo simulation are shown in Figure 4.14 for comparison.



**Figure 4.12 Comparison of MSAM with other methods of Example 4.5**

### 4.3.5 Discussion

#### 4.3.5.1 Accuracy of the MSAM

The only means of evaluating the accuracy of the proposed Modified SAM algorithm is with a large sample of Monte Carlo simulation. Such simulations were performed for all of the examples illustrated in this chapter. From Figures 4.6, 4.8, 4.10, 4.12 and 4.14, it can be observed that in most of the cases the comparisons correspond well, indicating that the proposed MSAM method gives a reliable project completion-time probability for the entire range of probabilities of interest. With the results from both MSAM and MCS methods (reported in Appendix C and D), a comparison table was produced for all of the examples (Table 4.10).

However, in Figure 4.14, a degree of difference between MSAM and Monte Carlo simulation can be observed. The inaccuracy of MSAM is inherent from the use of Clark's results which replace non-Normal distributions with Normal distributions. Therefore, MSAM is only an approximate method when the variables are not Normally distributed.

There are other approximations, such as methods by Mendell and Elston<sup>[99]</sup>, Kamakura<sup>[100]</sup>, and Langdon<sup>[101,102]</sup>, which work better than Clark's in this situation, and which could be used to replace Clark's approximation in MSAM in order to improve accuracy. However, this is discussed elsewhere and is beyond the scope of this thesis.

The accuracy of Clark's results, the Normal approximation of non-Normal variables, has been discussed in great detail in the article by Clark<sup>[88]</sup>. He pointed out that in many cases the moments of  $\max(\xi_1, \xi_2)$  are adequately approximated by the moments of  $\max(\eta_1, \eta_2)$ , in which  $\xi_1$  and  $\xi_2$  are non-Normal variables and  $\eta_1$  and  $\eta_2$  are Normal variables with the same expected value and variance corresponding to  $\xi_1$  and  $\xi_2$  respectively. If the difference  $E(\xi_1) - E(\xi_2)$  is large relative to the greater of  $V^{1/2}(\xi_1)$  and  $V^{1/2}(\xi_2)$ , the random variable  $\max(\xi_1, \xi_2)$  is practically identical with  $\xi_1$ . In such a case, no computations are required in order to approximate the first two moments of  $\max(\xi_1, \xi_2)$ . Certainly, there would be no significant error involved in replacing  $\xi_1$

and  $\xi_2$  by Normal approximations. However, if  $E(\xi_1)-E(\xi_2)$  is small relative to the standard deviations,  $\max(\xi_1, \xi_2)$  differs considerably from both  $\xi_1$  and  $\xi_2$ . In this case, the use of Normal approximations for  $\xi_1$  and  $\xi_2$  could conceivably produce larger errors in the approximation of the expected value and variance of the greater of the two variables.

**Table 4.8 Comparison of MSAM and MCS for all examples**

		MSAM Approximate	MCS Exact	Accuracy % $\left(\frac{\text{Appr.}-\text{Exa.}}{\text{Exa.}}\right) * 100$
Example 1 Activities: 10	Mean	67.88	67.85	0.04
	Variance	22.35	22.98	-2.76
	S.D.	4.73	4.79	-1.39
	5%	60.11	60.48	-0.62
	95%	75.66	76.19	-0.70
Example 2 Activities: 30	Mean	85.89	85.90	-0.01
	Variance	9.68	9.64	0.44
	S.D.	3.11	3.10	0.22
	5%	80.78	81.27	-0.61
	95%	91.01	91.40	-0.43
Example 3 Activities: 52	Mean	180.84	180.97	-0.07
	Variance	7.16	7.19	-0.35
	S.D.	2.68	2.68	-0.19
	5%	176.44	176.52	-0.05
	95%	185.24	185.33	-0.05
Example 4 Activities: 28	Mean	63.39	63.63	-0.37
	Variance	27.98	29.31	-4.55
	S.D.	5.29	5.41	-2.30
	5%	54.69	55.32	-1.14
	95%	72.09	72.91	-1.13
Example 5 Activities: 38	Mean	83.30	81.77	1.87
	Variance	66.67	92.34	-27.80
	S.D.	8.16	9.61	-15.03
	5%	69.87	67.64	3.29
	95%	96.73	98.44	-1.74

In the present study, the variance of duration of Example 4.5 is large when compared to the mean (the ratio of variance with mean is 0.8). The difference of the

approximations of variance can be seen to be up to 28%. To look at the example more closely, Table 4.11 was produced to show the percentile values of the project completion-time from 5 to 95%. It can be seen that the differences are within  $\pm 5\%$ . The largest difference between the MSAM and Monte Carlo simulation method is 3.37 days (4.62%) in the percentile value of 20%. This approximation should be acceptable when compared to the methods of data collection and the assumptions made.

**Table 4.9 Accuracy of percentile values of MSAM in Example 4.5**

<b>Percentile Values</b>	<b>MSAM Approximate</b>	<b>MCS Exact</b>	<b>Accuracy % <math>\left(\frac{\text{Appr.} - \text{Exa.}}{\text{Exa.}}\right) * 100</math></b>
0.05	69.87	67.64	3.29
0.10	72.83	69.75	4.42
0.15	74.83	71.54	4.60
0.20	76.42	73.05	4.62
0.25	77.79	74.48	4.45
0.30	79.01	75.84	4.19
0.35	80.15	77.07	3.99
0.40	81.23	78.36	3.66
0.45	82.27	79.69	3.24
0.50	83.30	81.04	2.78
0.55	84.32	82.38	2.36
0.60	85.36	83.71	1.98
0.65	86.44	85.00	1.70
0.70	87.58	86.59	1.14
0.75	88.80	88.32	0.55
0.80	90.17	90.08	0.10
0.85	91.76	92.01	-0.28
0.90	93.76	94.66	-0.95
0.95	96.73	98.44	-1.74



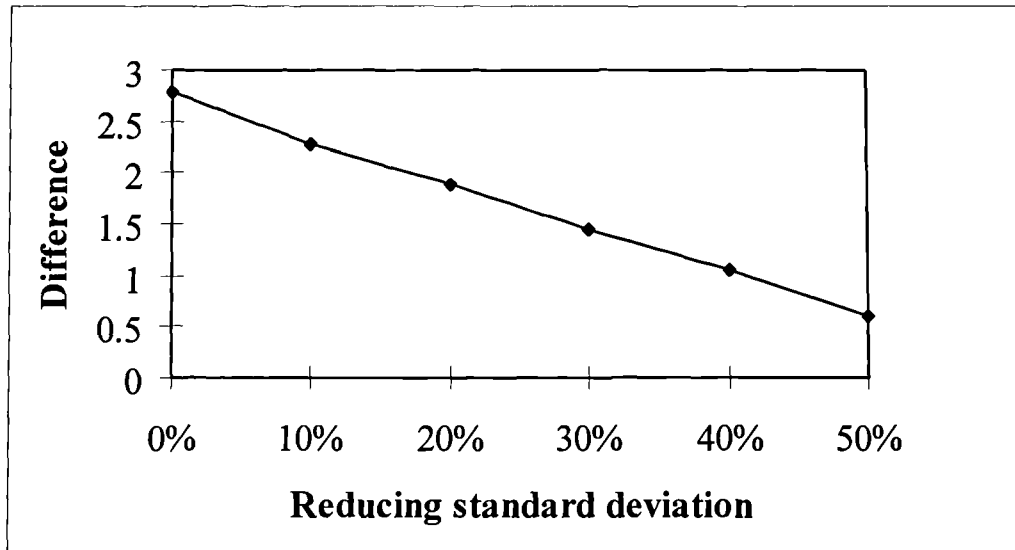
The most important information in the risk analysis of a project, is the probability for a prescribed target time being achieved or the required project time at a specific probability. If this criteria (percentile value) is used, the MSAM approximation is sufficiently accurate for all of the examples discussed.

From the study of the examples in this chapter, it can be concluded that MSAM method produces very good results for the entire range of probabilities of interest when compared to Monte Carlo simulation. However, it should be accepted that MSAM is an approximation method, especially when the case is that the variance is large compared to the mean value and thus a closer check with Monte Carlo simulation is recommended.

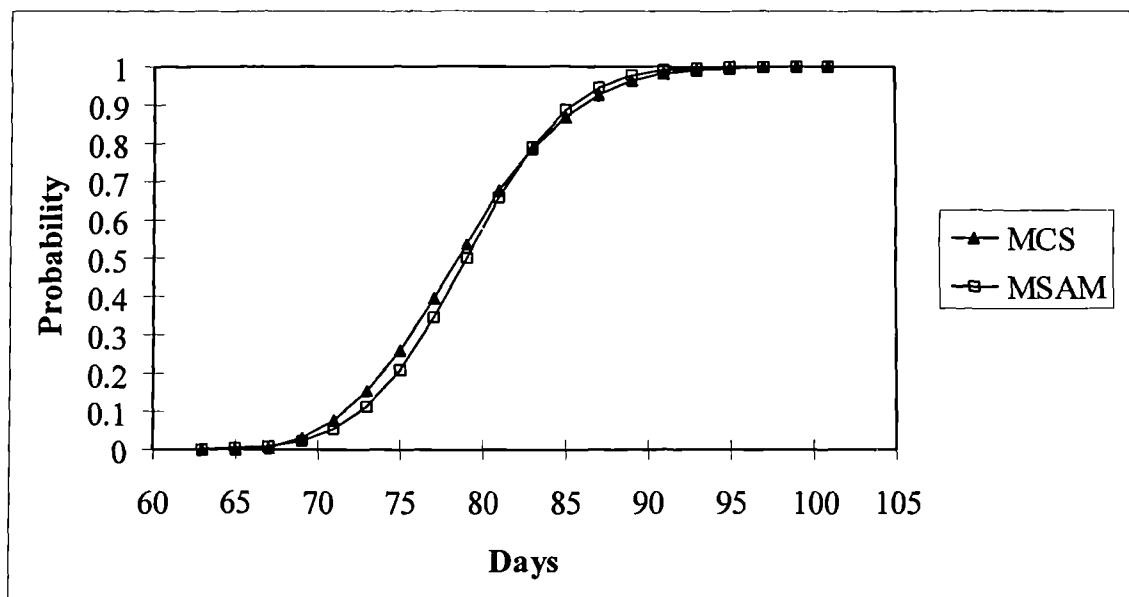
A sensitivity study of the ratio between variance (standard deviation) with mean which affects the accuracy of the MSAM when compared to MCS was carried out. In Example 4.5 (industrial building project), the means of each individual activity were kept constant while the standard deviations of them were reduced 10% to 50% respectively. Figure 4.13 shows the difference of the MSAM compared to MCS in the 50<sup>th</sup> percentile value is steadily reduced when the standard deviations of each individual activity are reduced. This indicates the accuracy of the MSAM is improving.

Figure 4.14 shows the comparison of the MSAM and MCS methods when the standard deviations of each individual activity are reduced to 50% of the original data (Table 6). It can be seen that the results of MSAM are in excellent agreement with large sample (10,000) Monte Carlo simulations.

The observations are that for a general network, if the variance of a project duration is small when compared to the mean, such as the ratio of the variance with mean is less than 0.5, the MSAM produces excellent results when compared to those of MCS. If the ratio is larger than 0.5, the MSAM is a more approximation method and produces more pessimistic results.



**Figure 4.13 Improving accuracy of the MSAM as the standard deviations reduced**



**Figure 4.14 Comparison of MSAM with MCS when the standard deviations are reduced to 50% of the original data**

#### 4.3.5.2 Comparison of the MSAM with other methods

Examples 4.4 and 4.5 were used to study the comparison of MSAM with other methods, such as PERT, PNET and Monte Carlo simulations. Figures 4.12 and 4.14

provide a comparisons of these methods and the following observations can be obtained:

Under certain conditions, if the variance is small when compared to the mean of the project duration, MSAM produces excellent results when contrasted to those of Monte Carlo simulation and PNET methods. If the ratio is large, MSAM produces more pessimistic results when compared to those of other methods. PERT always gives the most optimistic estimate of a project duration.

In contrast to the methods of PERT and CPM where only the critical path is considered, MSAM takes total account of the correlations between paths in a network. PNET and MSAM are both approximate analytical methods and consider path correlations. Unlike PNET, in which all paths need enumeration, MSAM is a one - pass method which can be more easily understood and implemented.

MSAM is a direct analytical procedure and is very efficient when compared to the Monte Carlo simulation method in which repeated sample solutions are required. For all the examples in this chapter, the calculations were completed in a matter of seconds using the MSAM program whereas the Monte Carlo simulations took approximately 4-5 hours for each example using the @Risk Monte Carlo simulation package in the same computer.

#### **4.4 Summary of findings**

A simple method for evaluating the project duration probabilities of project networks has been presented. The proposed new method, MSAM, is a direct application of Clark's results and is modified from the SAM, which is a method previously used solely in traffic networks. A computer program has been written to implement the MSAM algorithm. The method can be used to estimate completion times of projects in the presence of uncertainty and thus permits the consideration of network decision problems in terms of risk and probability.

The MSAM method is developed for analysing general construction networks. Five example projects chosen randomly from published work have been studied, and the validity of the method has been verified by results of large sample Monte Carlo simulations. From the results of these examples, the following conclusions can be drawn:

1. In all of the cases studied, MSAM produces very good results when compared to MCS for the entire range of probabilities of interest. However, MSAM is an approximate method, and in the case of the variance of project duration being large when compared to the mean (Example 4.5), MSAM produces more pessimistic results when compared to other methods (PERT, PNET and MCS).
2. PERT gives the most optimistic estimate of a project duration.
3. When the ratio of the standard deviation and the mean is reduced, the accuracy of the MSAM improves.
4. It is possible to use other approximations, such as methods by Mendell and Elston [99], Kamakura [100], and Langdon [101,102], to replace Clark's results in order to improve accuracy and still use the MSAM mechanism.
5. When compared to PERT and CPM, the MSAM has the advantage of taking total account of the correlations between paths in a network.
6. When compared to PNET, the MSAM is a one-pass method which does not need path enumeration and thus can be more easily implemented.
7. When compared to MCS, the MSAM is a direct analytical procedure and is very efficient in saving computational time. The calculations take only few seconds whereas the MCS take 4-5 hours in the same PC in all of the examples.
8. In its present form, the MSAM method yields the probability of project completion within a prescribed target time, or the required project time at a specific

probability. The method may be extended to consider the effects of uncertainty in general activity and decision networks.

## **CHAPTER 5 THE FIRST ORDER SECOND MOMENT METHOD (FOSM)**

### **5.1 Introduction**

As stated in Chapter 1, the overall objective of this PhD is to develop risk analysis methods for quantifying risk in construction project management. One of the objectives of this PhD is to develop a statistical method, the FOSM method (a methodology previously used solely in structural reliability analysis) as a method of risk analysis to aid decision making in project management.

The FOSM methods have been developed over the last thirty years and have been applied in other branches of engineering such as “Reliability and reliability-based design of engineering systems” [16,103,104,105,106]. These methods were specially developed for estimation of structural failure probability. It has been shown that the methods are computationally fast and provide a measure of risk with a desired degree of accuracy. However, the FOSM methods have not been previously used in construction project management.

Unlike the Modified Stochastic Assignment Model (MSAM) developed in Chapter 4, which is specifically used in network analysis for prediction of project completion time, the FOSM is a more general method, which can be applied for risk assessment. It is especially suited to the conceptual stage of a project but can be used to assess the failure probability throughout a project life cycle.

This chapter aims to show why and how the FOSM methods can be applied for risk analysis in construction project management. Since only the latest developed FOSM methods are applied to construction economics, these methods are expressed as one method, FOSM method, in the present study. Firstly, the definition of the FOSM method is given and a detailed mathematical treatment of the method is described. The methodology of using FOSM in construction project management is explained. Eight examples are analysed using both the FOSM method and Monte Carlo simulations to

show the applicability and the degree of accuracy of the method. These examples are chosen in a systematic way covering different aspects of common project management problems (with linear or non-linear performance functions) and the distributions of the variables are assumed with standard distributions commonly used, such as Normal, Log-Normal, Triangular, Uniform and Beta. Finally, two further examples, a probabilistic cost modelling example and a realistic plant hire rate setting example, are illustrated to show further practical applications of the method.

## **5.2 The FOSM method**

### **5.2.1 The basic problem of reliability of engineering systems**

The philosophical basis of FOSM methods were presented in the works of Cornell <sup>[107]</sup>, Ang <sup>[108]</sup>, Ang and Cornell <sup>[109]</sup>, Hasofer and Lind <sup>[110]</sup>, Rackwitz and Fiessler <sup>[111]</sup> and Chen and Lind <sup>[112]</sup>. The FOSM methods were initially applied in engineering relating to the reliability of engineering systems. The problem of reliability of engineering systems may be cast essentially as a problem of supply versus demand. Defining the following random variables:

$X$  = The supply capacity.

$Y$  = The demand requirement.

The objective of reliability analysis is to insure the event  $(X > Y)$  throughout the useful life of the engineering system. This assurance is possible only in terms of the probability  $P(X > Y)$ . This probability, therefore, represents a realistic measure of the reliability of the system (denoted as  $p_s$ ); conversely, the probability of the complimentary event  $(X < Y)$  is the corresponding measure of unreliability (failure probability denoted as  $p_F$ ).

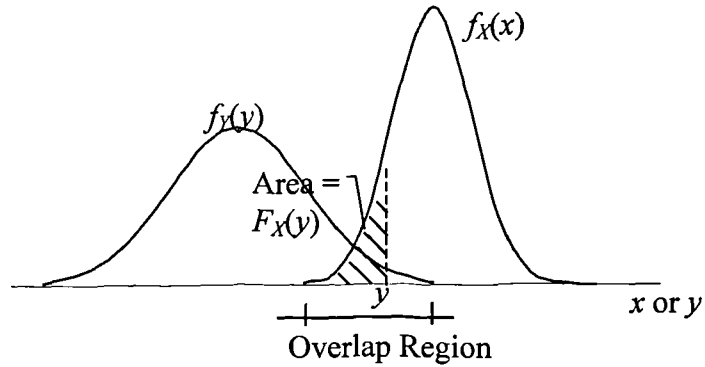
It can be assumed that the necessary probability distributions of  $X$  and  $Y$  are available, that is,  $F_X(x)$  or  $f_X(x)$  and  $F_Y(y)$  or  $f_Y(y)$  are known. The required probabilities may then be formulated as follows:

$$p_F = P(X < Y) = \sum_{all\ y} P(X < Y|Y = y)P(Y = y) \quad (5.1)$$

If the supply and demand,  $X$  and  $Y$ , are statistically independent and continuous, the above equation then becomes (Freudenthal, Garrelt and Shinozuka<sup>[113]</sup>):

$$p_F = \int_0^{\infty} F_X(y)f_Y(y)dy \quad (5.2)$$

Equation 5.2 is the convolution with respect to  $y$  and may be explained with reference to Figure 5.1 as follows: if  $Y = y$ , the conditional probability of failure would be  $F_X(y)$ , but since  $Y = y$  (or more precisely  $y < Y \leq y + dy$ ) is associated with probability  $f_Y(y)dy$ , integration over all values of  $Y$  yields Equation 5.2.



**Figure 5.1** PDFs  $f_X(x)$  and  $f_Y(y)$

As portrayed graphically in Figure 5.1, the overlapping of the curves  $f_X(x)$  and  $f_Y(y)$  represents a qualitative measure of the failure probability  $p_F$ . In this regard, it can be observed that any measure of safety or reliability properly ought to be a function of the relative positions of  $f_X(x)$  and  $f_Y(y)$  as well as of the degree of dispersions.

Theoretically, the failure probability  $p_F$  will also depend on the form of  $f_X(x)$  and  $f_Y(y)$ . In practice, however, information is often limited; available information may be sufficient only to evaluate the main statistics (or first few moments) of  $X$  and  $Y$ , such as the mean values,  $\mu_X$  and  $\mu_Y$ , and the corresponding c.o.v.'s (coefficient of variation),  $\delta_X$



and  $\delta_Y$ . The quantitative evaluation of the true  $p_F$  often poses major problems, for example, the determination of the correct forms of  $f_X(x)$  and  $f_Y(y)$  would be necessary, which may not be a simple task.

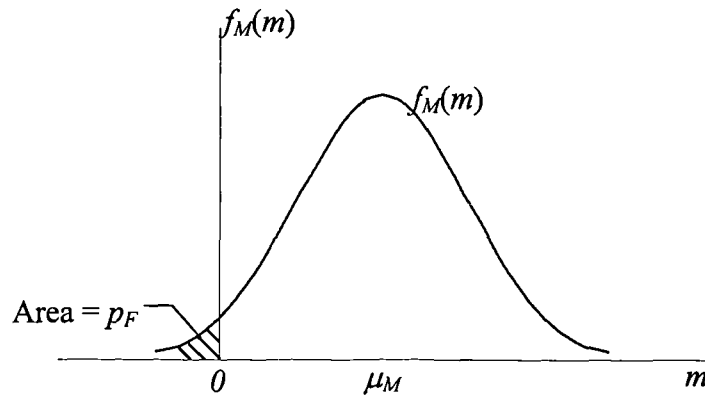
In Equation 5.2,  $X$  and  $Y$  are assumed to be statistically independent random variables. In general, however, these variables may be correlated. In such cases, the probability of failure may be expressed in terms of the joint PDF as follows:

$$p_F = \int_0^\infty \left[ \int_0^y f_{X,Y}(x,y) dx \right] dy \quad (5.3)$$

The above supply versus demand problem may be formulated in terms of the safety margin,  $M = X - Y$ . As  $X$  and  $Y$  are random variables,  $M$  is also a random variable with corresponding PDF  $f_M(m)$ . In this case, failure is clearly the event  $(M < 0)$ , and thus the probability of failure is:

$$p_F = \int_{-\infty}^0 f_M(m) dm = F_M(0) \quad (5.4)$$

Graphically, this is represented by the area under  $f_M(m) < 0$ , as shown in Figure 5.2.



**Figure 5.2 PDF of safety margin  $M$**

### 5.2.2 Second-moment formulation

The calculation of the probability of safety, or probability of failure, requires the knowledge of the distributions  $f_X(x)$  and  $f_Y(y)$ , or the joint distribution  $f_{X,Y}(x,y)$ . In

practice, this information is often unavailable or difficult to obtain for reasons of insufficient data. Furthermore, even when the required distributions can be specified, the exact evaluation of the probabilities, generally requiring the numerical integration of Equations 5.1 - 5.4, may be impractical; as a practical alternative, equivalent Normal distributions may be resorted to in approximation.

Not infrequently, the available information or data may be sufficient only to evaluate the first and second moments; namely, the mean values and variances of the respective random variables (and perhaps the covariances between pairs of variables). Practical measures of safety or reliability, therefore, must often be limited to functions of these first two moments. Under this condition, the implementation of reliability concepts must necessarily be limited to a formulation based on the first and second moments of the random variables - that is, restricted to the second-moment formulation (Cornell <sup>[107]</sup>, Ang and Cornell <sup>[109]</sup>).

With the second-moment approach, the reliability may be measured entirely with a function of the first and second moments of the design variables; namely, the reliability index,  $\beta$ , when there is no information on the probability distributions; whereas, if the appropriate forms of the distributions are prescribed, the corresponding probability may be evaluated on the basis of equivalent Normal distributions.

Recall the safety margin  $M = X - Y$ . In this term, the “safe state” of a system may be defined as ( $M > 0$ ), whereas the “failure state” is ( $M < 0$ ). The boundary separating the safe and failure states is the “limit-state” defined by the equation  $M = 0$ .

Cornell defined a safety index  $\beta$ , which is given by:

$$\beta = \mu_m / \sigma_m \quad (5.5)$$

where  $\mu_m$ , and  $\sigma_m$  represents the mean and standard deviation of  $M$  respectively. If  $X$  and  $Y$  are Normally distributed, then  $M$  is also Normally distributed. In this case, the safety index  $\beta$  is uniquely related to the failure probability by the expression:

$$P_f = \Phi(-\beta) \quad (5.6)$$

where  $\Phi(-)$  is the standardised Normal cumulative distribution function.

The reliability of an engineering system may involve multiple variables. In particular, the supply and demand may respectively, be functions of several other variables. A performance function can be defined as:

$$g(X) = g(X_1, X_2, \dots, X_n) \quad (5.7)$$

where  $X = X_1, X_2, \dots, X_n$  is a vector of basic state (or design) variables of the system, and the function  $g(X)$  determines the performance or state of the system. Accordingly, the limiting performance requirement may be defined as  $g(X) = 0$ , which is the “limit-state” of the system. It follows, therefore, that:

$$[g(X) > 0] = \text{the “safe state”}$$

and

$$[g(X) < 0] = \text{the “failure state”}$$

Geometrically, the limit-state equation,  $g(X) = 0$ , is an  $n$ -dimensional surface that may be called the “failure surface”. One side of the failure surface is the safe state,  $g(X) > 0$ , whereas the other side of the failure surface is the failure state,  $g(X) < 0$ .

Hence, if the joint PDF of the design variables  $X_1, X_2, \dots, X_n$  is  $f_{X_1, \dots, X_n}(x_1, \dots, x_n)$ , the probability of the failure state is:

$$p_F = \int_{g(X) < 0} \int f_{X_1, \dots, X_n}(x_1, \dots, x_n) dx_1 \dots dx_n \quad (5.8)$$

which may be written, for brevity, as:

$$p_F = \int_{g(X) < 0} f_X(X) dX \quad (5.9)$$

The evaluation of the probability  $p_F$  through the above equation, however, is generally a formidable task.

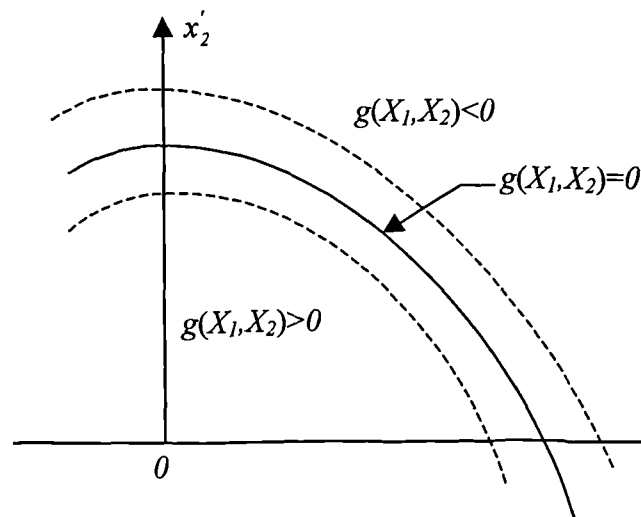
For practical purposes, alternative methods of evaluating  $p_S$  or  $p_F$  (or its equivalent) are necessary and these have been studied by a number of authors.

Consider firstly the case of uncorrelated variables. The set of uncorrelated reduced variates are introduced (Freudenthal <sup>[114]</sup>):

$$X'_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \quad i = 1, 2, \dots, n \quad (5.10)$$

Obviously, the safe state and failure state may also be portrayed in the space of the above reduced variates, separated by the appropriate limit-state equation. The two - variable case would be shown in Figure 5.3. In terms of the reduced variates,  $X'_i$ , the limit-state equation would be:

$$g(\sigma_{X_1} X'_1 + \mu_{X_1}, \dots, \sigma_{X_n} X'_n + \mu_{X_n}) = 0 \quad (5.11)$$



**Figure 5.3 Safe and failure states in space of reduced variates**

Observe from Figure 5.3 that as the limit-state surface (or failure surface),  $g(X)=0$ , moves further or closer to the origin, the safe region,  $g(X)>0$ , increases or decreases

accordingly. Therefore, the position of the failure surface relative to the origin of the reduced variates should determine the safety or reliability of the system. The position of the failure surface may be represented by the minimum distance from the surface  $g(X)=0$  to the origin of the reduced variates (Hasofer and Lind <sup>[110]</sup>, Ditlevsen <sup>[70]</sup>); indeed, Shinozuka <sup>[115]</sup> has shown that the point on the failure surface with minimum distance to the origin is the most probable failure point (Freudenthal <sup>[114]</sup>). Thus, in some approximate sense, this minimum distance may be used as a measure of reliability; the precise nature of this approximation is discussed in Section 5.2.4 for a general (non-linear) performance function.

Following Shinozuka <sup>[115]</sup>, the required minimum distance may be determined as follows. The distance from a point  $X' = (X'_1, X'_2, \dots, X'_n)$  on the failure surface  $g(X)=0$  to the origin of  $X'$  is:

$$D = \sqrt{X'^2_1 + \dots + X'^2_n} = (X'^t X')^{1/2} \quad (5.12)$$

The point on the failure surface,  $(x'^*_1, x'^*_2, \dots, x'^*_n)$ , having the minimum distance to the origin may be determined by minimising the function  $D$ , subject to the constraint  $g(X)=0$ ; that is,

$$\begin{aligned} &\text{Minimise } D \\ &\text{subject to } g(X)=0. \end{aligned}$$

For this purpose, the method of Lagrange's multiplier may be used. Let

$$L = D + \lambda g(X) \quad (5.13)$$

or

$$L = (X'^t X')^{1/2} + \lambda g(X) \quad (5.14)$$

In scalar notation,

$$L = \sqrt{X'^2_1 + X'^2_2 + \dots + X'^2_n} + \lambda g(X_1, X_2, \dots, X_n) \quad (5.15)$$

in which,  $X_i = \sigma_{X_i} X'_i + \mu_{X_i}$ .

Minimising  $L$ , the following set of  $n+1$  equations with  $n+1$  unknowns are obtained:

$$\frac{\partial L}{\partial X'_i} = \frac{X'_i}{\sqrt{X'^2_1 + X'^2_2 + \dots + X'^2_n}} + \lambda \frac{\partial g}{\partial X'_i} = 0 \quad i = 1, 2, \dots, n \quad (5.16)$$

and

$$\frac{\partial L}{\partial \lambda} = g(X_1, X_2, \dots, X_n) = 0 \quad (5.17)$$

The solution of the above set of equations should yield the most probable failure point  $(x'^*_1, x'^*_2, \dots, x'^*_n)$ .

Introduce the gradient vector:

$$G = \left( \frac{\partial g}{\partial X'_1}, \frac{\partial g}{\partial X'_2}, \dots, \frac{\partial g}{\partial X'_n} \right) \quad (5.18)$$

in which:

$$\frac{\partial g}{\partial X'_i} = \frac{\partial g}{\partial X_i} \frac{dX_i}{dX'_i} = \sigma_{X_i} \frac{\partial g}{\partial X_i} \quad (5.19)$$

The above set of equations, Equation 5.16, can then be written in matrix notation as:

$$\frac{X'}{(X'' X')'^2} + \lambda G = 0 \quad (5.20)$$

from which:

$$X' = -\lambda DG \quad (5.21)$$

Therefore,

$$D = \left[ (\lambda D G') (\lambda D G) \right]^{1/2} = \lambda D (G' G)^{1/2} \quad (5.22)$$

and thus,

$$\lambda = (G' G)^{-1/2} \quad (5.23)$$

Using this last result in Equation 5.21 yields:

$$X' = \frac{-GD}{(G' G)^{1/2}} \quad (5.24)$$

Conversely (pre-multiplying Equation 5.24 by  $G'$ ),

$$D = \frac{-G' X'}{(G' G)^{1/2}} \quad (5.25)$$

Substituting Equation 5.24 in Equation 5.17 results in a single equation with the unknown  $D$ ; solution of the resulting equation then yields the minimum distance  $d_{min} = \beta$ , thus:

$$\beta = \frac{-G'^* X'^*}{(G'^* G^*)^{1/2}} \quad (5.25a)$$

in which  $G^*$  is the gradient vector at the most probable failure point  $(x_1^*, x_2^*, \dots, x_n^*)$ . In scalar form, Equation 5.25a is:

$$\beta = \frac{-\sum_i x_i'^* \left( \frac{\partial g}{\partial X_i'} \right)_*}{\sqrt{\sum_i \left( \frac{\partial g}{\partial X_i'} \right)_*^2}} \quad (5.25b)$$

where the derivatives  $\left(\partial g / \partial X_i\right)_*$  are evaluated at  $\left(x_1^*, x_2^*, \dots, x_n^*\right)$ . Using the above Equation 5.24, the most probable point on the failure surface becomes:

$$X^* = \frac{-G^* \beta}{\left(G^{*t} G^*\right)^{1/2}} \quad (5.24a)$$

the scalar form, the components of  $X^*$ , Equation 5.24a, are:

$$x_i^* = -\alpha_i^* \beta; i = 1, 2, \dots, n \quad (5.26a)$$

in which

$$\alpha_i^* = \frac{\left(\frac{\partial g}{\partial X_i}\right)_*}{\sqrt{\sum_i \left(\frac{\partial g}{\partial X_i}\right)_*^2}} \quad (5.26b)$$

are the direction cosines along the axes  $x_i^*$ .

The results derived above, Equations 5.24a and 5.25a may be interpreted on the basis of first - order approximations for the function  $g(X)$  as follows.

Expand the performance function  $g(X)$  in a Taylor series at a point  $x^*$ , which is in the failure surface  $g(x^*)=0$ ; that is,

$$g(X_1, X_2, \dots, X_n) = g(x_1^*, x_2^*, \dots, x_n^*) + \sum_{i=1}^n (X_i - x_i^*) \left(\frac{\partial g}{\partial X_i}\right)_* + \sum_{j=1}^n \sum_{i=1}^n (X_i - x_i^*) (X_j - x_j^*) \left(\frac{\partial^2 g}{\partial X_i \partial X_j}\right)_* + \dots \quad (5.27)$$

where the derivatives are evaluated at  $(x_1^*, x_2^*, \dots, x_n^*)$ . But  $g(x_1^*, x_2^*, \dots, x_n^*) = 0$  in the failure surface; therefore,



$$g(X_1, X_2, \dots, X_n) = \sum_{i=1}^n (X_i - x_i^*) \left( \frac{\partial g}{\partial X_i} \right)_* + \sum_{j=1}^n \sum_{i=1}^n (X_i - x_i^*) (X_j - x_j^*) \left( \frac{\partial^2 g}{\partial X_i \partial X_j} \right)_* + \dots \quad (5.28)$$

Recall that

$$X_i - x_i^* = (\sigma_{X_i} X'_i + \mu_{X_i}) - (\sigma_{X_i} x'_i + \mu_{X_i}) = \sigma_{X_i} (X'_i - x'_i) \quad (5.29)$$

and

$$\frac{\partial g}{\partial X_i} = \frac{\partial g}{\partial X'_i} \left( \frac{dX'_i}{dX_i} \right) = \frac{1}{\sigma_{X_i}} \left( \frac{\partial g}{\partial X'_i} \right) \quad (5.30)$$

Then,

$$g(X_1, X_2, \dots, X_n) = \sum_{i=1}^n (X'_i - x'_i) \left( \frac{\partial g}{\partial X'_i} \right)_* + \dots \quad (5.31)$$

In first - order approximation, that is, truncating the above series at the first - order term, the mean value of the function  $g(X)$ , therefore, is:

$$\mu_g \approx - \sum_{i=1}^n x'_i \left( \frac{\partial g}{\partial X'_i} \right)_* \quad (5.32)$$

whereas the corresponding first - order approximate variance (for uncorrelated variates is:

$$\sigma_g^2 \approx \sum_{i=1}^n \sigma_{X'_i}^2 \left( \frac{\partial g}{\partial X'_i} \right)_*^2 = \sum_{i=1}^n \left( \frac{\partial g}{\partial X'_i} \right)_*^2 \quad (5.33)$$

From Equations 5.32 and 5.33, the ratio:

$$\frac{\mu_g}{\sigma_g} = \frac{- \sum_{i=1}^n x'_i \left( \frac{\partial g}{\partial X'_i} \right)_*}{\sqrt{\sum_{i=1}^n \left( \frac{\partial g}{\partial X'_i} \right)_*^2}} \quad (5.34)$$

Comparing this with Equation 5.25b we see that the above ratio is the same as Equation 5.25b, and thus  $\mu_g/\sigma_g$  is also the distance from the tangent plane of the

failure surface at  $x^*$  to the origin of the reduced variates. Therefore, the reliability index is also:

$$\beta = \mu_g / \sigma_g \quad (5.35)$$

It may be emphasised that the first-order approximation of  $\mu_g$  and  $\sigma_g$  derived above must be evaluated at a point on the failure surface  $g(X)=0$ .

### 5.2.3 Linear performance functions

Consider a specialised class of performance functions, namely the linear performance function. Aside from its own usefulness, certain aspects of the linear case would be the basis for an approximation to non-linear performance functions, as will be discussed later in section 5.2.4.

A linear performance function may be represented as:

$$g(X) = a_0 + \sum_i a_i X_i \quad (5.36)$$

where  $a_0$  and  $a_i$ 's are constants. The corresponding limit-state equation, therefore, is:

$$a_0 + \sum_i a_i X_i = 0 \quad (5.37)$$

In terms of the reduced variates, Equation 5.10, the limit - state equation becomes:

$$a_0 + \sum_i a_i (\sigma_{x_i} X'_i + \mu_{x_i}) = 0 \quad (5.37a)$$

The distance of the failure plane, Equation 5.37a, to the origin of the reduced variates  $X'$  is:

$$\beta = \frac{a_0 + \sum_i a_i \mu_{x_i}}{\sqrt{\sum_i (a_i \sigma_{x_i})^2}} \quad (5.38)$$

### 5.2.3.1 Equivalent Normal distributions

If the probability distributions of the random variables  $X_1, X_2, \dots, X_n$  are not Normal, the probability  $p_F$  and  $p_S$  may be evaluated through Equations 5.8 and 5.9 (invariably, numerical integration would be necessary), where in the case of linear performance functions,  $g(X) = a_0 + \sum a_i X_i$ . However,  $p_S$  may be evaluated also using equivalent Normal distributions (Paloheimo <sup>[116]</sup> and Rackwitz <sup>[117]</sup>). Theoretically, such equivalent Normal distributions may be obtained through the Rosenblatt transformation <sup>[118]</sup>. With such equivalent Normal distributions, the calculation of  $p_S$  follows the same procedure as that for Normal variates; that is through Equation 5.38 for linear performance functions.

For an individual variate, the equivalent Normal distribution for a non-Normal variate may be obtained such that the cumulative probability as well as the probability density ordinate of the equivalent Normal distribution are equal to those of the corresponding non-Normal distribution at the appropriate point  $x_i^*$ , on the failure surface.

Equating the cumulative probabilities as described above at the failure point  $x_i^*$ , we have:

$$\Phi\left(\frac{x_i^* - \mu_{x_i}^N}{\sigma_{x_i}^N}\right) = F_{x_i}(x_i^*) \quad (5.39)$$

where:

$\mu_{x_i}^N, \sigma_{x_i}^N$  = The mean value and standard deviation, respectively, of the equivalent Normal distribution for  $X_i$ .

$F_{x_i}(x_i^*)$  = The original CDF of  $X_i$  evaluated at  $x_i^*$ .

$\Phi(-)$  = The CDF of standard Normal distribution.

The above equality then yields:

$$\mu_{x_i}^N = x_i^* - \sigma_{x_i}^N \Phi^{-1} \left[ F_{x_i}(x_i^*) \right] \quad (5.40)$$

whereas equating the corresponding probability density ordinates at  $x_i^*$  means:

$$\frac{1}{\sigma_{x_i}^N} \phi \left( \frac{x_i^* - \mu_{x_i}^N}{\sigma_{x_i}^N} \right) = f_{x_i}(x_i^*) \quad (5.41)$$

where  $\phi(-)$  = the PDF of the standard Normal distribution; from this we obtain:

$$\sigma_{x_i}^N = \frac{\phi \left\{ \Phi^{-1} \left[ F_{x_i}(x_i^*) \right] \right\}}{f_{x_i}(x_i^*)} \quad (5.42)$$

In the case of a linear performance function, the appropriate point on the failure surface is given by Equation 5.26, where the direction cosines,  $\alpha_i$ , Equation 5.26b are:

$$\alpha_i = \frac{a_i}{\sqrt{\sum_i a_i^2}} \quad (5.43)$$

and according to Equation 5.38, the safety index is:

$$\beta = \frac{a_0 + \sum_i a_i \mu_{x_i}^N}{\sqrt{\sum_i (a_i \sigma_{x_i}^N)^2}} \quad (5.44)$$

where the superscript N denotes the statistics for the equivalent Normal distribution. Therefore, the failure point is:

$$x_i^* = \sigma_{x_i}^N x_i'^* + \mu_{x_i}^N = -\alpha_i \beta \sigma_{x_i}^N + \mu_{x_i}^N \quad (5.45)$$

It may be emphasised that replacing the actual distribution with an equivalent Normal distribution requires replacing the actual mean and standard deviation with those of the

equivalent Normal distribution, that is, Equations 5.40 and 5.42. Using these in Equation 5.44, we obtain the safety index  $\beta$ .

### 5.2.3.2 Correlated variates

The procedure described above for evaluating the probability of safety or failure is tacitly based on the assumption that the random variables  $X_1, X_2, \dots, X_n$  are uncorrelated or statistically independent. For random variables that are correlated, the original variates may be transformed to a set of uncorrelated variables. The procedure described above, namely Equation 5.25a, may then be applied to the uncorrelated set of transformed variables. Indeed, this has been explicitly shown by Shinozuka <sup>[115]</sup>.

The required transformation is necessarily dependent on the covariances, or covariance matrix, of the original variates and may be obtained as follows.

Suppose the covariance matrix of the original variates  $X_1, X_2, \dots, X_n$  is:

$$[C] = \begin{bmatrix} \sigma_{X_1}^2 & Cov(X_1, X_2) & Cov(X_1, X_3) & \cdots & Cov(X_1, X_n) \\ Cov(X_2, X_1) & \sigma_{X_2}^2 & Cov(X_2, X_3) & \cdots & Cov(X_2, X_n) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ Cov(X_n, X_1) & Cov(X_n, X_2) & Cov(X_n, X_3) & \cdots & \sigma_{X_n}^2 \end{bmatrix} \quad (5.46)$$

where the elements,  $Cov(X_i, X_j)$  are the respective covariances between the pairs of variables  $X_i$  and  $X_j$ . The corresponding covariance between a pair of reduced variates  $X'_i$  and  $X'_j$  is:

$$\begin{aligned} Cov(X'_i, X'_j) &= E[(X'_i - \mu_{X'_i})(X'_j - \mu_{X'_j})] \\ &= \frac{E[(X_i - \mu_{X_i})(X_j - \mu_{X_j})]}{\sigma_{X_i} \sigma_{X_j}} \\ &= \frac{Cov(X_i, X_j)}{\sigma_{X_i} \sigma_{X_j}} \\ &= \rho_{X_i, X_j} \end{aligned} \quad (5.47)$$

which means that the covariance between a pair of reduced variates,  $X'_i$  and  $X'_j$ , is equal to the correlation coefficient between the corresponding pair of original variates  $X_i$  and  $X_j$ . Therefore, the covariance matrix of the reduced variates  $(X'_1, X'_2, \dots, X'_n)$  is the corresponding correlation matrix of the original variates  $(X_1, X_2, \dots, X_n)$ ; that is, the covariance matrix of  $X'_1, X'_2, \dots, X'_n$  is:

$$[C'] = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \rho_{23} & \cdots & \rho_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \rho_{n1} & \rho_{n2} & \rho_{n3} & \cdots & 1 \end{bmatrix} \quad (5.48)$$

The required set of (uncorrelated) transformed variates can be obtained from  $X'$  through the following orthogonal transformation:

$$Y = T' X' \quad (5.49)$$

in which:

$$X' = \{X'_1, X'_2, \dots, X'_n\}.$$

$Y = \{Y_1, Y_2, \dots, Y_n\}$  is the required set of uncorrelated transformed variates.

$T$  = An orthogonal transformation matrix (superscript  $t$  represents the *transpose*).

$T$  will be an orthogonal matrix if it is composed of the eigenvectors corresponding to the eigenvalues of the correlation matrix  $[C']$ . Specifically,  $T$  is such that:

$$T'[C']T = [\lambda] \quad (5.50)$$

in which  $[\lambda]$  is a diagonal matrix of the eigenvalues of  $[C']$ . It may be emphasised that the matrix  $[C']$  is real and symmetric, as  $\rho_{ij} = \rho_{ji}$ ; and thus the eigenvectors are mutually orthogonal.

With the orthogonal transformation of Equation 5.49, it can be shown (Shinozuka <sup>[115]</sup>) that the safety index of Equation 5.25a becomes:

$$\beta = \frac{-G^* X'^*}{\left(G^* [C'] G^*\right)^{1/2}} \quad (5.25c)$$

The reduced variates  $X'$  and original variate  $X$  are related to  $Y$  as follows.

Since  $T$  is orthogonal,  $T^{-1} = T'$ ; inversion of Equation 5.49 yields:

$$X' = TY \quad (5.51)$$

and

$$\begin{aligned} X &= [\sigma_X] X' + \mu_X \\ &= [\sigma_X] TY + \mu_X \end{aligned} \quad (5.52)$$

in which:

$$[\sigma_X] = \begin{bmatrix} \sigma_{X_1} & 0 & \cdots & 0 \\ 0 & \sigma_{X_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{X_n} \end{bmatrix} \quad (5.53)$$

and

$$\mu_X = \begin{Bmatrix} \mu_{X_1} \\ \mu_{X_2} \\ \vdots \\ \mu_{X_n} \end{Bmatrix} \quad (5.54)$$

Observe that the covariance matrix of  $Y$  is:

$$\begin{aligned} [C_Y] &= E(YY') = E(T' X' X' T) \\ &= T' E(X' X') T \end{aligned} \quad (5.55)$$

but

$$E(X' X') = [C'] \quad (5.56)$$

Thus, with Equation 5.50,

$$[C_Y] = T' [C'] T = [\lambda] \quad (5.57)$$

Hence, the eigenvalues of  $[C']$  are also the variances of the respective variates  $Y_1, Y_2, \dots, Y_n$ .

In the space of the transformed variates  $Y$ , the derivatives may be obtained through the chain rule of partial differentiation:

$$\frac{\partial g}{\partial Y_i} = \sum_{j=1}^n \frac{\partial g}{\partial X'_j} \frac{\partial X'_j}{\partial Y_i} \quad (5.58)$$

Moreover,

$$\frac{\partial g}{\partial X'_j} = \frac{\partial g}{\partial X_j} \frac{dX_j}{dX'_j} = \sigma_{X_j} \left( \frac{\partial g}{\partial X_j} \right) \quad (5.59)$$

The transformation of Equation 5.49 represents a rotation of the co-ordinates from  $X'$  to  $Y$ ; The origin of the  $Y$  axes remains the same as that of the  $X'$  axes.

The above transformation obviously applies also to linear performance functions. In this case, the partial derivatives of Equation 5.58 are independent of the variables, and thus the failure point  $y^*$  and  $x^*$  can be determined directly; that is, one iteration of the numerical algorithm is sufficient. Alternatively, for linear performance functions of correlated (Normal) variates, the safety index may also be determined directly from Equation 5.25c or on the basis of Equation 5.35, yielding:

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{a_0 + \sum_{i=1}^n a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \rho_{ij} \sigma_{X_i} \sigma_{X_j}}} \quad (5.60)$$

in which  $\rho_{ij}$  is the correlation coefficient between  $X_i$  and  $X_j$ .

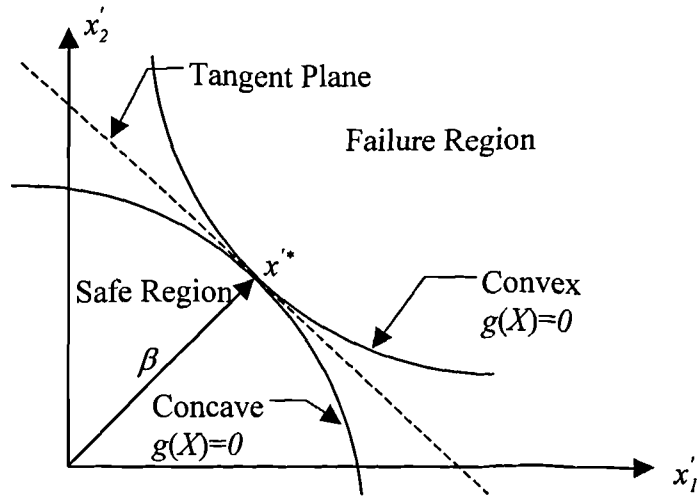


Again, if the distributions of the original random variables are non-Normal, the corresponding probability of safety or failure may be evaluated using equivalent Normal distributions; in such a case, the mean values and standard deviations of the equivalent Normal distributions,  $\mu_{x_i}^N$  and  $\sigma_{x_i}^N$ , must be used in place of  $\mu_{x_i}$  and  $\sigma_{x_i}$ , in Equation 5.60.

#### 5.2.4 Non-linear performance functions

For performance functions,  $g(X)$ , that are non-linear, the evaluation of the exact probability of safety or failure will generally be involved. The limit-state equation,  $g(X)=0$ , will also be non-linear as shown in Figure 5.3; unlike the linear case, there is no unique distance from the failure surface to the origin of the reduced variates. As indicated in Section 5.2.2, the evaluation of the exact probability of safety will involve the integration of the joint probability density function over the non-linear region  $g(X)>0$ ; generally, this will require multiple numerical quadrature.

For practical purpose, approximation to the exact probability will be necessary. According to the results of Section 5.2.2, the point  $(x_1^*, x_2^*, \dots, x_n^*)$  on the failure surface with minimum distance to the origin of the reduced variates is the most probable failure point (Shinozuka [115]). The tangent plane to the failure surface at  $(x_1^*, x_2^*, \dots, x_n^*)$  may then be used to approximate the actual failure surface, and the required reliability index or probability of safety may be evaluated as in the linear case of Section 5.2.3. Depending on whether the exact non-linear failure surface is convex or concave toward the origin, this approximation will be on the safe side or unsafe side, respectively, as may be seen in Figure 5.4 for the two-variable case.



**Figure 5.4 Tangent plane to  $g(X)=0$  at  $x'^*$**

The pertinent tangent plane at  $X'^* = (x'_1, x'_2, \dots, x'_n)$  is:

$$\sum_{i=1}^n (X'_i - x'^*_i) \left( \frac{\partial g}{\partial X'_i} \right)_* = 0 \quad (5.61)$$

where the partial derivatives  $(\partial g / \partial X'_i)_*$  are evaluated at  $(x'_1, x'_2, \dots, x'_n)$ .

On the basis of the above approximation, the distance from the “minimum” tangent plane, Equation 5.61, to the origin of the reduced variates is the appropriate reliability index, which may be used to represent the measure of reliability.

In the present case (in which the performance function is non-linear, the pertinent point of tangency on the failure surface is not, a priori, known. Consequently, the determination of the required reliability index would not be as simple as in the linear case (Section 5.2.3), even though linear approximation is invoked. The “minimum” point of tangency on the failure surface may be determined through the Lagrange multiplier method as described in Section 5.2.2. The relevant results of Section 5.2.2 may be summarised as follows.

The most probable failure point of Equation 5.26a is:

$$x_i'^* = -\alpha_i^* \beta \quad (5.26a)$$

in which  $\alpha_i$  are the direction cosines of Equation 5.26b:

$$\alpha_i^* = \frac{\left( \frac{\partial g}{\partial X_i'} \right)_*}{\sqrt{\sum_i \left( \frac{\partial g}{\partial X_i'} \right)_*^2}} \quad (5.26b)$$

where the derivatives are evaluated at  $(x_1'^*, x_2'^*, \dots, x_n'^*)$ . Then,

$$x_i^* = \sigma_{x_i} x_i'^* + \mu_{x_i} = \mu_{x_i} - \alpha_i^* \sigma_{x_i} \beta \quad (5.62)$$

The solution of the limit-state equation:

$$g(x_1^*, x_2^*, \dots, x_n^*) = 0 \quad (5.63)$$

then yields  $\beta$ .

#### 5.2.4.1 Numerical algorithm

The results summarised above would suggest the following simple algorithm (Rackwitz<sup>[117]</sup>):

1. Assume initial values of  $x_i^*$ ;  $i = 1, 2, \dots, n$  and obtain  $x_i'^* = \frac{x_i^* - \mu_{x_i}}{\sigma_{x_i}}$ .
2. Evaluate  $(\partial g / \partial X_i')$  and  $\alpha_i^*$  at  $x_i'$ .
3. Form  $x_i^* = \mu_{x_i} - \alpha_i^* \sigma_{x_i} \beta$ .
4. Substitute above  $x_i^*$  in  $g(x_1^*, x_2^*, \dots, x_n^*) = 0$  and solve for  $\beta$ .
5. Using the  $\beta$  obtained in Step 4, reevaluate  $x_i'^* = -\alpha_i^* \beta$ .

6. Repeat Steps 2 through 5 until convergence is obtained.

#### 5.2.4.2 Accuracy of linear approximation

The “linear” approximation of non-linear performance functions is tantamount to replacing an  $n$ -dimensional failure surface (a hyper-surface) with a hyper-plane tangent to the failure surface at the “most probable failure point”. In effect, this changes the boundary between the safe state,  $g(X) > 0$ , and the failure state,  $g(X) < 0$ , from a general curvilinear surface to a plane surface; the failure probability,  $p_F$ , is then the generalised volume integral of the joint PDF over the failure region  $g(X) < 0$ . As observed earlier from Figure 5.4, the reliability  $p_S$  estimated on the basis of this approximate planar failure surface will be on the conservative or un-conservative side depending on whether the actual failure surface is convex or concave toward the origin of the reduced variates. The accuracy may be improved through quadratic or polynomial approximation (e.g. Fiessler, Neumann and Rackwitz <sup>[119]</sup>) at the cost of mathematical and computational complications.

For a concave failure surface, the safe state,  $g(X) > 0$ , is furthermore bounded between the half-space with the tangent plane (of distance  $\beta$ ) and the hyper-sphere of radius  $\beta$  (as illustrated in Figure 5.5 for two dimensions). The failure equation corresponding to the hyper-sphere is (Hasofer and Lind <sup>[120]</sup>):

$$\sum_{i=1}^n X_i'^2 - \beta^2 = 0 \quad (5.64)$$

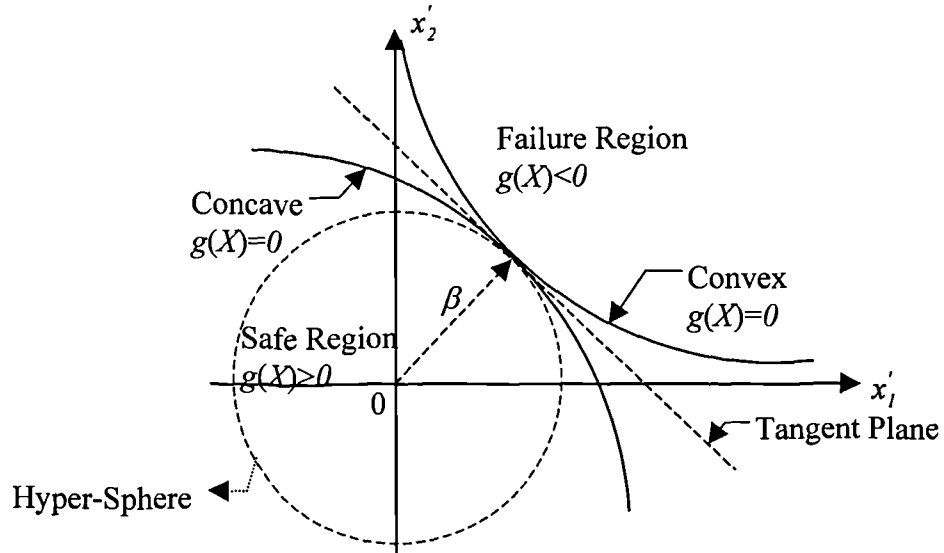
If the variates  $X_i'$  are un-correlated standard Normal variates, the sum of squares  $\sum_{i=1}^n X_i'^2$  has a chi-square distribution with  $n$  degrees-of-freedom. Therefore, the probability of failure becomes:

$$p_F = 1 - x_n^2(\beta^2) \quad (5.65)$$

where  $x_n^2(-)$  is the CDF of the chi-square distribution with  $n$  degrees of freedom.

Accordingly, for concave failure surfaces the failure probability is bounded as follows:

$$\Phi(-\beta) < p_F < 1 - x_n^2(\beta^2) \quad (5.66)$$



**Figure 5.5 Implications of various failure surfaces**

In general, the accuracy of the second - moment linear approximation is difficult to assess; this will depend on the degree of non-linearity of the function  $g(X)$ . Obviously, the method is mathematically exact if  $g(X)$  is linear. For a general non-linear  $g(X)$ , the accuracy may only be appraised numerically for specific forms of non-linear performance functions.

For a general non-linear performance function, the “correct” probability of failure may be evaluated through large-sample Monte Carlo calculations. In the next section, results of Monte Carlo calculations for all of the examples, which involving non-linear  $g(X)$ , are compared to those of FOSM method. On this basis, the accuracy of the second-moment approximate method may be inferred. The results (albeit limited) provide evidence of the validity and accuracy of the second-moment approximation. Moreover, all the non-linear performance functions examined are typical of those found in practical engineering economics problems.

### 5.2.5 Concluding remarks

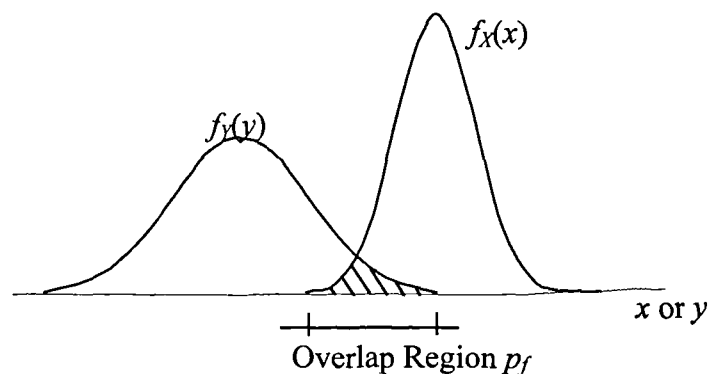
It can be seen that the FOSM is an analytical and approximate method which only uses the first two moments, namely the mean values and variances, of each variable. From these, the probability of failure of the system can be found. It is logical that FOSM can be applied in construction economics with only notational changes. Section 5.3 shows how the method can be applied in construction economics.

## 5.3 Use of FOSM for risk analysis in construction economics

### 5.3.1 Applicability of the FOSM in construction economics

In assessing the reliability of engineering systems, the problems of supply versus demand (strength versus load) may equate to that of revenues versus costs in construction economics. In considering whether to invest in a project, investors and managers are concerned with ensuring that the revenues are sufficient to recover all capital and interest, and earn a required rate of return on investment.

In economic analysis, if  $X$  represents the anticipated distribution, e.g. NPV (Net Present Value) of the revenues from a project, and  $Y$  corresponding to the NPV of the estimated costs. The failure probability,  $p_F$ , will be the probability of the project being a loss maker. Recall Figure 5.1 in section 5.2.1, the overlapping of the curves  $f_X(x)$  and  $f_Y(y)$  represents a qualitative measure of the failure probability  $p_F$ . This area would therefore represent the probability of a project being a loss in project economics (Figure 5.6).



**Figure 5.6** The loss marker  $p_{f(loss)}$

The revenues  $X$  and costs  $Y$  may be modelled as random variables. Each of them also can be a combination of several random variables. In these terms, therefore, the financial risk of a project may be more realistically measured in terms of probability. The objective of risk analysis is to insure the event  $(X > Y)$  throughout the project life cycle.

Since the revenues and cost are random variables or a combination of several random variables. The performance function  $M = X - Y$  is also a random variable. The probability density function of  $M$  is the integration of the joint probability density function of  $X$  and  $Y$ . As stated above, the FOSM method is a method which does not require the joint probability density function of the basic variables but require only the first two moments, namely, the mean values and variances, of each variable. Therefore, it is logical that the FOSM can be used in construction economic analysis with only notational changes.

The probability of loss making  $p_{loss}$  in construction economics, thus is:

$$p_{loss} = p_f = \Phi(-\beta) \quad (5.67)$$

### 5.3.2 Validations and applications

As previously stated in section 5.1, the following examples show different typical aspects of construction economics with hypothetical numerical data used to aid analysis and discussion of the results.

#### 5.3.2.1 Example 5.1 – Linear, uncorrelated Normals

Consider the following mutually exclusive investment proposals. (See Table 5.1)

**Table 5.1 The present values (PV) of Example 5.1**

	Present values (£)	
	Project A	Project B
Mean	3000	6000
Standard deviation	1000	2000
PV distribution	Normal	Normal

**Risk analysis objective:** To find out which project is the better project to invest in. This can be done by computing the probability of failure, in which  $\{PV_A > PV_B\}$ .

Using the FOSM method:

The performance function:  $M = PV_A - PV_B$

$$\beta = \frac{\mu_M}{\sigma_M} = \frac{\mu_A - \mu_B}{\sqrt{\sigma_A^2 + \sigma_B^2}} = \frac{3000 - 6000}{\sqrt{1000^2 + 2000^2}} = -1.34$$

$$p_F = 1 - \Phi(\beta) = 1 - \Phi(-1.34) = 0.909877$$

Using the Monte Carlo simulation, the summary of the results are shown in Table 5.2.

**Table 5.2 Results of MCS for Example 5.1**

Iterations	Failure probability
1,000	0.8917
5,000	0.9023
10,000	0.9073

**Interpreting the results:** The result of FOSM gives a quantitative answer that the failure probability of  $\{PV_A > PV_B\}$  is 90.99%, this means  $PV_A$  has only a 9% chance of being larger than  $PV_B$ . Therefore, Project B is a better project to invest in.

### 5.3.2.2 Example 5.2 – Linear, uncorrelated non-Normals

The material costs of a kitchen unit are estimated as follows in Table 5.3 (values are in £). Assuming these costs are Triangular distributed and are statistically independent.



**Table 5.3 The material costs of Example 5.2**

Material	Low	Most likely	High
A	1800	2250	2800
B	50	75	120
C	80	150	170
D	4000	4800	6000
E	350	380	420

**Risk analysis objective:** To find out the probability that the total cost will exceed £8500.

Using the FOSM method:

The performance function:  $X = A + B + C + D + E - 8500$

Since the material costs are non-Normal distributed, equivalent Normal distributions may be used to calculate the corresponding probability of failure. An iterative solution, using Equations 5.40 and 5.42, is illustrated below.

For the first iteration, assume:

$$a^* = 2250$$

$$b^* = 75$$

$$c^* = 150$$

$$d^* = 4800$$

$$e^* = 380$$

For a Triangular distribution, the density function and the cumulative probability distribution are:

$$f(x) = \begin{cases} \frac{2 \cdot (x - L)}{(H - L)(M - L)} & L \leq x \leq M \\ \frac{2 \cdot (H - x)}{(H - L)(H - M)} & M \leq x \leq H \end{cases}$$

$$F(x) = \begin{cases} \frac{(x-L)^2}{(H-L)(M-L)} & L \leq x \leq M \\ 1 - \frac{(H-x)^2}{(H-L)(H-M)} & M \leq x \leq H \end{cases}$$

For material A, assuming Triangular distributed and thus:

$$f_A(a^*) = \frac{2 \cdot (H - a^*)}{(H-L)(H-M)} = 0.002$$

$$F_A(a^*) = 1 - \frac{(H - a^*)^2}{(H-L)(H-M)} = 0.45$$

Thus Equations 5.40 and 5.42 yield:

$$\sigma_A^N = \frac{\phi\{\Phi^{-1}[F_A(a^*)]\}}{f_A(a^*)} = 197.9$$

$$\mu_A^N = a^* - \sigma_A^N \Phi^{-1}[F_A(a^*)] = 2274.9$$

Same can be done to material B, C, D and E. Then, according to Equation 5.44:

$$\beta = \frac{a_0 + \sum_i a_i \mu_{X_i}^N}{\sqrt{\sum_i (a_i \sigma_{X_i}^N)^2}} = \frac{\mu_A^N + \mu_B^N + \mu_C^N + \mu_D^N + \mu_E^N - 8500}{\sqrt{(\sigma_A^N)^2 + (\sigma_B^N)^2 + (\sigma_C^N)^2 + (\sigma_D^N)^2 + (\sigma_E^N)^2}} = -1.6684$$

The direction cosine of A:

$$\alpha_A^* = \frac{\sigma_A^N}{\sqrt{(\sigma_A^N)^2 + (\sigma_B^N)^2 + (\sigma_C^N)^2 + (\sigma_D^N)^2 + (\sigma_E^N)^2}} = 0.4453$$

Thus the failure point is:

$$a^* = \mu_A^N - \alpha_A^* \beta \sigma_A^N = 2425.2$$

The results of iterations are summarised in Table 5.4:

Based on the results of the above 4<sup>th</sup> iteration:

$$p_F = 1 - \Phi(\beta) = 1 - \Phi(-1.3682) = 0.9144$$

Using the Monte Carlo simulation, the summary of the results are shown in Table 5.5.

**Interpreting the results:** The result of FOSM shows that there is only 91.44% chance that the total cost will less than £8500.

**Table 5.4 Results of the iterations of FOSM for Example 5.2**

<b>Iteration No.</b>	<b>Assumed Failure Point</b>	$\sigma^N$	$\mu^N$	$\alpha^*$	$\beta$
1	2250	197.9	2274.9	0.4553	-1.6684
	75	13.1	79.8	0.0300	
	150	13.4	139.8	0.0308	
	4800	386.3	4897.9	0.8888	
	380	13.7	382.5	0.0316	
2	2425	235.8	2270.1	0.4676	-1.3680
	80	15.9	80.3	0.0315	
	140	20.4	136.3	0.0404	
	5470	444.7	4940.8	0.8819	
	383	15.2	382.6	0.0301	
3	2421	235.9	2270.1	0.4696	-1.3682
	81	16.1	80.3	0.0320	
	137	21.8	136.1	0.0434	
	5477	442.4	4943.6	0.8807	
	383	15.2	382.6	0.0302	
4	2422	235.9	2270.1	0.4694	-1.3682
	81	16.1	80.3	0.0320	
	137	21.8	136.1	0.0435	
	5477	442.6	4943.3	0.8808	
	383	15.2	382.6	0.0302	

**Table 5.5 Results of MCS for Example 5.2**

<b>Iterations</b>	<b>Failure probability</b>
<b>1,000</b>	0.9244
<b>5,000</b>	0.9242
<b>10,000</b>	0.9239

### 5.3.2.3 Example 5.3 – Linear, correlated Normals

Consider a project whose net cash flow streams,  $F_n$ , are as follows in Table 5.6. In addition, the correlation coefficients among the  $F_n$ 's are known to be  $\rho_{01} = \rho_{02} = \rho_{03} = 0.5$  and  $\rho_{12} = \rho_{23} = \rho_{13} = 1$ . Assuming the cash flow streams are Normally distributed and the interest,  $i = 10\%$ .

**Table 5.6 The net cash flow streams of Example 5.3**

Year ( $n$ )	$E(F_n)$	$Var(F_n)$
0	-10	9
1	3	4
2	8	16
3	10	25

**Risk analysis objective:** To find out the probability of the net present value being negative

Using the FOSM method:

The performance function:

$$PV = F_0 + \frac{F_1}{1+i} + \frac{F_2}{(1+i)^2} + \frac{F_3}{(1+i)^3}$$

$$\beta = \frac{\mu_{PV}}{\sigma_{PV}} = \frac{a_0 + \sum_{i=1}^n a_i \mu_{X_i}}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n a_i a_j \rho_{ij} \sigma_{X_i} \sigma_{X_j}}} = 0.640329$$

$$p_F = 1 - \Phi(\beta) = 1 - \Phi(0.640329) = 0.260979$$

Using the Monte Carlo simulation, the summary of the results are shown in Table 5.7.

**Table 5.7 Results of MCS for Example 5.3**

Iterations	Failure probability
1,000	0.2569
5,000	0.2565
10,000	0.2595

**Interpreting the results:** The result of FOSM gives a quantitative answer that the failure probability is only 26.1%. It means that the chance of PV being negative is small.

#### 5.3.2.4 Example 5.4 – Linear, correlated non-Normals

Consider the following investment cash flows over a 2-year life (Table 5.8). Assuming that  $F_1$  and  $F_2$  are Beta distributed and partially correlated with  $\rho_{12} = 0.3$ , also  $i = 10\%$ .

**Table 5.8 The cash flows of Example 5.4**

	$\mu$	$\sigma$	<i>Min.(a)</i>	<i>Max.(b)</i>
$F_0$	-500	0		
$F_1$	200	50	100	300
$F_2$	500	50	300	650

**Risk analysis objective:** To find out the probability of the net present value being negative.

Using the FOSM method:

The performance function:

$$PV = g(X) = F_0 + \frac{F_1}{1+i} + \frac{F_2}{(1+i)^2}$$

For both  $F_1$  and  $F_2$ , the parameters of the Beta distribution are

$$\mu_X = a + \frac{q}{q+r} \cdot (b-a)$$

$$\sigma_X^2 = \frac{q \cdot r}{(q+r)^2 \cdot (q+r+1)} \cdot (b-a)^2$$

Thus:

$$q_{F_1} = r_{F_1} = 1.5$$

$$q_{F_2} = 6.286, \quad r_{F_2} = 4.714$$

And:

$$f_x(x) = \frac{1}{B(q,r)} \cdot \frac{(x-a)^{q-1} \cdot (b-x)^{r-1}}{(b-a)^{q+r-1}} \quad a \leq x \leq b$$

$$= 0 \quad \text{elsewhere}$$

$$F_x(x) = \int_a^x f_x(u) du \quad a \leq x \leq b$$

$$= 0 \quad \text{elsewhere}$$

The correlation matrix can be expressed as:

$$[C'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.3 \\ 0 & 0.3 & 1 \end{bmatrix}$$

The diagonal matrix of eigenvalues and orthogonal transformation matrix are:

$$[\lambda] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.3 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -0.707 & -0.707 \\ 0 & -0.707 & 0.707 \end{bmatrix}$$

For the first iteration, assume

$$f_{F_1}^* = 200$$

$$f_{F_2}^* = 500$$

For cash flow  $F_1$

$$f_{F_1}(f_{F_1}^*) = \frac{1}{B(q,r)} \cdot \frac{(x-a)^{q-1} \cdot (b-x)^{r-1}}{(b-a)^{q+r-1}} = 0.006366$$

$$F_{F_1}(f_{F_1}^*) = \int_a^b f_{F_1}(u) du = 0.5$$

Thus Equations 5.40 and 5.42 yield:

$$\sigma_{F_1}^N = \frac{\phi\{\Phi^{-1}[F_{F_1}(f_{F_1}^*)]\}}{f_{F_1}(f_{F_1}^*)} = 62.677$$

$$\mu_{F_1}^N = f_{F_1}^* - \sigma_{F_1}^N \Phi^{-1}[F_{F_1}(f_{F_1}^*)] = 200$$

Same can be done for  $F_2$ .

According to Equations 5.51 and 5.52, the original variates can be obtained:

$$X = [\sigma_X^N]TY + \mu_X^N$$

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 62.677 & 0 \\ 0 & 0 & 53.432 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.707 & -0.707 \\ 0 & -0.707 & 0.707 \end{pmatrix} \cdot \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} + \begin{pmatrix} -500 \\ 200 \\ 501.603 \end{pmatrix}$$

Expand the above matrix:

$$X = \begin{bmatrix} -500 \\ -44.305 \cdot Y_2 - 44.305 \cdot Y_3 + 200 \\ -37.776 \cdot Y_2 + 37.776 \cdot Y_3 + 501.603 \end{bmatrix}$$

The limit-state equation then becomes

$$g(X) = 96.124 - 71.476 \cdot Y_2 - 9.070 \cdot Y_3 = 0$$

Since the variables  $Y_1$ ,  $Y_2$  and  $Y_3$  are uncorrelated, the safety index

$$\beta = \frac{96.124}{\sqrt{(-71.476 \cdot \sqrt{1.3})^2 + (-9.070 \cdot \sqrt{0.7})^2}} = 1.174$$

The partial derivatives are

$$\left( \frac{\partial g}{\partial Y_2'} \right)_* = -71.476$$



$$\left( \frac{\partial g}{\partial Y_3'} \right)_* = -9.070$$

The direction cosines of Equation 5.27b, therefore are:

$$\alpha_{Y_2}^* = \frac{-71.476 \cdot \sqrt{1.3}}{\sqrt{(-71.476 \cdot \sqrt{1.3})^2 + (-9.070 \cdot \sqrt{0.7})^2}} = -0.996$$

$$\alpha_{Y_3}^* = \frac{-9.070 \cdot \sqrt{0.7}}{\sqrt{(-71.476 \cdot \sqrt{1.3})^2 + (-9.070 \cdot \sqrt{0.7})^2}} = -0.093$$

Hence, the components of the failure point in the  $Y$  space, are:

$$Y_2^* = -\alpha_{Y_2}^* \beta \sqrt{1.3} = -(-0.996) \cdot \beta \cdot \sqrt{1.3} = 1.333$$

$$Y_3^* = -\alpha_{Y_3}^* \beta \sqrt{0.7} = -(-0.093) \cdot \beta \cdot \sqrt{0.7} = 0.091$$

Whereas in the space of the original variates,

$$f_{F_1}^* = -44.305 \cdot Y_2^* - 44.305 \cdot Y_3^* + 200 = 136.91$$

$$f_{F_2}^* = -37.776 \cdot Y_2^* + 37.776 \cdot Y_3^* + 501.603 = 454.685$$

The results of iterations are summarised in Table 5.9.

Based on the results of the above 4<sup>th</sup> iteration:

$$p_F = 1 - \Phi(\beta) = 1 - \Phi(1.241) = 0.107$$

**Table 5.9 Results of the iterations of FOSM for Example 5.4**

Ite. No	$X_i$	Assume d $x_i^*$	$\sigma_{X_i}^N$	$\mu_{X_i}^N$	$\left(\frac{\partial g}{\partial Y_i}\right)_*$	$\alpha_{Y_i}^*$	New $y_i^*$
1	$F_0$	-500					
	$F_1$	200	62.677	200.000	-71.476	-0.996	$-(-0.996) \cdot \beta \cdot \sqrt{1.3}$
	$F_2$	500	53.432	501.603	-9.070	-0.093	$-(-0.093) \cdot \beta \cdot \sqrt{0.7}$
	$g(X) = 96.124 - 71.476 \cdot Y_2 - 9.070 \cdot Y_3 = 0$ $\beta = 1.174$						
2	$F_0$	-500					
	$F_1$	136.910	42.114	184.962	-58.233	-0.999	$-(-0.999) \cdot \beta \cdot \sqrt{1.3}$
	$F_2$	454.685	53.372	501.148	4.103	0.052	$-0.052 \cdot \beta \cdot \sqrt{0.7}$
	$g(X) = 82.078 - 58.233 \cdot Y_2 - 4.103 \cdot Y_3 = 0$ $\beta = 1.234$						
3	$F_0$	-500					
	$F_1$	144.707	47.134	190.238	-61.110	-1.000	$-(-1) \cdot \beta \cdot \sqrt{1.3}$
	$F_2$	446.056	52.774	500.527	0.528	0.006	$-0.006 \cdot \beta \cdot \sqrt{0.7}$
	$g(X) = 86.361 - 61.110 \cdot Y_2 + 0.528 \cdot Y_3 = 0$ $\beta = 1.239$						
4	$F_0$	-500					
	$F_1$	143.370	46.312	189.404	-60.539	-1.000	$-(-1) \cdot \beta \cdot \sqrt{1.3}$
	$F_2$	447.561	52.701	500.624	1.013	0.012	$-0.012 \cdot \beta \cdot \sqrt{0.7}$
	$g(X) = 85.684 - 60.539 \cdot Y_2 + 1.013 \cdot Y_3 = 0$ $\beta = 1.241$						

Using the Monte Carlo simulation, the summary of the results are shown in Table 5.10.

**Table 5.10 Results of MCS for Example 5.4**

Iterations	Failure probability
1,000	0.0910
5,000	0.0972
10,000	0.0965

**Interpreting the results:** The result of FOSM gives a quantitative answer that the failure probability is only 10.7%. It means that the chance of PV being negative is small.

#### 5.3.2.5 Example 5.5 – Non-linear, uncorrelated Normals

Suppose that an uncertain lump-sum return  $F$  is expected shortly after termination of a project. Because of current uncertain market conditions, the earning interest rate ( $i$ ) seems to fluctuate for the foreseeable future. It is believed, however, that both the lump sum and the interest rate are Normally distributed but each with a unique range:

$F$       Normal (150, 30)

$i$       Normal (10%, 2%)

Assuming that this uncertain lump sum is to be reinvested at an interest rate  $i$  over the next 3 years. The future worth would be:

$$Z = FV(i) = F(1 + i)^3 = FY^3$$

where  $Y = 1 + i$ . Assume that the  $F$  and  $V(i)$  are uncorrelated.

**Risk analysis objective:** If the target of the future worth is £200, find the probability that the investment does not meet the target.

Using the FOSM method:

The performance function:  $g(x) = FY^3 - 200$

The partial derivatives are:

$$\left( \frac{\partial g}{\partial F'} \right) = \sigma_F \cdot Y^3$$

$$\left(\frac{\partial g}{\partial Y'}\right) = F \cdot 3 \cdot Y^2 \cdot \sigma_i$$

For the first iteration, assume:  $f^* = 150, i^* = 10\%$

Then

$$\left(\frac{\partial g}{\partial F'}\right)_* = \sigma_F \cdot Y^3 = 30 \cdot (1 + 0.01)^3 = 39.93$$

$$\left(\frac{\partial g}{\partial Y'}\right)_* = F \cdot 3 \cdot Y^2 \cdot \sigma_i = 150 \cdot 3 \cdot (1 + 0.01)^2 \cdot 0.02 = 10.89$$

The direction cosines of Equation 5.26b, therefore, are:

$$\alpha_{F'}^* = \frac{\left(\frac{\partial g}{\partial X'_i}\right)_*}{\sqrt{\sum_i \left(\frac{\partial g}{\partial X'_i}\right)_*^2}} = \frac{39.93}{\sqrt{39.93^2 + 10.89^2}} = 0.965$$

$$\alpha_{Y'}^* = \frac{10.89}{\sqrt{39.93^2 + 10.89^2}} = 0.263$$

Hence, the components of the failure point are:

$$f^* = 150 - \alpha_{F'}^* \cdot \beta \cdot 30 = 150 - 0.965 \cdot \beta \cdot 30$$

$$y^* = 1.1 - \alpha_{Y'}^* \cdot \beta \cdot 0.02 = 1.1 - 0.263 \cdot \beta \cdot 0.02$$

Substituting these into the limit-state equation:  $f^* y^{*3} - 200 = 0$  yields the following equation:

$$(150 - 0.965 \cdot \beta \cdot 30)(1.1 - 0.263 \cdot \beta \cdot 0.02)^3 - 200 = 0$$

From which the solution is obtained

$$\beta = -0.00846$$

The revised failure point then becomes

$$f^* = 150 - 0.965 \cdot \beta \cdot 30 = 150.2447$$

$$y^* = 1.1 - 0.263 \cdot \beta \cdot 0.02 = 1.100044$$

Repeating the procedure for the results are summarised in Table 5.11.

**Table 5.11 Results of the iterations of FOSM for Example 5.5**

Ite. No	$X_i$	Assumed $x_i^*$	$\left(\frac{\partial g}{\partial X_i'}\right)_*$	$\alpha_{x_i}^*$	New $x_i^*$
1	$F$	150	39.93	0.965	$150 - 0.965 \cdot \beta \cdot 30$
	$Y$	1.1	10.89	0.263	$1.1 - 0.263 \cdot \beta \cdot 0.02$
	$\beta = -0.00846$				
2	$F$	150.2447	39.93	0.965	$150 - 0.965 \cdot \beta \cdot 30$
	$Y$	1.100044	10.91	0.264	$1.1 - 0.264 \cdot \beta \cdot 0.02$
	$\beta = -0.00845$				
3	$F$	150.2447	39.93	0.965	$150 - 0.965 \cdot \beta \cdot 30$
	$Y$	1.100045	10.91	0.264	$1.1 - 0.264 \cdot \beta \cdot 0.02$
	$\beta = -0.00845$				

Based on the results of the above 3<sup>th</sup> iteration:

$$p_F = 1 - \Phi(\beta) = 1 - \Phi(-0.00845) = 0.5034$$

Using the Monte Carlo simulation, the summary of the results are shown in Table 5.12.

**Table 5.12 Results of MCS for Example 5.5**

Iterations	Failure probability
1,000	0.5439
5,000	0.5108
10,000	0.5081

**Interpreting the results:** The result of FOSM gives a quantitative answer that the failure probability is 50.34%. It means that the investment has half of the chance to meet the target.

### 5.3.2.6 Example 5.6 – Non-linear, uncorrelated non-Normals

Suppose the random variables in Example 5.5 are non-Normal, with the following distributions:

$F$  Uniform (120, 180)

$i$  Uniform (8%, 12%)

Assume that the  $F$  and  $V(i)$  are uncorrelated.

**Risk analysis objective:** If the target of the future worth is also £200, find the probability that the investment does not meet the target.

Using the FOSM method:

The performance function:  $g(x) = FY^3 - 200$

where  $Y = 1 + i$ .

The partial derivatives are:

$$\left( \frac{\partial g}{\partial F'} \right) = \sigma_F \cdot Y^3$$

$$\left( \frac{\partial g}{\partial Y'} \right) = F \cdot 3 \cdot Y^2 \cdot \sigma_i$$

For the Uniform distribution, the probability density function and cumulative probability distribution are:

$$f_x(x) = \frac{1}{b-a} \quad (a < x < b)$$
$$= 0 \quad (\text{elsewhere})$$

$$F_x(x) = 0 \quad (x < a)$$
$$= \frac{x-a}{b-a} \quad (a \leq x < b)$$
$$= 1 \quad (x \geq b)$$

For the first iteration, assume:  $f^* = 150$ ,  $i^* = 10\%$

According to Equations 5.40 and 5.42,

$$\sigma_F^N = \frac{\phi\{\Phi^{-1}[F_F(f^*)]\}}{f_F(f^*)} = 23.9365$$

$$\mu_F^N = 150 - \sigma_F^N \Phi^{-1}[F_F(f^*)] = 150$$

Same as for Y:

$$\sigma_Y^N = \frac{\phi\{\Phi^{-1}[F_Y(y^*)]\}}{f_Y(y^*)} = 0.015958$$

$$\mu_Y^N = 1.1 - \sigma_Y^N \Phi^{-1}[F_Y(y^*)] = 1.1$$

The partial derivatives are:

$$\left(\frac{\partial g}{\partial F'}\right)_* = \sigma_F^N \cdot Y^3 = 23.9365 \cdot 1.1^3 = 31.85949$$

$$\left(\frac{\partial g}{\partial Y'}\right)_* = F \cdot 3 \cdot Y^2 \cdot \sigma_Y^N = 150 \cdot 3 \cdot 1.1^2 \cdot 0.015958 = 8.688953$$

The direction cosines of Equation 5.26b, therefore, are:

$$\alpha_{F'}^* = \frac{\left(\frac{\partial g}{\partial X'_i}\right)_*}{\sqrt{\sum_i \left(\frac{\partial g}{\partial X'_i}\right)_*^2}} = \frac{31.85949}{\sqrt{31.86^2 + 8.689^2}} = 0.965$$

$$\alpha_{Y'}^* = \frac{8.689}{\sqrt{31.86^2 + 8.689^2}} = 0.263$$

Hence, the components of the failure point are:

$$f^* = 150 - \alpha_{F'}^* \cdot \beta \cdot \sigma_F^N = 150 - 0.965 \cdot \beta \cdot 23.9365$$

$$y^* = 1.1 - \alpha_{Y'}^* \cdot \beta \cdot 0.015958 = 1.1 - 0.263 \cdot \beta \cdot 0.015958$$

Substituting these into the limit-state equation:  $f^* y^{*3} - 200 = 0$  yields the following equation:

$$(150 - 0.965 \cdot \beta \cdot 23.9365)(1.1 - 0.263 \cdot \beta \cdot 0.015958)^3 - 200 = 0$$

From which the solution is obtained

$$\beta = -0.0106$$

The revised failure point then becomes:

$$f^* = 150 - 0.965 \cdot \beta \cdot 30 = 150.2447$$

$$y^* = 1.1 - 0.263 \cdot \beta \cdot 0.02 = 1.100044$$

Repeating the procedure for the results are summarised in Table 5.13.

**Table 5.13 Results of the iterations of FOSM for Example 5.6**

Ite. No	$X_i$	Assume $d x_i^*$	$\sigma_{X_i}^N$	$\mu_{X_i}^N$	$\left( \frac{\partial g}{\partial X_i} \right)_*$	$\alpha_{X_i}^*$	New $x_i^*$
1	$F$	150	23.9365	150	31.859	0.965	$150 - 0.965 \cdot \beta \cdot 23.937$
	$Y$	1.1	0.01596	1.1	8.6890	0.263	$1.1 - 0.263 \cdot \beta \cdot 0.01596$
	$\beta = -0.0106$						
2	$F$	150.245	23.9353	150	31.862	0.965	$150 - 0.965 \cdot \beta \cdot 23.935$
	$Y$	1.10004	0.01596	1.1	8.7038	0.264	$1.1 - 0.264 \cdot \beta \cdot 0.01596$
	$\beta = -0.0106$						
3	$F$	150.245	23.9353	150	31.862	0.965	$150 - 0.965 \cdot \beta \cdot 23.935$
	$Y$	1.10005	0.01596	1.1	8.7038	0.264	$1.1 - 0.264 \cdot \beta \cdot 0.01596$
	$\beta = -0.0106$						

Based on the results of the above 3<sup>th</sup> iteration:

$$p_F = 1 - \Phi(\beta) = 1 - \Phi(-0.0106) = 0.5042$$

Using the Monte Carlo simulation, the summary of the results are shown in Table 5.14.

**Table 5.14 Results of MCS for Example 5.6**

Iterations	Failure probability
1,000	0.5383
5,000	0.5095
10,000	0.5081



**Interpreting the results:** The result of FOSM gives a quantitative answer that the failure probability is 50.42%. It means that the investment has half of the chance to meet the target.

### 5.3.2.7 Example 5.7 – Non-linear, correlated Normals

Consider the break-even equation of a property development

$$Z = (M - C) \cdot V - K$$

where M is the unit sales price, C is unit variable cost, V is sales volume, K is a fixed cost, and Z is profit realised. Assuming that K is known with certainty ( $K = £2500 \times 10^3$ ) but M, C and V are Normally distributed dependent random variables with the following statistics (within the parentheses are mean and standard deviation respectively):

M	Normal ( $100 \times 10^3$ , 7000)	$\rho_{MC}$	0.5
C	Normal ( $50 \times 10^3$ , 3500)	$\rho_{MV}$	0.3
V	Normal (100, 10)	$\rho_{CV}$	0.1

**Risk analysis objective:** The target profit is set to  $£2000 \times 10^3$ , find the probability of the target profit not being achieved.

Using the FOSM method:

The performance function:

$$\begin{aligned} g(X) &= Z - 2000 \cdot 10^3 \\ &= (M - C) \cdot V - K - 2000 \cdot 10^3 \\ &= (M - C) \cdot V - 4500 \cdot 10^3 \end{aligned}$$

with the notation  $X = \{M, C, V\}$ , the pertinent correlation matrix, therefore, is:

$$[C'] = \begin{bmatrix} 1 & 0.5 & 0.3 \\ 0.5 & 1 & 0.1 \\ 0.3 & 0.1 & 1 \end{bmatrix}$$

The diagonal matrix of eigenvalues and orthogonal transformation matrix are:

$$[\lambda] = \begin{bmatrix} 0.457 & 0 & 0 \\ 0 & 0.912 & 0 \\ 0 & 0 & 1.631 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.73 & -0.084 & 0.678 \\ -0.619 & -0.502 & 0.604 \\ -0.289 & 0.861 & 0.418 \end{bmatrix}$$

Thus, the transformed variates  $Y$  of Equation 5.51 are:

$$Y = T'X' \text{ and } X' = TY$$

Also,

$$X = [\sigma_x]TY + \mu_x$$

$$X = \begin{pmatrix} 7000 & 0 & 0 \\ 0 & 3500 & 0 \\ 0 & 0 & 10 \end{pmatrix} \cdot \begin{pmatrix} 0.73 & -0.084 & 0.678 \\ -0.619 & -0.502 & 0.604 \\ -0.289 & 0.861 & 0.418 \end{pmatrix} \cdot \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} + \begin{pmatrix} 100 \times 10^3 \\ 50 \times 10^3 \\ 100 \end{pmatrix}$$

Expand the above matrix:

$$X = \begin{bmatrix} 5110.0 \cdot Y_1 - 588.0 \cdot Y_2 + 4746.0 \cdot Y_3 + 100000 \\ -2166.5 \cdot Y_1 - 1757.0 \cdot Y_2 + 2114.0 \cdot Y_3 + 50000 \\ -2.89 \cdot Y_1 + 8.61 \cdot Y_2 + 4.18 \cdot Y_3 + 100 \end{bmatrix}$$

The limit-state equation then becomes

$$\begin{aligned} g(X) &= -21029.085 \cdot Y_1^2 + 59272.255 \cdot Y_1 \cdot Y_2 + 22809.29 \cdot Y_1 \cdot Y_3 + 583150.0 \cdot Y_1 \\ &\quad + 10065.09 \cdot Y_2^2 + 27547.94 \cdot Y_2 \cdot Y_3 + 547400.0 \cdot Y_2 + \\ &\quad + 11001.76 \cdot Y_3^2 + 472200.0 \cdot Y_3 + 500000 \\ &= 0 \end{aligned}$$

The partial derivatives are

$$\frac{\partial g}{\partial Y_1} = -42058.17 \cdot Y_1 + 59272.255 \cdot Y_2 + 22809.29 \cdot Y_3 + 583150.0$$

$$\frac{\partial g}{\partial Y_2} = 59272.255 \cdot Y_1 + 20130.18 \cdot Y_2 + 27547.94 \cdot Y_3 + 547400.0$$

$$\frac{\partial g}{\partial Y_3} = 22809.29 \cdot Y_1 + 27547.94 \cdot Y_2 + 22003.52 \cdot Y_3 + 472200.0$$

For the first iteration, assume:

$$x_1^* = \bar{M} = 100 \times 10^3$$

$$x_2^* = \bar{C} = 50 \times 10^3$$

$$x_3^* = \bar{V} = 100$$

Since the corresponding mean values of  $Y$  are zero,

$$\left( \frac{\partial g}{\partial Y_1} \right)_* = 583150.0$$

$$\left( \frac{\partial g}{\partial Y_2} \right)_* = 547400.0$$

$$\left( \frac{\partial g}{\partial Y_3} \right)_* = 472200.0$$

The direction cosines of Equation 5.26b, therefore are:

$$\alpha_{Y_1}^* = \frac{583150 \cdot \sqrt{0.457}}{\sqrt{(583150 \cdot \sqrt{0.457})^2 + (547400 \cdot \sqrt{0.912})^2 + (472200 \cdot \sqrt{1.631})^2}} = 0.443$$

$$\alpha_{Y_2}^* = \frac{547400 \cdot \sqrt{0.912}}{\sqrt{(583150 \cdot \sqrt{0.457})^2 + (547400 \cdot \sqrt{0.912})^2 + (472200 \cdot \sqrt{1.631})^2}} = 0.587$$

$$\alpha_{Y_3}^* = \frac{472200 \cdot \sqrt{1.631}}{\sqrt{(583150 \cdot \sqrt{0.457})^2 + (547400 \cdot \sqrt{0.912})^2 + (472200 \cdot \sqrt{1.631})^2}} = 0.677$$

Hence, the components of the failure point in the  $Y$  space, are:

$$Y_1^* = -\alpha_{Y_1}^* \beta \sqrt{0.457} = -0.443 \cdot \beta \cdot \sqrt{0.457}$$

$$Y_2^* = -\alpha_{Y_2}^* \beta \sqrt{0.912} = -0.587 \cdot \beta \cdot \sqrt{0.912}$$

$$Y_3^* = -\alpha_{Y_3}^* \beta \sqrt{1.631} = -0.677 \cdot \beta \cdot \sqrt{1.631}$$

Substituting these into the failure equation,  $g(X)=0$ , we obtain:

$$\beta = 0.576$$

Thus the new components of the failure point in the  $Y$  space, are:

$$Y_1^* = -\alpha_{Y_1}^* \beta \sqrt{0.457} = -0.443 \cdot \beta \cdot \sqrt{0.457} = -0.172$$

$$Y_2^* = -\alpha_{Y_2}^* \beta \sqrt{0.912} = -0.587 \cdot \beta \cdot \sqrt{0.912} = -0.323$$

$$Y_3^* = -\alpha_{Y_3}^* \beta \sqrt{1.631} = -0.677 \cdot \beta \cdot \sqrt{1.631} = -0.498$$

The results of iterations are summarised as follows in Table 5.15:

**Table 5.15 Results of the iterations of FOSM for Example 5.7**

Ite. No	Variable $Y_i$	Assumed $y_i^*$	$\left( \frac{\partial g}{\partial Y_i} \right)_*$	$\alpha_{Y_i}^*$	New $y_i^*$
1	$Y_1$	0	583150.0	0.443	$-0.443 \cdot \beta \cdot \sqrt{0.457}$
	$Y_2$	0	547400.0	0.587	$-0.587 \cdot \beta \cdot \sqrt{0.912}$
	$Y_3$	0	472200.0	0.677	$-0.677 \cdot \beta \cdot \sqrt{1.631}$
	$\beta = 0.576$				
2	$Y_1$	-0.173	559890.9	0.448	$-0.448 \cdot \beta \cdot \sqrt{0.457}$
	$Y_2$	-0.323	516935.4	0.584	$-0.584 \cdot \beta \cdot \sqrt{0.912}$
	$Y_3$	-0.498	448395.9	0.677	$-0.677 \cdot \beta \cdot \sqrt{1.631}$
	$\beta = 0.576$				
3	$Y_1$	-0.174	560075.0	0.448	$-0.448 \cdot \beta \cdot \sqrt{0.457}$
	$Y_2$	-0.321	516870.5	0.584	$-0.584 \cdot \beta \cdot \sqrt{0.912}$
	$Y_3$	-0.498	448408.1	0.677	$-0.677 \cdot \beta \cdot \sqrt{1.631}$
	$\beta = 0.576$				

Based on the results of the above 3<sup>th</sup> iteration:

$$p_F = 1 - \Phi(\beta) = 1 - \Phi(0.576) = 0.2823$$

Using the Monte Carlo simulation, the summary of the results are shown in Table 5.16.

**Table 5.16 Results of MCS for Example 5.7**

Iterations	Failure probability
<b>1,000</b>	0.2795
<b>5,000</b>	0.2911
<b>10,000</b>	0.2889

**Interpreting the results:** The result of FOSM gives a quantitative answer that the failure probability is 28.23%. It means that the chance of not achieving the target profit is small.

### 5.3.2.8 Example 5.8 – Non-linear, correlated non-Normals

Consider the same problem as Example 5.7, except that the variable are now distributed as follows

M	Log-Normal
C	Log-Normal
V	Normal

Otherwise, the same statistics apply, namely (within the parentheses are mean and standard deviation respectively):

M	Log-Normal ( $100 \times 10^3$ , 7000)	$\rho_{MC}$	0.5
C	Log-Normal ( $50 \times 10^3$ , 3500)	$\rho_{MV}$	0.3
V	N (100, 10)	$\rho_{CV}$	0.1

**Risk analysis objective:** The target profit is also set to  $\text{£}2000 \times 10^3$ , find the probability of the target profit not being achieved.

Using the FOSM method:

The performance function:

$$\begin{aligned}
 g(X) &= Z - 2000 \cdot 10^3 \\
 &= (M - C) \cdot V - K - 2000 \cdot 10^3 \\
 &= (M - C) \cdot V - 4500 \cdot 10^3
 \end{aligned}$$

with the notation  $X = \{M, C, V\}$ , the pertinent correlation matrix, therefore, is:

$$[C'] = \begin{bmatrix} 1 & 0.5 & 0.3 \\ 0.5 & 1 & 0.1 \\ 0.3 & 0.1 & 1 \end{bmatrix}$$

The diagonal matrix of eigenvalues and orthogonal transformation matrix are

$$[\lambda] = \begin{bmatrix} 0.457 & 0 & 0 \\ 0 & 0.912 & 0 \\ 0 & 0 & 1.631 \end{bmatrix}$$

$$T = \begin{bmatrix} 0.73 & -0.084 & 0.678 \\ -0.619 & -0.502 & 0.604 \\ -0.289 & 0.861 & 0.418 \end{bmatrix}$$

Since M and C are Log-Normally distributed, the parameters of the Log-Normal distribution for M are:

$$\zeta_M \approx \frac{\sigma}{\mu} \approx \frac{7000}{100000} \approx 0.07$$

$$\lambda_M = \ln \mu - \frac{1}{2} \zeta^2 = 11.51$$

and

$$F_M(m) = \Phi\left(\frac{\ln m - \lambda_M}{\zeta_M}\right)$$

$$f_M(m) = \frac{1}{m\zeta} \Phi\left(\frac{\ln m - \lambda_M}{\zeta_M}\right)$$

Then, Equations 5.40 and 5.42, respectively, yield:

$$\sigma_M^N = \frac{1}{f(m^*)} \phi\left\{\Phi^{-1}\left[\Phi\left(\frac{\ln m - \lambda_M}{\zeta_M}\right)\right]\right\} = \frac{1}{f(m^*)} \phi\left(\frac{\ln m - \lambda_M}{\zeta_M}\right) = m^* \zeta_M$$

$$\mu_M^N = m^* - \sigma_M^N \Phi^{-1}\left[\Phi\left(\frac{\ln m - \lambda_M}{\zeta_M}\right)\right] = m^* - m^* \zeta_M \left(\frac{\ln m - \lambda_M}{\zeta_M}\right) = m^* (1 - \ln m^* + \lambda_M)$$

and so as C:

$$\zeta_C \approx \frac{\sigma}{\mu} \approx \frac{3500}{50000} \approx 0.07$$

$$\lambda_C = \ln \mu - \frac{1}{2} \zeta^2 = 10.82$$

$$\sigma_C^N = c^* \zeta_C$$

$$\mu_c^N = c^* (1 - \ln c^* + \lambda_c)$$

For the first iteration, assume

$$x_1^* = \overline{M} = 100 \times 10^3$$

$$x_2^* = \overline{C} = 50 \times 10^3$$

$$x_3^* = \overline{V} = 100$$

Thus,

$$\sigma_M^N = m^* \zeta_M = 0.07 \cdot 100 \times 10^3 = 7000$$

$$\mu_M^N = m^* (1 - \ln m^* + \lambda_M) = 100 \times 10^3 \cdot (1 - \ln(100 \times 10^3) + 11.51) = 9.976 \cdot 10^4$$

$$\sigma_c^N = c^* \zeta_c = 3500$$

$$\mu_c^N = c^* (1 - \ln c^* + \lambda_c) = 4.988 \cdot 10^4$$

Thus, the transformed variates  $Y$  of Equation 5.51 are:

$$Y = T' X' \text{ and } X' = TY$$

Also,

$$X = [\sigma_X^N] TY + \mu_X^N$$

$$X = \begin{pmatrix} 7000 & 0 & 0 \\ 0 & 3500 & 0 \\ 0 & 0 & 10 \end{pmatrix} \cdot \begin{pmatrix} 0.73 & -0.084 & 0.678 \\ -0.619 & -0.502 & 0.604 \\ -0.289 & 0.861 & 0.418 \end{pmatrix} \cdot \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} + \begin{pmatrix} 99760 \\ 49880 \\ 100 \end{pmatrix}$$

Expand the above matrix:

$$X = \begin{bmatrix} 5110.0 \cdot Y_1 - 588.0 \cdot Y_2 + 4746.0 \cdot Y_3 + 99760 \\ -2166.5 \cdot Y_1 - 1757.0 \cdot Y_2 + 2114.0 \cdot Y_3 + 49880 \\ -2.89 \cdot Y_1 + 8.61 \cdot Y_2 + 4.18 \cdot Y_3 + 100 \end{bmatrix}$$

The limit-state equation then becomes

$$\begin{aligned} g(X) = & -21029.085 \cdot Y_1^2 + 59272.255 \cdot Y_1 \cdot Y_2 + 22809.29 \cdot Y_1 \cdot Y_3 + 583496.8 \cdot Y_1 \\ & + 10065.09 \cdot Y_2^2 + 27547.94 \cdot Y_2 \cdot Y_3 + 546366.8 \cdot Y_2 + \\ & + 11001.76 \cdot Y_3^2 + 471698.4 \cdot Y_3 + 488000.0 \\ = & 0 \end{aligned}$$

The partial derivatives are:

$$\frac{\partial g}{\partial Y_1} = -42058.17 \cdot Y_1 + 59272.255 \cdot Y_2 + 22809.29 \cdot Y_3 + 583496.8$$

$$\frac{\partial g}{\partial Y_2} = 59272.255 \cdot Y_1 + 20130.18 \cdot Y_2 + 27547.94 \cdot Y_3 + 546366.8$$

$$\frac{\partial g}{\partial Y_3} = 22809.29 \cdot Y_1 + 27547.94 \cdot Y_2 + 22003.52 \cdot Y_3 + 471698.4$$

The reduced variates at the failure point:

$$x_i^{\prime*} = \frac{x_i^* - \mu_{x_i}^N}{\sigma_{x_i}^N}$$

Thus:

$$m^* = \frac{100000 - 9.976 \cdot 10^4}{7000} = 0.034$$

$$c^* = \frac{50000 - 4.988 \cdot 10^4}{3500} = 0.034$$

The transformed variates are:

$$\begin{pmatrix} y_1^* \\ y_2^* \\ y_3^* \end{pmatrix} = \begin{pmatrix} 0.73 & -0.084 & 0.678 \\ -0.619 & -0.502 & 0.604 \\ -0.289 & 0.861 & 0.418 \end{pmatrix} \cdot \begin{pmatrix} 0.034 \\ 0.034 \\ 0 \end{pmatrix} = \begin{pmatrix} 2.1964 \cdot 10^{-2} \\ -3.8114 \cdot 10^{-2} \\ 1.9448 \cdot 10^{-2} \end{pmatrix}$$

$$\left( \frac{\partial g}{\partial Y_1} \right)_* = 5.808 \cdot 10^5$$

$$\left( \frac{\partial g}{\partial Y_2} \right)_* = 5.474 \cdot 10^5$$

$$\left( \frac{\partial g}{\partial Y_3} \right)_* = 4.716 \cdot 10^5$$

The direction cosines of Equation 5.26b, therefore are:

$$\alpha_{y_1}^* = \frac{5.808 \cdot 10^5 \cdot \sqrt{0.457}}{\sqrt{(5.808 \cdot 10^5 \cdot \sqrt{0.457})^2 + (5.474 \cdot 10^5 \cdot \sqrt{0.912})^2 + (4.716 \cdot 10^5 \cdot \sqrt{1.631})^2}} = 0.442$$



$$\alpha_{Y_2}^* = \frac{5.474 \cdot 10^5 \cdot \sqrt{0.912}}{\sqrt{(5.808 \cdot 10^5 \cdot \sqrt{0.457})^2 + (5.474 \cdot 10^5 \cdot \sqrt{0.912})^2 + (4.716 \cdot 10^5 \cdot \sqrt{1.631})^2}} = 0.588$$

$$\alpha_{Y_3}^* = \frac{4.716 \cdot 10^5 \cdot \sqrt{1.631}}{\sqrt{(5.808 \cdot 10^5 \cdot \sqrt{0.457})^2 + (5.474 \cdot 10^5 \cdot \sqrt{0.912})^2 + (4.716 \cdot 10^5 \cdot \sqrt{1.631})^2}} = 0.678$$

Hence, the components of the failure point in the  $Y$  space, are

$$Y_1^* = -\alpha_{Y_1}^* \beta \sqrt{0.457} = -0.442 \cdot \beta \cdot \sqrt{0.457}$$

$$Y_2^* = -\alpha_{Y_2}^* \beta \sqrt{0.912} = -0.588 \cdot \beta \cdot \sqrt{0.912}$$

$$Y_3^* = -\alpha_{Y_3}^* \beta \sqrt{1.631} = -0.678 \cdot \beta \cdot \sqrt{1.631}$$

Substituting these into the failure equation,  $g(X)=0$ , we obtain

$$\beta = 0.562$$

Thus the new components of the failure point in the  $Y$  space, are

$$Y_1^* = -\alpha_{Y_1}^* \beta \sqrt{0.457} = -0.442 \cdot \beta \cdot \sqrt{0.457} = -0.168$$

$$Y_2^* = -\alpha_{Y_2}^* \beta \sqrt{0.912} = -0.588 \cdot \beta \cdot \sqrt{0.912} = -0.316$$

$$Y_3^* = -\alpha_{Y_3}^* \beta \sqrt{1.631} = -0.678 \cdot \beta \cdot \sqrt{1.631} = -0.487$$

The original variables

$$X = \begin{bmatrix} 5110.0 \cdot Y_1 - 588.0 \cdot Y_2 + 4746.0 \cdot Y_3 + 99760 \\ -2166.5 \cdot Y_1 - 1757.0 \cdot Y_2 + 2114.0 \cdot Y_3 + 49880 \\ -2.89 \cdot Y_1 + 8.61 \cdot Y_2 + 4.18 \cdot Y_3 + 100 \end{bmatrix}$$

$$m^* = 5110.0 \cdot y_1^* - 588.0 \cdot y_2^* + 4746.0 \cdot y_3^* + 99760 = 9.678 \cdot 10^4$$

$$c^* = 4.977 \cdot 10^4$$

$$v^* = 95.734$$

The results of iterations are summarised in Table 5.17.

**Table 5.17 Results of the iterations of FOSM for Example 5.8**

Ite. No	$X_i$	Assume $d x_i^*$	$\sigma_{X_i}^N$	$\mu_{X_i}^N$	$y_i^*$	$\left(\frac{\partial g}{\partial Y_i}\right)_*$	$\alpha_{Y_i}^*$	New $y_i^*$
1	$M$	100000	7000	99760	0.02196	580800	0.442	$-0.299 \cdot \beta$
	$C$	50000	3500	49880	-0.0381	547400	0.588	$-0.562 \cdot \beta$
	$V$	100	10	100	0.01945	471600	0.678	$-0.866 \cdot \beta$
	$\beta = 0.562$							
2	$M$	96780	6775	99710	-0.6021	590600	0.472	$-0.319 \cdot \beta$
	$C$	49770	3484	49880	0.02556	510600	0.576	$-0.550 \cdot \beta$
	$V$	95.734	10	100	-0.0812	442600	0.668	$-0.853 \cdot \beta$
	$\beta = 0.568$							
3	$M$	96770	6774	99710	-0.5990	590900	0.471	$-0.318 \cdot \beta$
	$C$	49800	3486	49880	0.02712	511300	0.576	$-0.550 \cdot \beta$
	$V$	95.808	10	100	-0.0695	442800	0.667	$-0.852 \cdot \beta$
	$\beta = 0.569$							
4	$M$	96770	6774	99710	-0.5996	590900	0.472	$-0.319 \cdot \beta$
	$C$	49800	3486	49880	0.02651	511200	0.576	$-0.550 \cdot \beta$
	$V$	95.803	10	100	-0.0699	442800	0.668	$-0.853 \cdot \beta$
	$\beta = 0.568$							

Based on the results of the above 4<sup>th</sup> iteration:

$$p_F = 1 - \Phi(\beta) = 1 - \Phi(0.568) = 0.285$$

Using the Monte Carlo simulation, the summary of the results are shown in Table 5.18.

**Table 5.18 Results of MCS for Example 5.8**

Iterations	Failure probability
1,000	0.2829
5,000	0.2928
10,000	0.2926

**Interpreting the results:** The result of FOSM gives a quantitative answer that the failure probability is 28.5%. It means that the chance of not achieving the target profit is small.

### 5.3.3 Practical applications

#### 5.3.3.1 Example 5.9 – Elemental cost analysis

This elemental cost model chosen to illustrate the potential application of FOSM is based on published data by Chau <sup>[121]</sup>. In the paper subjective estimates of the construction costs of the ten major subsystems of electrical services contracts of government clinics in Hong Kong are examined. The data is summarised in Table 5.19.

All of the subsystem costs are extracted from seven experienced estimators in conventional three-point estimates (maximum, most likely and minimum) and the average of these estimates are listed on the right hand side of the table. Since there is no information on the form of distribution for the subsystems, the PERT procedures (Equations 5.68 and 5.69) have been used to assess the means and standard deviations for the subsystems. All of the subsystem costs are then assumed to have a Normal distribution:

$$Mean = \frac{Max. + 4 \times Mos. + Min.}{6} \quad (5.68)$$

$$S.D. = \frac{Max. - Min.}{6} \quad (5.69)$$

**Risk analysis objective:** Based on the subjective estimates of elemental costs, calculating the probability of the total cost does not exceed the expected revenue, which is assumed Normally distributed with mean 900 and standard deviation 100 (Figures are in Hong Kong dollars). The expected revenue maybe uncertain depending on the contractual arrangement between the client and the contractor.

**Table 5.19 Probabilistic estimation of subsystem costs for Example 5.9**

(Figures are in Hong Kong dollars, expressed per square metre of gross floor area.)

Subsystems		Estimator							A.E.
		1	2	3	4	5	6	7	
<b>1. LV switchboard</b>	Max.	143	170	180	190	160	300	250	199
	Mos.	87	124	120	140	100	110	100	112
	Min.	50	55	50	60	60	50	65	56
<b>2. Main and submain distribution</b>	Max.	174	250	270	200	160	110	300	209
	Mos.	111	112	150	130	110	75	120	115
	Min.	56	85	60	80	60	60	70	67
<b>3. General final circuit and equipment</b>	Max.	367	411	540	420	500	600	400	463
	Mos.	297	335	300	340	350	350	320	327
	Min.	185	230	220	210	200	250	250	221
<b>4. Special final circuit and equipment</b>	Max.	43	62	50	80	50	72	70	61
	Mos.	36	44	40	40	40	45	45	41
	Min.	27	30	25	30	28	25	35	29
<b>5. Conduit trunking system for other services</b>	Max.	21	45	28	40	28	30	55	35
	Mos.	15	25	15	30	22	25	25	22
	Min.	11	12	10	15	10	14	11.5	12
<b>6. Power supplies to air conditioning and ventilation</b>	Max.	60	150	130	180	80	80	70	107
	Mos.	41	71	60	60	45	50	40	52
	Min.	12	20	11	20	15	12	20	16
<b>7. Earthing</b>	Max.	5.5	16	8	15	5	10	6	9
	Mos.	3.5	7	5	6	3.3	5.5	4.5	5
	Min.	2	3	2	2	1.5	2	2.5	2
<b>8. Labour for fixing items supplied by</b>	Max.	3.5	3.3	4	4	7.5	6	3.5	5
	Mos.	2.8	3	3.5	3	3	4	2.5	3
	Min.	1.8	2.2	2.4	2	2.5	2	2	2
<b>9. Testing and commissioning</b>	Max.	27	19	18	20	18	30	23	22
	Mos.	16	13	10	10	14	15	10	13
	Min.	3	4	5	5	4.5	5	5	5
<b>10. Equipotential &amp; supplementary bonding</b>	Max.	39	81	50	72	60	45	70	60
	Mos.	31	47	35	40	35	35	30	36
	Min.	22	17	15	15	15	20	20	18

Using the FOSM method:

The performance function:  $g(X) = C_{target} - \sum_{n=1}^{10} C_{subcosts}$

$$\beta = \frac{\mu_g}{\sigma_g} = \frac{900 - \sum_{n=1}^{10} \mu_{subcosts}}{\sqrt{100^2 + \sum_{n=1}^{10} (\sigma_{subcosts})^2}} = 1.302$$

$$P_f = 1 - \Phi(\beta) = 1 - \Phi(1.302) = 0.0965$$

Using the Monte Carlo simulation, the summary of the results are shown in Table 5.20.

**Table 5.20 Results of MCS for Example 5.9**

Iterations	Failure probability
1,000	0.1111
5,000	0.0988
10,000	0.0949

**Interpreting the results:** The result of FOSM gives a quantitative answer that the failure probability is 9.65%. It means that the chance of total cost will exceed the expected revenue is small.

### 5.3.3.2 Example 5.10 – Setting realistic plant hire rate

This example is chosen from a text book, “Management and Investment Decisions Construction Plant” (Harris and McCaffer <sup>[122]</sup>).

An excavator, crawler mounted, 1½ m³ capacity, is purchased new for £46000. Its estimated life is 10 years, with a historical resale value of £4000. Other information:

Fixed overhead	£4000
Road tax and licences	£100
Insurance premium	£200
Consumables	£400
Maintenance	£4600
Total	£9300

The targeted plant hire rate for year 5 is set by the required rate of return on the investment and the inflation being 15% and 10% respectively per annum over 10 years. In reality, both the interest rate of return and inflation rate can not be predicted over time, hence, it has been assumed that both rate are variables Normally distributed with the following statistics per year over ten years (within the parentheses are mean and standard deviation respectively):

Interest rate of return: Normal (15%, 3%)

Inflation rate: Normal (8%, 4%)

**Analysis objective:** Calculating the probability of the plant hire rate for year 5 exceeding the targeted plant hire rate.

The targeted plant hire rate for year 5 can be set as following steps:

1. The cost of capital and depreciation are calculated in Table 5.21, in which  $i_d$  is the inflation rate.

**Table 5.21 Calculation of depreciation over time**

Year ( <i>n</i> )	Index	(£) Replacement price	(£) Accumulated historical depreciation	(£) Accumulated inflated depreciation	(£) Book value
	<i>A</i> $100*(1+i_d)^n$	<i>B</i> $46000*A/100$	<i>C</i> $(46000-4000)/10*n$	<i>D</i> $C*A/100$	<i>E</i> <i>B-D</i>
0	100.0	46000	0	0	46000
1	110.0	50600	4200	4620	45980
2	121.0	55660	8400	10164	45496
3	133.1	61226	12600	16771	44455
4	146.4	67349	16800	24597	42752
5	161.1	74083	21000	33821	40263
6	177.2	81492	25200	44643	36848
7	194.9	89641	29400	57292	32349
8	214.4	98605	33600	72025	26581
9	235.8	108466	37800	89130	19335
10	259.4	119312	42000	108937	10375

2. Interest on finance is calculated as follows:

With inflation at 10% the apparent rate of return must be used in the calculations:

$$(1 + i_a) = (1 + i_r)(1 + i_d)$$

where:  $i_a$  = apparent rate of return,  $i_r$  = real rate of return,  $i_d$  = rate of inflation.

Therefore,

$$(1 + i_a) = (1 + i_r)(1 + i_d) = (1 + 0.15)(1 + 0.1) = 1.265$$

$$i_a = 1.265 - 1 = 0.265 = 26.5\%$$

The capital recovery factor:

$$\frac{i_a(1 + i_a)^n}{(1 + i_a)^n - 1} = \frac{0.265 \times 1.265^{10}}{1.265^{10} - 1} = 0.2929$$

Thus interest on finance using the capital recovery factor:

$$\frac{46000 \times 0.2929 \times 10 - 46000}{10} = £8873$$

3. If inflation continued as shown by the indices then the targeted plant hire rate for year 5 should be:

Depreciation for year 5:	(33821-24597)	£9224
Interest on finance:	(step 2)	£8873
Other items:	(9300*161.1/100)	£14860
Total:		£32957

Using the FOSM method to calculate the probability of the plant hire rate for year 5 exceeding the targeted plant hire rate when both the interest rate of return and inflation rate are considered as variables:

The performance function:

$$g(M) = X + Y + Z - 32957$$

where  $X$  is the depreciation for year 5,  $Y$  is the interest on finance, and  $Z$  is other items.

$$\begin{aligned}
X &= (1+i_d)^5 \times 5 \times 4200 - (1+i_d)^4 \times 4 \times 4200 \\
&= (1+i_d)^5 \times 21000 - (1+i_d)^4 \times 16800
\end{aligned}$$

$$\begin{aligned}
Y &= \frac{46000 \times 10 \times [(1+i_r)(1+i_d) - 1] - 46000}{10} \\
&= 46000 \times [(1+i_r)(1+i_d) - 1] - 4600 \\
&= 46000 \times (1+i_r)(1+i_d) - 50600
\end{aligned}$$

$$Z = 9300 \times (1+i_d)^5$$

Thus the performance function becomes:

$$\begin{aligned}
g(M) &= X + Y + Z - 32957 \\
&= (1+i_d)^5 \times 21000 - (1+i_d)^4 \times 16800 + 46000 \times (1+i_r)(1+i_d) - 50600 + \\
&\quad + 9300 \times (1+i_d)^5 - 32957 \\
&= (1+i_d)^5 \times 21000 - (1+i_d)^4 \times 16800 + 46000 \times (1+i_r)(1+i_d) + 9300 \times (1+i_d)^5 - 83557
\end{aligned}$$

where

$$i_r = \text{Normal}(15\%, 0.03)$$

$$i_d = \text{Normal}(8\%, 0.04)$$

Since the calculation involving powers of the  $i_r$  and  $i_d$ , the performance function is non-linear and an iterative solution is needed.

The partial derivatives are:

$$\begin{aligned}
\left( \frac{\partial g}{\partial i_d} \right)_* &= 5 \times (1+i_d^*)^4 \times 21000 \times \sigma_{i_d} - 4 \times (1+i_d^*)^3 \times 16800 \times \sigma_{i_d} + 46000 \times (1+i_r^*) \times \sigma_{i_d} \\
&\quad + 9300 \times 5 \times (1+i_d^*)^4 \times \sigma_{i_d}
\end{aligned}$$

$$\left( \frac{\partial g}{\partial i_r} \right)_* = 46000 \times (1+i_d^*) \times \sigma_{i_r}$$



For the first iteration, assume that:  $i_d^* = \bar{i}_d = 0.08$ , and  $i_r^* = \bar{i}_r = 0.15$ , then

$$\begin{aligned}\left(\frac{\partial g}{\partial \bar{a}_d'}\right)_* &= 5 \times (1 + 0.08)^4 \times 21000 \times 0.04 - 4 \times (1 + 0.08)^3 \times 16800 \times 0.04 + 46000 \times (1 + 0.15) \times 0.04 \\ &+ 9300 \times 5 \times (1 + 0.08)^4 \times 0.04 \\ &= 6974.457\end{aligned}$$

$$\left(\frac{\partial g}{\partial \bar{a}_r'}\right)_* = 46000 \times (1 + 0.08) \times 0.03 = 1490.4$$

The direction cosines are:

$$\alpha_{i_d}^* = \frac{\left(\frac{\partial g}{\partial \bar{a}_d'}\right)_*}{\sqrt{\left(\frac{\partial g}{\partial \bar{a}_d'}\right)_*^2 + \left(\frac{\partial g}{\partial \bar{a}_r'}\right)_*^2}} = \frac{6974.457}{\sqrt{6974.457^2 + 1490.4^2}} = 0.9779$$

$$\alpha_{i_r}^* = \frac{\left(\frac{\partial g}{\partial \bar{a}_r'}\right)_*}{\sqrt{\left(\frac{\partial g}{\partial \bar{a}_d'}\right)_*^2 + \left(\frac{\partial g}{\partial \bar{a}_r'}\right)_*^2}} = \frac{1490.4}{\sqrt{6974.457^2 + 1490.4^2}} = 0.2090$$

Hence the components of the failure point are:

$$i_d^* = 0.08 + 0.9779 \times 0.04 \times \beta$$

$$i_r^* = 0.15 + 0.2090 \times 0.03 \times \beta$$

Substituting these into the limit-state equation,  $g(M) = 0$ , solve for  $\beta$

$$\beta = -0.6428$$

The revised failure point then becomes:

$$i_d^* = 0.08 + 6974.457 \times 0.04 \times (-0.6428) = 0.105144$$

$$i_r^* = 0.15 + 1490.4 \times 0.03 \times (-0.6428) = 0.15403$$

Repeating the procedure for subsequent iterations, the results are summarised in the following Table 5.22:

**Table 5.22 Results of the iterations of FOSM for Example 5.10**

Iteration No.	Variable	Assumed Failure point	$\left(\frac{\partial g}{\partial M_i}\right)$	$\alpha_{M_i}^*$	New $m_i^*$
1	$i_d$	0.08	6974.457	0.9779	0.105144
	$i_r$	0.15	1490.4	0.2090	0.15403
$\beta = -0.6428$					
2	$i_d$	0.105144	7534.844	0.9801	0.105199
	$i_r$	0.15403	1525.099	0.1984	0.153825
$\beta = -0.64276$					
3	$i_d$	0.105199	7535.73	0.9801	0.1052
	$i_r$	0.153825	1525.175	0.1984	0.153825
$\beta = -0.64276$					

Based on the results of the above 3<sup>th</sup> iteration, therefore, the underlying probability of failure is:

$$P_F = 1 - \Phi(-0.64276) = 0.7398 = 73.98\%$$

Using the Monte Carlo simulation method, the summary results are shown in Table 5.23:

**Table 5.23 Results of MCS for Example 5.10**

Iteration	Probability
1000	0.7578
5000	0.7485
10000	0.7462

**Interpreting the results:** When both the required rate of return and inflation are considered as being uncertain, the probability of exceeding the targeted hire rate for year 5 is 73.98%.

### 5.3.4 Discussion

#### 5.3.4.1 Discussion of individual examples

Example 5.1 is a typical example of selecting a project or alternatives using the FOSM method.

The results from the FOSM for Example 5.1 are exact when the performance function is linear and the variables are Normally distributed. The explanation for this can be referred back to the definition of the reliability index (safety index), which is defined as the minimum distance from the tangent plane of the failure surface to the origin of the reduced variates (Equations 5.10-5.38). It can also be seen from Figure 5.3, that when the performance function is linear,  $g(X_1, X_2) = 0$  is a straight line, thus the minimum distance from the failure surface,  $g(X_1, X_2) = 0$ , to the origin of the reduced variates is unique. This is same for Example 5.9.

Example 5.2 and Example 5.9 represent a typical task in economic analysis, elemental cost analysis, the subsystem costs have been assumed to be Triangular and Normally distributed respectively in the two examples.

Example 5.2 used the three point parameters (minimum, most likely and maximum) estimates and assumed the Triangular distribution for the subsystem costs. This approach (using the three point parameters estimates) is very common when objective data is unavailable and subjective data must be acquired by the use of experienced estimators. (Wilson <sup>[26]</sup>, Raftery <sup>[123]</sup>, Newton <sup>[63]</sup> and Chau <sup>[121]</sup>).

Example 5.2 involves non-Normal distribution and as stated in section 5.2.3.1, for an individual variate, the equivalent Normal distribution for a non-Normal variate may be obtained such that the cumulative probability as well as the probability density ordinate of the equivalent Normal distribution are equal to those of the corresponding non-Normal distribution at the appropriate point  $x_i^*$ , on the failure surface. This can be done by using Equations 5.40 and 5.42. The procedures have been applied to all of examples involving non-Normal distribution (Examples 5.2, 5.4, 5.6, and 5.8).

Although the performance function of Example 5.2 is linear, the mean values and standard deviations required in Equation 5.38 are unknown as these are now functions of the respective failure point values. An iterative solution, using Equations 5.40 and 5.42, is therefore necessary.

The first iteration is assumed using the most likely values. Four iterations were carried out. It can be seen that the third and fourth iterations are identical and thus three iterations are sufficient for the result to be convergent in this case.

Example 5.3 involves correlated variables and despite of this, since the performance function is linear and the variables are Normally distributed, the result of the FOSM from Equation 5.60 is exact. This is because as stated in section 5.2.3.2, when random variables are correlated, the original variates may be transformed to a set of uncorrelated variables. Equation 5.49 indicates that a rotation of the co-ordinates from  $X'$  to  $Y$  and the origin of the  $Y$  axes remains the same as that of the  $X'$  axes.

When the performance function is linear, it can be seen from Figure 5.3 but now the axes  $x'$  have rotated to  $Y$ ,  $g(X_1, X_2) = 0$  is a straight line, thus the minimum distance from the failure surface,  $g(X_1, X_2) = 0$ , to the origin of the reduced variates is still unique.

Alternatively, results of the FOSM of Example 5.3 can be obtained using Equations 5.46-5.59 (as those in Example 5.4), in such one iteration of the algorithm is sufficient.

Examples 5.3 and 5.4 represent another typical economic analysis, net cash flow streams problems including net present value and future worth. Both examples are linear cases as the interest rate is constant.

The algorithm of Examples 5.4 is a combination of Examples 5.2 and 5.3. It can be seen that in the linear cases, equivalent Normal distribution using Equations 5.40 and

5.42 are proceeded first and Equation 5.60 is then used for solving the correlation problem.

The first iteration uses the mean values of the variables and it can be seen that four iterations are sufficient for this example.

Examples 5.5 and 5.6 also represent a typical economic analysis of net cash flow streams as Examples 5.3 and 5.4. The difference is that when the interest and/or cash flows are considered as variables, these examples become non-linear cases.

When the performance functions are non-linear, the limit state equation,  $g(X_1, X_2) = 0$ , will also be non-linear as shown in Figure 5.3 and unlike the linear case, there is no unique distance from the failure surface to the origin of the reduced variates. Thus the results from the FOSM are approximate.

As stated in section 5.2.4, the point  $(x_1^*, x_2^*, \dots, x_n^*)$  on the failure surface with minimum distance to the origin of the reduced variates is the most probable failure point (Shinozuka, 1983). The tangent plane to the failure surface at  $(x_1^*, x_2^*, \dots, x_n^*)$  may then be used to approximate the actual failure surface, and the required reliability index or probability of safety may be evaluated as in the linear case of Section 5.2.3. Depending on whether the exact non-linear failure surface is convex or concave toward the origin, this approximation will be on the safe side or unsafe side, respectively.

In the case of the performance function is non-linear, the pertinent point of tangency on the failure surface is not, a priori, known. Consequently, the determination of the required reliability index would not be as simple as in the linear case (Section 5.2.3), even though linear approximation is invoked. The “minimum” point of tangency on the failure surface may be determined through the iterative algorithm as described in Section 5.2.4.1. These iterative procedures have been applied to the examples with non-linear performance function (Examples 5.5-5.8).

Example 5.6 is similar to Example 5.5, except that both variables are considered to be Uniformly distributed instead of Normally distributed. In these two particular cases, the effect of the form of distribution is small as the failure probabilities are 0.5034 and 0.5042 respectively for Examples 5.5 and 5.6. The reason for this is because the variables of the two examples and their standard deviations are relatively small. This may also be the reason for the accuracy, the results of the FOSM of these examples are deemed to be accurate after only two iterations.

Examples 5.7 and 5.8 are again considered to be the same problems with the only difference being that the variables are Normally and non-Normally (Log-Normal) distributed.

Both examples involve correlated variables and the performance functions are non-linear. Therefore, the iterative solutions have to be used.

From both examples, the third iteration is sufficient for an accurate result and the failure probabilities are 0.2823 and 0.2850 respectively for Examples 5.7 and 5.8. The results again show the effect of the form of the distribution is small in these cases. The reason for this is believed same as in Example 5.5 and 5.6, that the variables of the two examples and their standard deviations are relatively small.

Example 5.10 shows that the FOSM can be applied to a typical problem in construction, that of setting plant hire rate. Traditionally, plant hire rates are set deterministically and a contingency is then used to offset the perceived uncertainty. The example used here shows that the FOSM can provide additional probabilistic information for decision making.

In this example, the interest rate of return and inflation are considered as variables. Since the standard deviation of both rates is small, it can be seen that the results of the second and third iteration are identical and as such the second iteration was sufficient for this example.

### 5.3.4.2 Accuracy of the FOSM

The results from the FOSM are exact when the performance function is linear and the variables are Normally distributed, such as already discussed in Examples 5.1, 5.3 and 5.9.

However, when the variables are non-Normal and/or the performance function is non-linear, the FOSM uses an iterative solution and thus the results are approximate.

For a general performance function, the “correct” probability of failure can be evaluated through large sample of Monte Carlo simulations.

For assessing the accuracy of the FOSM methods, all of the examples were studied also using the Monte Carlo simulations method. The accuracy of the MCS results can be seen in Table 5.24, where the standard error is calculated by:

$$\langle \mu \rangle_{1-\alpha} = \left( \bar{x} - k_{\alpha/2} \frac{\sigma}{\sqrt{n}}; \bar{x} + k_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right) \quad (5.70)$$

where  $(1-\alpha)$  is the specified confidence level,  $k_{\alpha/2} = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$  is the value of the standard Normal variate with cumulative probability level  $\alpha/2$ ,  $\bar{x}$  is the sample mean,  $\sigma$  is the standard deviation and  $n$  is the sample size.

**Table 5.24 The standard error (95 % intervals) for the means in MCS**

Iterations	1000	5000	10000
<b>Example 5.1</b>	-2945.74 ± 146.35	-2950.43 ± 61.806	-2984.37 ± 43.669
<b>Example 5.2</b>	-715.581 ± 29.013	-696.902 ± 12.862	-690.917 ± 9.0206
<b>Example 5.3</b>	6.89553 ± 0.6675	6.85393 ± 0.2964	6.86533 ± 0.2090
<b>Example 5.4</b>	96.0179 ± 4.2774	93.9599 ± 1.9314	95.0143 ± 1.3748
<b>Example 5.5</b>	-3.73617 ± 2.5658	-0.34529 ± 1.1642	-0.52390 ± 0.8165
<b>Example 5.6</b>	-2.35412 ± 1.4731	-0.40026 ± 0.6661	-0.49229 ± 0.4697
<b>Example 5.7</b>	514471 ± 54635	523407 ± 24721	517219 ± 17507
<b>Example 5.8</b>	514621 ± 54575	523375 ± 24800	516957 ± 17549
<b>Example 5.9</b>	145.074 ± 6.9801	148.434 ± 3.2073	148.576 ± 2.2530
<b>Example 5.10</b>	-4844.70 ± 457.59	-4504.15 ± 200.05	-4427.32 ± 140.60

Table 5.24 shows that the results at 1000 iterations of the MCS provide sufficiently accurate results for all of the examples. For a further increase in confidence, results at 10000 iterations are used for comparison.

Table 5.25 shows a comparison of the results from both FOSM and Monte Carlo simulations for all of the examples. The results of the MCS are generated at 10000 iterations and the 95% intervals for the  $p_F$  are calculated using the following equation:

$$\langle p \rangle_{1-\alpha} = \left( \hat{p} - k_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}; \hat{p} + k_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right) \quad (5.71)$$

where  $(1-\alpha)$  is the specified confidence level, and  $\pm k_{\alpha/2}$  is values of the standard Normal variate with cumulative probability levels  $\alpha/2$  and  $(1-\alpha/2)$ , respectively,  $\hat{p}$  is the calculated failure probability,  $n$  is the sample size which in the current case is 10000.

**Table 5.25 Comparison of calculated failure probabilities of FOSM with MCS**

	FOSM	MCS	95% Confidence interval	
			Lower	upper
<b>Example 5.1</b>	0.9099#	0.9073	0.9016	0.9130
<b>Example 5.2</b>	0.9144*	0.9239	0.9187	0.9291
<b>Example 5.3</b>	0.2610#	0.2595	0.2509	0.2681
<b>Example 5.4</b>	0.1070*	0.0965	0.0907	0.1023
<b>Example 5.5</b>	0.5034	0.5081	0.4983	0.5179
<b>Example 5.6</b>	0.5042	0.5081	0.4983	0.5179
<b>Example 5.7</b>	0.2823	0.2889	0.2800	0.2978
<b>Example 5.8</b>	0.2850	0.2926	0.2837	0.3015
<b>Example 5.9</b>	0.0965#	0.0949	0.0892	0.1006
<b>Example 5.10</b>	0.7398	0.7462	0.7377	0.7547

# denotes the result of FOSM is exact.

\* denotes the result of FOSM is not included in the 95% confidence intervals of the MCS.



Table 5.25 shows that only two examples out of ten that the results of FOSM are slightly not included in the 95% confidence intervals of those of MCS. It can be seen that the results of the FOSM are very accurate for the examples studied when compared to MCS. This is believed in the present study, the numbers of the variables in the examples is relatively small as most of the examples have two to five variables. Example 5.9 has ten variables but the distribution of these variables are Normal and the performance function is linear, thus the results are exact. Also in most of the examples studied the standard deviations of the variables are relatively small. Therefore, the approximations tend to be very accurate.

These results (although limited) provide evidence of the validity and accuracy of the FOSM method. Moreover, all the examples examined are typical of those found in practical construction economics.

#### **5.3.4.3 The algorithm of the FOSM**

The methodology of the FOSM has been illustrated in detail through ten examples and from these the complete algorithm of the FOSM can be summarised in Figures 5.7-5.15.

Form Figures 5.8 and 5.12, it can be seen that when the performance function is linear and the variables are Normally distributed, the FOSM is very straight forward - the safety index can be calculated in an equation and the probability of failure can then be found (Examples 5.1, 5.3 and 5.9).

Figures 5.9, 5.11, 5.13 and 5.15 show when the performance function is non-linear, an iterative solution is needed using the algorithm described in section 5.2.4.1 (Examples 5.5, 5.6, 5.7, 5.8 and 5.10).

Figures 5.10, 5.11, 5.14 and 5.15 show when the variables involving non-Normal distribution, equivalent Normal distribution (Equations 5.40 and 5.42) are used and the

calculation of the failure probability then follows the same procedure as that for Normal variates (Examples 5.2, 5.6, 5.4 and 5.8).

Figures 5.12 - 5.15 show when the variables are correlated, the original variates have to be transformed to a set of uncorrelated variables (Examples 5.3, 5.4, 5.7 and 5.8). However, when the performance function is linear and the variables are Normally distributed (Figure 5.12), one iteration of the numerical algorithm is sufficient (Example 5.3).

These algorithms indicate that there are no limits to the number of variables and they can have different probability distribution forms within a performance function.

These Figures also show that the algorithms of the FOSM can be easily written to a computer code.

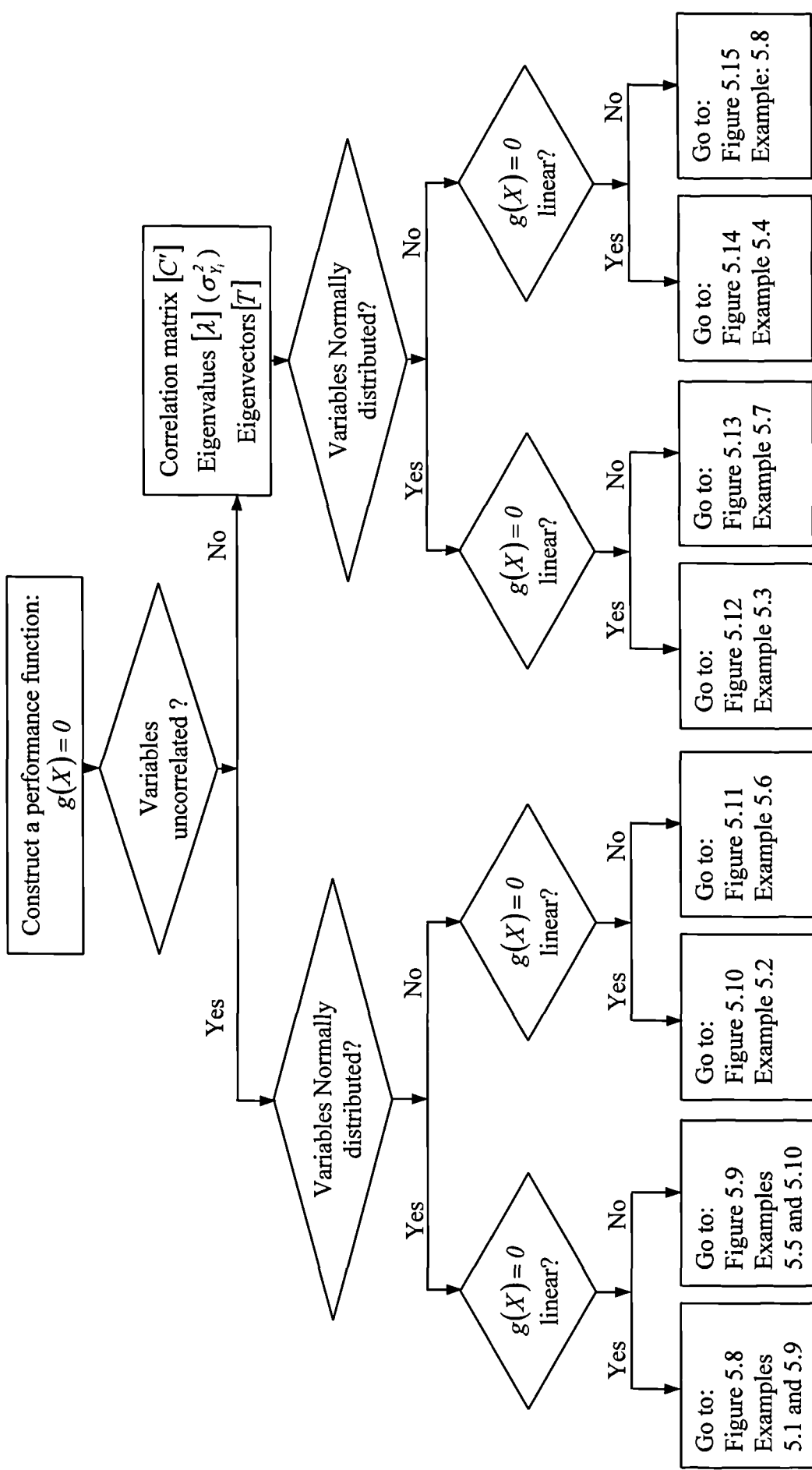
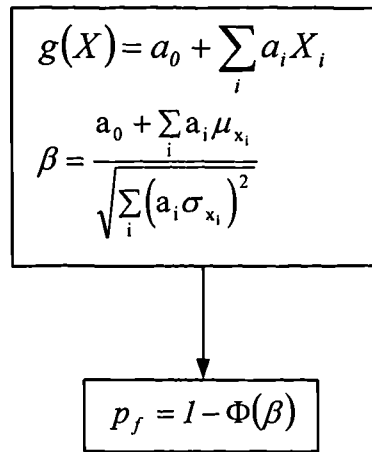
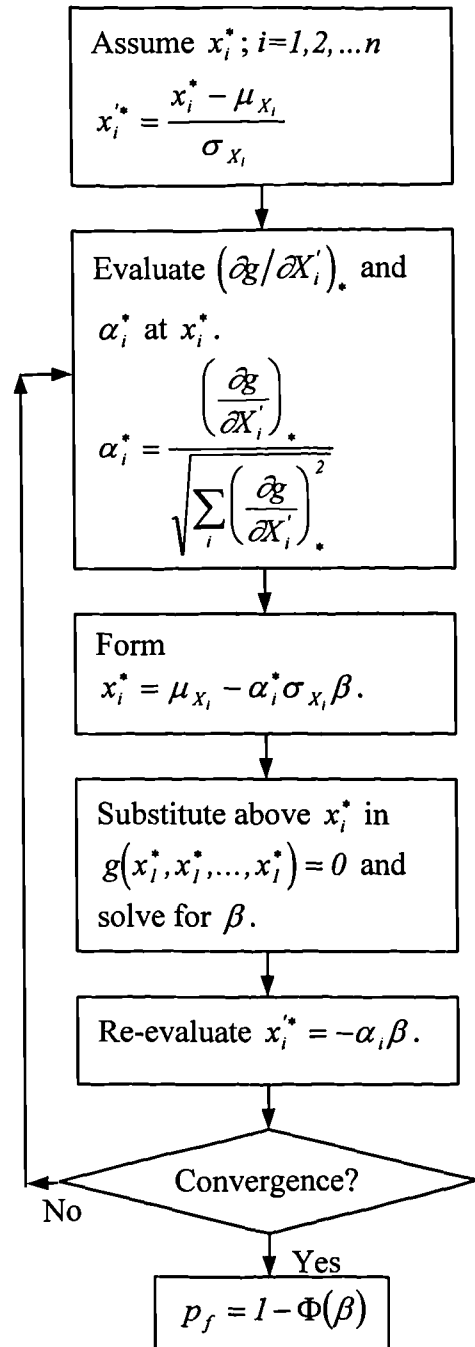


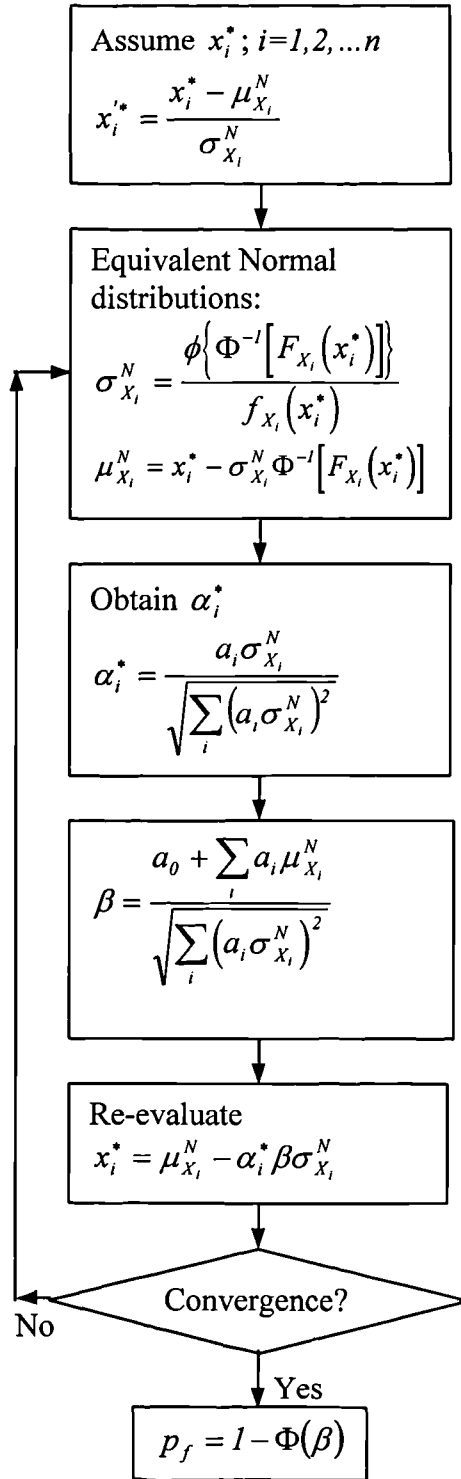
Figure 5.7 Algorithm logic of the FOSM method



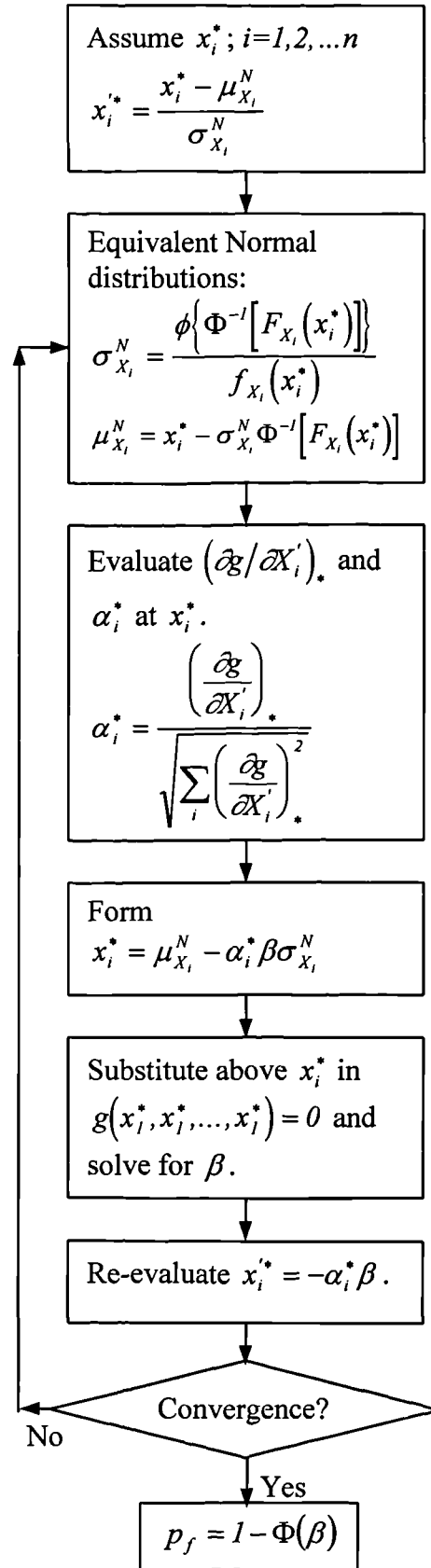
**Figure 5.8 Logic for linear, uncorrelated Normals**



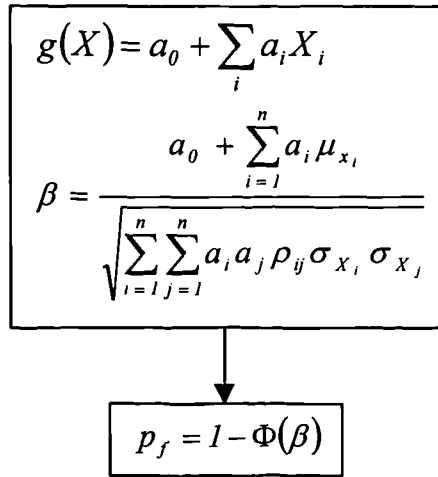
**Figure 5.9 Logic for non-linear, uncorrelated Normals**



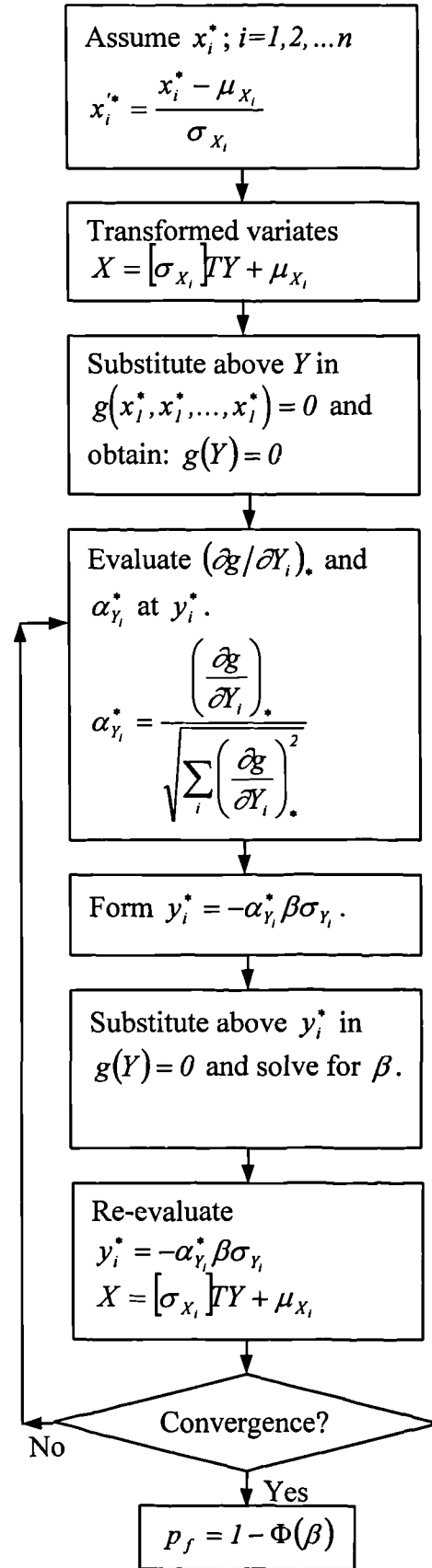
**Figure 5.10 Logic for linear, uncorrelated non-Normals**



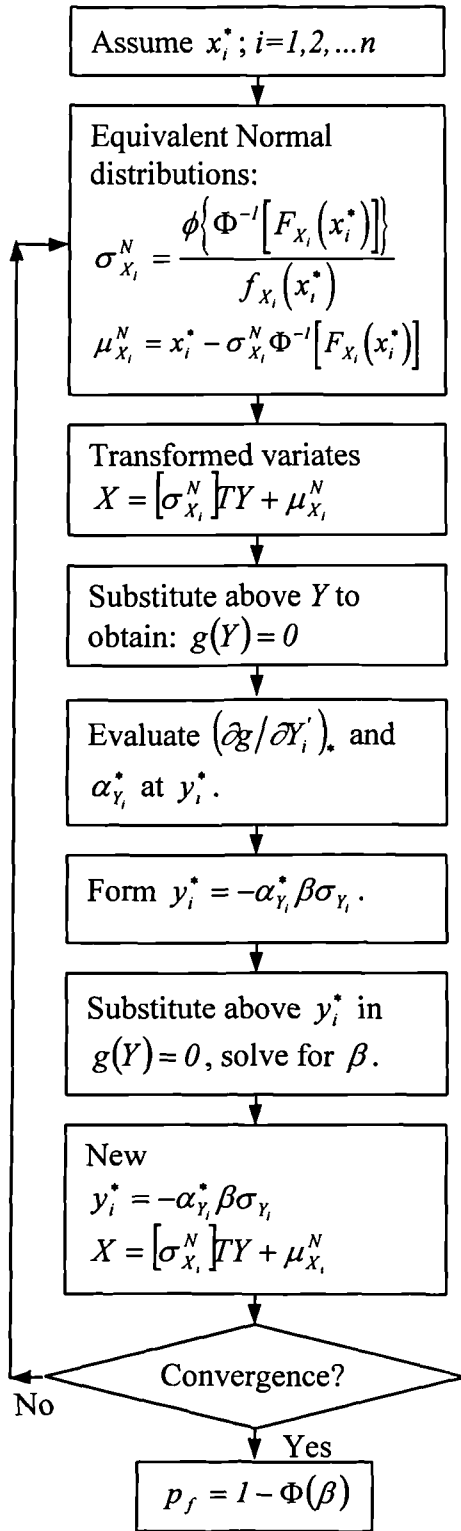
**Figure 5.11 Logic for non-linear, uncorrelated non-Normals**



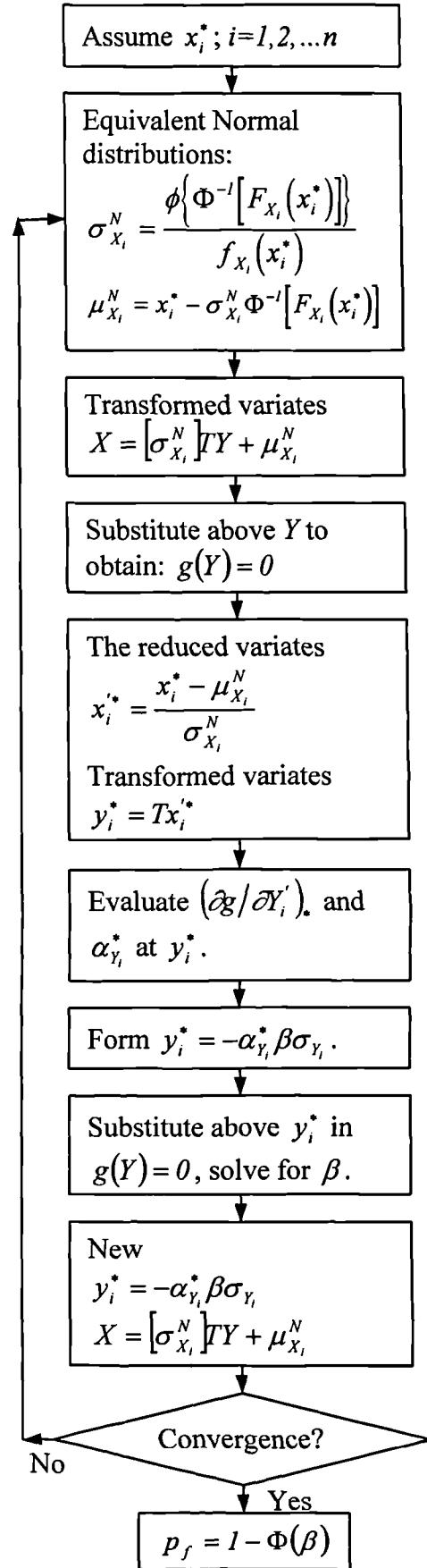
**Figure 5.12 Logic for linear, correlated Normals**



**Figure 5.13 Logic for non-linear, correlated Normals**



**Figure 5.14 Logic for linear, correlated non-Normals**



**Figure 5.15 Logic for non-linear, correlated non-Normals**

## 5.4 Summary of findings

The current research shows that it is possible to use the FOSM methods for risk analysis and decision-making in construction economics. From the results of the examples, the following conclusions can be drawn:

1. The results from the FOSM are exact when the performance function is linear and the variables are Normally distributed.
2. When the variables are not Normally distributed, the equivalent Normal distribution for a non-Normal variate is obtained such that the cumulative probability as well as the probability density ordinate of the equivalent Normal distribution are equal to those of the corresponding non-Normal distribution at the appropriate point  $x_i^*$ , on the failure surface. Thus an iterative procedure is needed.
3. When the variables are correlated, the original variates may be transformed to a set of uncorrelated variables. The procedure in Equation 5.25a can then be applied to the uncorrelated set of transformed variables.
4. When the performance functions are non-linear, the tangent plane to the failure surface at most probable failure point is used to approximate the actual failure surface. Since the pertinent point of tangency on the failure surface is not, a priori, known, the “minimum” point of tangency on the failure surface may be determined through the iterative algorithm. The results of the FOSM are approximate and depending on whether the exact non-linear failure surface is convex or concave toward the origin, this approximation will be on the safe side or unsafe side, respectively.
5. When iterative procedures are applied, for the first iteration, it is always assumed that the possible failure points are those mean values of the variables. In most of the cases, three or four iterations are sufficient to achieve accurate results.



6. In all of the cases studied, the results of 1000 iterations of MCS are sufficiently accurate.
7. The algorithm of the FOSM indicates that there are no limits to the number of variables. Furthermore, the variables can have different probability distribution forms within a performance function.
8. The FOSM methods are computationally fast and provide measures of risk with a desired degree of accuracy when compared to MCS. The algorithms of the FOSM have been summarised and it can be seen that they can easily be written into a computer code and once it is programmed, the calculation takes only a few iterations, unlike the MCS in which large samples are needed.
9. The FOSM methods are appropriate to practical problems of risk analysis such as selecting project alternatives, elemental cost analysis, cash flow streams and setting realistic plant hire rates.

However, if variables involved are non-normal, correlated, or a performance function is non-linear, the results of the FOSM methods are approximate and the accuracy would need further investigation.

# **CHAPTER 6 CONCLUSIONS AND FURTHER DEVELOPMENTS**

## **6.1 Introduction**

This thesis has developed original work in proposing a new method, the Modified Stochastic Assignment Model (MSAM) for analysis of project completion time and also applying the First Order Second Moment (FOSM) method in construction project management. Prior to this, the popular methods used for project duration analysis, the Programme Evaluation and Review Technique (PERT) and Monte Carlo Simulation (MCS) were studied and compared on the basis of two example projects.

This chapter presents general conclusions followed by specific conclusions arising from each strand of the research. By drawing them together in this manner it is hoped to eliminate unnecessary duplication and present clarity of thinking in the summing up of this thesis.

This chapter also provides recommendations for possible future research into the field of risk analysis in construction economics, particularly in project duration analysis and problems involving failure probability (probability of economic loss in project management).

## **6.2 Summary of conclusions**

### **6.2.1 Literature review conclusions**

The literature review showed that the Normal distribution is the most popular distribution used for developing probabilistic approximate analytical methods in risk analysis. This is due to the simplicity of the distribution (only two parameters can define the Normal distribution) and the Central Limit Theorem.

The literature review showed that MCS is the most popular method and PERT is the simplest probabilistic analytical method used in construction network analysis. It was

also found that more probabilistic approximate analytical methods are needed for network analysis.

### **6.2.2 PERT and MCS methods**

PERT is easy to understand, can be calculated manually and saves computational time. PERT gives the most optimistic results when they are compared to those of the MCS in both of the example projects.

MCS is also very easy to understand but requires considerable computational time. MCS can accommodate the different distribution forms for the duration of individual activities and with large samples it can provide more conservative results. Also MCS can determine critical indices for a given network whereas the PERT cannot.

When applying MCS, it can be seen that with an increasing number of iterations, a reduction of the sampling variability can be attained. For the two projects studied, with 30-52 activities, 1000 iterations was sufficient to provide accurate results.

The project duration range (maximum minus minimum) is increased as the number of simulations increase. For distributions without a boundary, such as the Normal and Log-Normal (one side bounded) distributions, the project duration range will tend to infinity. On the other hand, distributions with a boundary, such as the Triangular, Uniform and Beta distributions, project duration ranges will converge at certain points (the minimum and maximum values).

After a certain number of iterations (1000 iterations for the examples studied), the percentile values from 5% to 95% approximately tend to be similar and increasing the number of iterations only affects the minimum (less than 5%) and the maximum (greater than 95%) percentile values. This fact is irrespective of the type of probability distribution function specified for the activity durations.

For the parameters assessed in the present research, the Normal, Log-Normal or Beta distribution functions for individual activities give very similar predictions for the total project duration. It can be seen that between 5% and 95%, the percentile values of the

Normal, Log-Normal and Beta distributions share virtually identical values but before the 5% and after the 95%, the percentile values of each of the different distributions can be seen to be more varied. The Triangular distribution produces medium level durations which are proportionally similar to the Normal distribution results. The Uniform distribution always overestimates the probability of extremes and therefore gives conservative results.

When the mean is altered, that is all of the distributions are considered symmetric, Normal, Log-Normal and Beta distribution functions again give very similar predictions for the total project duration. Before a turn point (20%-25% percentile values), these three distributions (Normal, Log-Normal and Beta) provide more pessimistic results whereas a Uniform distribution provides the most optimistic results. After that point, the three distributions provide more optimistic results whereas a Uniform distribution provides the most pessimistic results. The Triangular distribution produces medium level durations which are proportionally similar to the Normal distribution results. This suggests the Triangular distribution is the best choice of distribution (not optimistic nor conservative) when using the subjective three point estimates (minimum, maximum and most likely) for risk analysis, particularly in the absence of objective data which is a common feature in project management.

The simulation results are not only dependent upon the shape of the distributions but are also dependent upon the manner of setting the mean and standard deviation. However, it can be seen that the effect of the choice of the distribution is greater than the effect of the manner of setting the mean and standard deviation in the two examples investigated. After altering the mean (the distributions are considered symmetric), the results of different distributions are all more pessimistic than those of before the alteration.

The overall properties of each distribution generated by the @Risk package show a characteristic probability distribution for the different distributions. The @Risk package executes in a way that correctly generates the random number according to the assigned distributions.

### 6.2.3 MSAM method

The proposed new method, MSAM, is a direct application of Clark's results and is modified from the SAM, which is a method previously used solely in traffic networks. A computer program has been written to implement the MSAM algorithm. The method can be used to estimate completion times of projects in the presence of uncertainty and thus permits the consideration of network decision problems in terms of risk and probability. It can be used in the planning and execution stages of a project.

The MSAM method has been developed for analysing general construction networks. Five example projects chosen randomly from published work have been studied, and the validity of the method has been verified by results of large sample Monte Carlo simulations. From the results of these examples, the following conclusions can be drawn.

In all of the cases studied, MSAM produces very good results when compared to MCS for the entire range of probabilities of interest. However, MSAM is an approximate method, and in the case of the variance of project duration being large when compared to the mean (Example 4.5), MSAM produces more pessimistic results when compared to other methods (PERT, PNET and MCS). PERT gives the most optimistic estimate of a project duration. When the ratio of the standard deviation and the mean is reduced, the accuracy of the MSAM improves. It is possible to use other approximations, such as methods by Mendell and Elston <sup>(24)</sup>, Kamakura <sup>(25)</sup>, and Langdon <sup>(26,27)</sup>, to replace Clark's results in order to improve accuracy and still use the MSAM mechanism.

When compared to PERT and CPM, the MSAM has the advantage of taking total account of the correlations between paths in a network. This enables the shorter but more uncertain paths to be considered thus increasing the accuracy of the prediction. When compared to PNET, the MSAM is a one-pass method which does not need path enumeration and thus can be more easily implemented and is also better to cope with large networks. When compared to MCS, the MSAM is a direct analytical procedure and is very efficient in saving computational time. The calculations take only few

seconds whereas the MCS take 4-5 hours on the same PC in all of the examples. This would enable considerable financial and time savings during the planning stage of a project where these aspects are usually critical.

In its present form, the MSAM method yields the probability of project completion within a prescribed target time, or the required project time at a specific probability. The method may be extended to consider the effects of uncertainty in general activity and decision networks.

#### **6.2.4 FOSM method**

The current research shows that it is readily possible to use the FOSM methods for risk analysis and decision-making in construction economics. This method is especially suited to the conceptual stage of a project but can be used to assess the failure probability throughout a project life cycle. From the results of the examples, the following conclusions can be drawn.

The results from the FOSM are exact when the performance function is linear and the variables are Normally distributed. When the variables are not Normally distributed, the equivalent Normal distribution for a non-Normal variate is obtained such that the cumulative probability as well as the probability density ordinate of the equivalent Normal distribution are equal to those of the corresponding non-Normal distribution at the design point on the failure surface. Thus an iterative procedure is needed.

When the variables are correlated, the original variates may be transformed to a set of uncorrelated variables. The procedure in Equation 5.25a can then be applied to the uncorrelated set of transformed variables.

When the performance functions are non-linear, the design point on the failure surface with minimum distance to the origin of the reduced variates is the most probable failure point. The tangent plane to the failure surface at the design point may then be used to approximate the actual failure surface. Since the pertinent point of tangency on the failure surface is not, a priori, known, the “minimum” point of tangency on the failure surface may be determined through the iterative algorithm as described in

Section 5.2.4.1. The results of the FOSM are approximate and depending on whether the exact non-linear failure surface is convex or concave toward the origin, this approximation will be on the safe side or unsafe side, respectively.

When iterative procedures are applied, for the first iteration, it is always assumed that the possible failure points are those mean values of the variables. In most of the cases, three or four iterations are sufficient to achieve accurate results and these are very close to the results of 1000 iterations of MCS. If variables involve non-normal, correlated, or a performance function which is non-linear, the results of the FOSM methods are more approximate than when dealing with Normal, uncorrelated variables and linear performance function.

The algorithm of the FOSM indicates that there is no limit to the number of the variables and they can have different probability distribution forms within a performance function.

The FOSM methods are computationally fast and provide measures of risk with an excellent degree of accuracy when compared to MCS. The algorithms of the FOSM have been summarised and it can be seen that they can be easily written into computer code and once it is programmed, the calculation takes only a few iterations and unlike the MCS in which large samples are needed. This significantly reduces computational time.

The FOSM methods are appropriate to practical problems of risk analysis such as selecting project alternatives, elemental cost analysis, cash flow streams and setting realistic plant hire rates.

### **6.3 Recommendations for further research**

PERT and MCS are well developed methods for project duration assessment. Future work should emphasise on the collection of objective data and based on this the distributions of activities and the dependent relationships of the activities can be determined.

The MSAM method has been written in FORTRAN 77 to implement its algorithm. Further work can be done by writing the program into different computer codes and developed into a more user friendly computer package, preferably in a windows environment.

The algorithm of the MSAM can be improved by using different approximate results instead of using Clark's results only.

The MSAM method should be tested in realistic situations, such as in a particular type of network, by which the practical accuracy of the method can be assessed.

In its present form, the MSAM method yields the probability of project completion within a prescribed target time, or the required project time at a specific probability. The method may be extended to consider the effects of uncertainty in general activity and decision networks.

The present research has shown the applicability of FOSM method in construction economics and the logic of the algorithm has been summarised. However, it has not been written as a computer program to implement its algorithm. Further work can be done by writing the program into different computer codes and developing into a more user friendly computer package, preferably in a windows environment.

The accuracy of the FOSM needs to be tested in realistic situations. Furthermore, where the performance function is non-linear or the variables are non-Normal, the accuracy of the FOSM method could be improved by including higher order approximation to the design point in determining the value of  $\beta$ , the reliability index. This area requires further investigation.

The FOSM method has a great potential for application into other areas of civil engineering where little quantitative probabilistic work has been carried out so far. Although the FOSM method was originally developed within the civil engineering industry, well presented and easy to understand computer software is still lacking for



practical use. It is also likely that both the MSAM and FOSM methods can be adjusted to apply to other industries.

In general, risk analysis and management methods are increasingly being accepted as providing a rational basis for decision making in the construction industry. Widespread use of risk analysis, such as within tender proposals, will only be possible if further research is undertaken to provide standardised solutions which can be easily adopted by practitioners in industry. Without such further development, some useful risk analysis methods will remain highly specialised and relatively unused tools.

## REFERENCES

- 1 Raftery, J. (1994). 'Risk Analysis in Project Management', E. & F. N. SPON (An imprint of Chapman & Hall), pp. 143.
- 2 Flanagan, R. and Norman, G. (1993). 'Risk Management and Construction', Blackwell Science, pp. 208.
- 3 Perry, J.G. and Hayes, R.W., 'Construction Projects – Know the Risks', Chartered Mechanical Engineer, 32 (2), February, 1985, pp 42-45.
- 4 Perry, J.G. and Hayes, R.W., 'Risk and Its Management in Construction Projects', Proceedings of the Institution of Civil Engineers, Part 1 (78), June, 1985, pp 499-521.
- 5 Perry, J.G., 'Risk Management – An Approach for Project Managers', Project Management, Vol. 4, No. 4, November, 1986, pp 211-216.
- 6 Thompson, P.A. and Perry, J.G. (1992). 'Engineering Construction Risks: A Guide to Project Risk Analysis and Risk Management', Thomas Telford, London.
- 7 Smith, N.J. (Editor) (1995). 'Engineering Project Management', Blackwell Science Ltd.
- 8 Ward, S.C. and Chapman, C.B. (1995). 'Risk Management and the Project Life Cycle', International Journal of Project Management, 13(3), pp. 145-149.
- 9 Chapman, C. and Ward, S. (1997). 'Project Risk Management: Processes, Techniques and Insights', John Wiley & Sons Ltd, England.

- 10 Healy, N. J. (1981). 'Risk Management in Giant Civil Engineering Projects', thesis presented to the University of Manchester, UMIST, in partial fulfilment of the requirements for the degree of Master of Science.
- 11 Wideman, R. M., 'Risk Management', Project Management Journal, Project Management Institute, September, 1986, pp. 20-26.
- 12 Al-Bahar, J.F. and Crandall, K.C., 'Systematic Risk Management Approach for Construction Projects', Journal of Construction Engineering and Management, Vol. 116, No. 3, September, 1990, pp. 533-546.
- 13 Park, C.S. and Sharp-Bette, G.P. (1990). 'Advanced Engineering Economics', John Wiley & Sons Ltd.
- 14 Canada, J.R., Sullivan, W.G. and White, J.A. (1996). 'Capital Investment Analysis for Engineering and Management', second edition, Prentice-Hall, Inc.
- 15 Dougherty, E.R. (1990). 'Probability and Statistics for the Engineering, Computing and Physical Sciences', Prentice-Hall international editions, Inc. A Division of Simon & Schuster Englewood Cliffs, New Jersey.
- 16 Ang, A.H.-S. and Tang, W.H. (1975). 'Probability Concepts in Engineering Planning and Design, Volume I - Basic Principles', John Wiley & Sons, Inc.
- 17 Hahn, G.J. and Shapiro, S.S. (1967). 'Statistical Models in Engineering', John Wiley & Sons, New York.
- 18 Pearson, K. (1934). 'Tables of the Incomplete B-Function', Cambridge University Press, Cambridge, England.
- 19 Pearson, E.S. and Johnson, N.L. (1968). 'Tables of the Incomplete Beta-Function', second edition, Cambridge University Press, Cambridge, England.

- 20 Pouliquen, L.Y. (1970). 'Risk Analysis in Project Appraisal', World Bank Staff Occasional Papers, No. 11.
- 21 Newton, S. (1991). 'An Agenda For Cost Modelling Research', Construction Management and Economics, 9, pp. 97-112.
- 22 Macaluso, P. 'A Risky Business - An Introduction to Monte Carlo Venture Analysis', Byte Magazine, McGraw-Hill, Vol. 9, No. 3, March, 1984, pp. 179-192.
- 23 Karni, R.A. (1987). 'Systematic Approach to Stochastic Scheduling', Construction Management and Economics, 5, pp. 187-198.
- 24 Bennet, J. and Ormerod, R.N. (1984). 'Simulation Applied to Construction Projects', Construction Management and Economics, 2, pp. 225-263.
- 25 Baxendale, T. (1984). 'Construction Resource Models by Monte Carlo Simulation', Construction Management and Economics, 2, pp. 201-217.
- 26 Wilson, A., 'Experiments in Probabilistic Cost Modelling. In Building Cost Techniques - New Directions: Transactions of Building Cost Research Conference', Portsmouth Polytechnic, 23-25, September, 1982, edited by P.S. Brandon. London, Spon, pp. 169-180.
- 27 Nunnally, S.W. (1981). 'Simulation in Construction Management', in Proceedings of the CIB Symposium on the Organisation and Management of Construction, Dublin, Eire, International Council for Building Research, Vol. 1, pp. 110-125.
- 28 Eppen, G.D. and Gould, F.J. (1985). 'Quantitative Concepts for Management: Decision Making Without Algorithms', second edition, New Jersey, Prentice Hall.

- 29 Pidd, M. (1992). 'Computer Simulation in Management Science', third edition, Chichester John Wiley & Sons.
- 30 '@Risk Risk Analysis for Spreadsheets: Guide to using @Risk for Project, Risk Analysis and Simulation Add-In for Microsoft Project', Window Version Release 1.0, January, 1994, Palisade Corporation.
- 31 '@Risk Risk Analysis for Spreadsheets: @Risk, Risk Analysis and Simulation Add-In for Microsoft Excel', Window or Apple Macintosh Version Release 1.12, User's Guide, July, 1993, Palisade Corporation.
- 32 O'Brien, J.J. (1971). 'CPM in Construction Management, Project Management with CPM', second edition, McGraw-Hill Book Company.
- 33 Lockyer, K. and Gordon, J. (1991). 'Critical Path Analysis and Other Project Network Techniques', fifth edition, Pitman.
- 34 Andrew, S.P.S., 'A Job Planning System for the Rapid Overhaul of Large Units of Plant', Report G.&P/SPSA/JD, 5th January 1956, I.C.I., Billingham Division.
- 35 Roy, B. (1962). 'Revue Francaise de Recherche Operationelle', 6, pp. 323.
- 36 Wille, H., 'Proceedings of NATO Conference on Digital Simulation', Hamburg, September, 1965, pp. 319.
- 37 Kelley, J.E.Jr. and Walker, M.R., 'Critical Path Planning and Scheduling', Proceedings of the Eastern Joint Computer Conference, Association for Computing Machinery, December, 1959, pp. 160-173.
- 38 'PERT, Program Evaluation Research Task, Phase 1 Summary Report', (1958), Special Projects Office, Bureau of Ordnance, Department of the Navy, Washington, D.C.

- 39 'PERT, Program Evaluation Research Task, Phase 2 Summary Report', (1958), Special Projects Office, Bureau of Ordnance, Department of the Navy, Washington, D.C.
- 40 Fondahl, J.W. (1961). 'A Non-Computer Approach to Critical Path Method for the Constructional Industry', the Department of Civil Engineering, Stanford University, California.
- 41 Moder, J.J., Phillips, C.R. and Davis, E.W. (1983) 'Project Management with CPM, PERT and Precedence Diagramming', third edition, Van Nostrand Reinhold New York & London.
- 42 Battersby, A. (1982). 'Network Analysis for Planning and Scheduling', third edition, the Macmillan Press Ltd, pp. 261-269.
- 43 Malcolm, D.G., Roseboom, J.H., Clark, C.E. and Fazar, W. (1959). 'Application of a Technique for Research and Development Program Evaluation', Operations Research, Vol. 7, No. 5, 646-669.
- 44 Elmaghraby, S.E. (1977). 'Probabilistic Activity Networks (PANs): A Critical Evaluation and Extension of the PERT Model - Chapter 4. In Activity Networks: Project Planning and Control by Network Models', John Wiley & Sons, New York, pp. 228-320.
- 45 Keefer, D.L. and Bodily, S.E. (1983). 'Three-Point Approximations for Continuous Random Variables', Management Science, 29, pp. 595-609.
- 46 Sasieni, M.W. (1986). 'A Note on PERT Times', Management Science, 32, pp. 1652-1653.
- 47 MacCrimmon, K.R. and Ryavec, C.A. (1964). 'An Analytical Study of the PERT Assumptions', Operations Research, 12, 16-37.

- 48 Ang, A.H.-S., Abdelnour, J. and Chaker, A.A., 'Analysis of Activity Networks Under Uncertainty', Journal of The Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 101, No. EM4, August, 1975, pp. 373-387.
- 49 Yen, J.Y., 'Finding the K Shortest Loopless Paths in a Network', Management Science, Vol. 17, No. 11, July 1971.
- 50 Al-sadek O. and Carmichael D.G. (1992). 'On Simulation in Planning Networks', Civil Engineering System, Vol. 9, pp. 59-68.
- 51 Disz C.F. and Hadipriono F.C., 'Nondeterministic Networking Methods', Journal of Construction Engineering and Management, Vol. 119, No.1, March, 1993, pp. 40-57.
- 52 Van Slyke R.M. (1963). 'Monte Carlo Methods and the PERT Problem', Operations Research, Vol. 10, No. 5, 839-860.
- 53 McGowan, L.L. (1964). 'Monte Carlo Techniques Applied to PERT Networks', Unpublished Master of Science thesis, Texas A&M University, College Station, Texas.
- 54 Hartley, H.O. and Wortham, A.W. (1966). 'A Statistical Theory for PERT Critical Path Analysis', Management Science, Vol. 12, No. 10, B469-B481.
- 55 Ringer, L.J., 'Numerical Operators for Statistical PERT Critical Path Analysis', Management Science, Vol. 16, No. 2, October, 1969.
- 56 Bowman, R. (1994). 'Stochastic Gradient-Based Time-Cost Tradeoffs in PERT Networks Using Simulation', Annals of Operations Research, 53, 533-551.
- 57 Arsham, H. (1996). 'Performance Extrapolation in Discrete-Event Systems Simulation', International Journal of Systems Science, 27, 863-869.

- 58 Diaz, C.F. (1989). 'Probabilistic Network Analyses for Construction Projects', MS thesis, Ohio State University, Columbus, Ohio.
- 59 Sculli, D. (1983). 'The Completion Time of PERT Networks', Journal of Operational Research Society. Vol. 34, No. 2, 155-158.
- 60 AbouRizk, S.M. and Halpin, D.W., 'Statistical Properties of Construction Duration Data', Journal of Construction Engineering and Management, Vol. 118, No. 3, September, 1992, 525-544.
- 61 AbouRizk, S.M., Halpin, D.W. and Wilson, J.R., 'Visual Interactive Fitting of Beta Distributions', Journal of Construction Engineering and Management, Vol. 117, No. 4, December, 1991, 589-605.
- 62 AbouRizk, S.M., Halpin, D.W. and Wilson, J.R., 'Fitting Beta Distributions Based on Sample Data', Journal of Construction Engineering and Management, Vol. 120, No. 2, June, 1994, 288-303.
- 63 Newton, S. (1992). 'Methods of Analysing Risk Exposure in the Cost Estimates of High Quality Office', Construction Management and Economics, 10, pp. 431-449.
- 64 Back, W.E., Boles, W.W. and Fry, G.T., 'Defining Triangular Probability Distributions From Historical Cost Data', Journal of Construction Engineering and Management, Vol. 126, No. 1, January/February, 2000.
- 65 Smith, S.D., Osborne, J.R. and Forde, M.C. (1996). 'The Use of a Discrete-Event Simulation Model with Erlang Probability Distributions in the Estimation of Earthmoving Production', Civil Engineering Systems, 13 (1), 25-44.
- 66 Ranasinghe, K.A.M.K. (1990). 'Analytical Method for Quantification of Economic Risks during Feasibility Analysis for Large Engineering Projects',



Thesis presented to the University of British Columbia, at Vancouver, Canada, in partial fulfilment of the requirements for the PhD.

- 67 Mehrotra, K., Chai, J. and Pillutla, S. (1996). 'A Study of Approximating the Moments of the Job Completion Time in PERT Networks', *Journal of Operations Management*, Vol. 14, pp 277–289.
- 68 Dawood, N. (1998). 'Estimating Project and Activity Duration: A Risk Management Approach Using Network Analysis', *Construction Management and Economic*, 16, 41-48.
- 69 Wall, D.M. (1997). 'Distributions and Correlation in Monte Carlo Simulation', *Construction Management and Economics*, Vol. 15, pp. 241-258.
- 70 Ditlevsen, O. (1979). 'Narrow Reliability Bounds for Structural Systems', *Journal of Structural Mechanics*, American Society of Civil Engineers, 7(4), 453-472.
- 71 Putcha, C.S. and Rao, S.J. (1991). 'Probabilistic Network Analysis for Project Completion Time', *Civil Engineering System*, Vol. 8, pp. 179-184.
- 72 Johnson, D.W. (1981). 'Linear Scheduling Method for Highway Construction', *Journal of Construction Division*, ASCE, 107(2), 247-261.
- 73 Harmelink, D.J. (1995). 'Linear Scheduling Model: The Development of a Linear Scheduling Model with Micro Computer Applications for Highway Construction Project Control', PhD thesis, Iowa State University, Ames. Iowa.
- 74 Laferriere, B.J. (1981). 'Personal Communication', Berkeley, California.
- 75 Rosenblueth, E., 'Two-Point Estimates in Probabilities', *Applied Mathematical Modelling*, 5, October, 1981, pp. 329-335.

- 76 Rosenblueth, E., 'Point Estimates for Probability Moments', Proceedings of the National Academy of Sciences, October, 1975, 72 (10), pp. 3812-4.
- 77 Lind, N.C., 'Modelling of Uncertainties in Complex Systems', Applied Mathematical Modelling.
- 78 Chrzanowski, E.N. and Johnston, D.W. (1988). 'Application of Linear Scheduling', Journal of Construction Engineering and Management, ASCE, Vol. 112, No. 4, pp. 476-491.
- 79 Harmelink, D.J. and Rowings, J.E., 'Linear Scheduling Model: Development of Controlling Activity Path', Journal of Construction Engineering and Management, Vol. 124, No. 4, July/August, 1998, pp. 263-268.
- 80 'Predict! Application Report - Project Risk', (1992). Risk Decisions Ltd. Oxford, pp. 3.
- 81 Ahuja, H.N., Dozzi, S.P. and AbouRizk, S.M. (1994). 'Project Management: Techniques in Planning and Controlling Construction Projects', John Wiley & Sons, New York, pp. 310.
- 82 Arsham, H. (1999). 'Stability Analysis of the Critical Path in Project Activity Networks', Civil Engineering and Environmental Systems, Vol. 15, 305-334.
- 83 O'Brien, J. (1984). 'CPM in Construction Management', McGraw-Hill Book Co., New York.
- 84 Fulkerson, D.R. (1962). 'Expected Critical Path Lengths in PERT Network', Operations Research, 10, 808-817.
- 85 Moder, J.J. and Rodgers, E.G. (1968). 'Judgement Estimates of the Moments of PERT Type Distributions', Management Science, 15, B76-83.

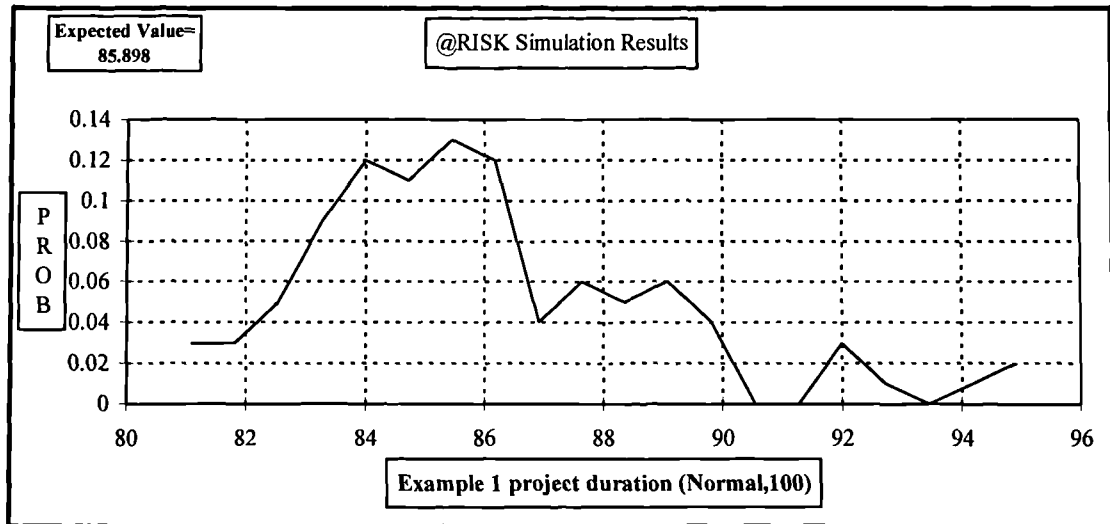
- 86 Ranasinghe, M. (1994). 'Quantification and Management of Uncertainty in Activity Duration Networks', *Construction Management and Economics*, 12 (1), 15-29.
- 87 Maher, M.J. (1992). 'SAM - A Stochastic Assignment Model', *Mathematics in Transport Planning and Control*, Griffiths, J.D., (Editor), Clarendon Press, Oxford, 121-132.
- 88 Clark, C.E. (1961). 'The Greatest of A Finite Set of Random Variables', *Operations Research*, Vol. 9, 145-162.
- 89 Maher, M.J. and Hughes, P.C. (1997). 'A Probit-Based Stochastic User Equilibrium Assignment Model', *Transportation Research*, Vol. 31, No. 4, 341-355.
- 90 Du, Z.P. and Nicholson, A.J. (1993). 'Degradable Transport Systems: Performance, Sensitivity and Reliability Analysis', *Research Report 93-8*, Department of Civil Engineering, University of Canterbury, Christchurch, New Zealand.
- 91 Dijkstra, E.W. (1959). 'A Note on Two Problems in Connection with Graphs', *Numerische Mathematik*, 1, pp. 269-271.
- 92 Hughes, P. (1998). 'Link-Based Methods for Stochastic User Equilibrium Traffic Assignment Model', Thesis presented to the Napier University, Edinburgh, UK, in partial fulfilment of the requirements for the PhD.
- 93 Bell, M.G.H. and Iida, I. (1997). 'Transportation Network Analysis', Chichester, John Wiley & Sons Ltd.
- 94 Phillips, D.T. and Garcia-Diaz, A. (1981). 'Fundamentals of Network Analysis', Prentice-Hall, Inc., Englewood Cliffs, N.J. 07632.
- 95 Cass, T. (1969). 'Statistical Methods in Management', Cassell, London.

- 96 Kamburowski, J. (1985). 'Normally Distributed Activity Durations in PERT Networks', *Journal of Operational Research Society*, Vol. 36, No. 11, 1051-1057.
- 97 Brook, A.C., Leahy, J.P. and Shaffer, L.R., 'A Man - Machine CPM System for Decision Making in the Construction Industry', *Civil Engineering Studies, Construction Research Series*, No. 9, University of Illinois, Urbana, Illinois, June, 1967.
- 98 Brand, J.D., Meyer, W.L. and Shaffer, L.R., 'The Resource Scheduling Problem in Construction', *Civil Engineering Studies, Construction Research Series*, No. 5, University of Illinois, Urbana, Illinois, June, 1964.
- 99 Mendell, N.R. and Elston, R.C. (1974). 'Multifactorial Qualitative Traits: Genetic Analysis and Prediction of Recurrence Risks', *Biometrics* 30, 41-57.
- 100 Kamakura, W.A. (1989). 'The Estimation of Multinomial Probit Models: A New Calibration Algorithm', *Transportation Science*, 23 (4), 253-265.
- 101 Langdon, M.G. (1984a). 'Improved Algorithms for Estimating Choice Probabilities in the Multinomial Probit Model', *Transportation Science*, 18 (3), 267-299.
- 102 Langdon, M.G. (1984b). 'Methods of Determining Choice Probability in Utility Maximizing Multiple Alternative Models', *Transportation Research*, B18 (3), 209-234.
- 103 Ang, A.H.-S. and Tang, W.H. (1984). 'Probability Concepts in Engineering Planning and Design, Volume II- Design, Risk, and Reliability', New York, John Wiley & Sons, Inc.
- 104 Madsen, H.O., Krenk, S. and Lind, N.C. (1986). 'Methods of Structural Safety', Englewood Cliffs, New Jersey, Prentice-Hall.

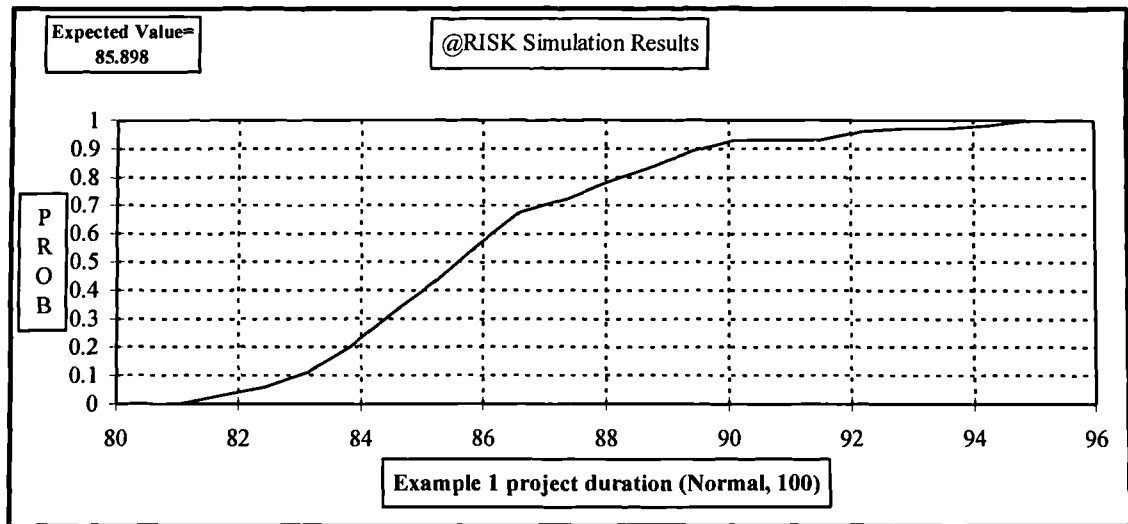
- 105 Melchers, R.E. (1987). 'Structural Reliability Analysis and Prediction', New York, Halsted Press.
- 106 Thoft-Christensen, P. and Baker, M.J. (1982). 'Structural Reliability Theory and Its Applications', New York, Springer-Verlag.
- 107 Cornell, C.A. (1969). 'A Probability-Based Structural Code', Journal of the American Concrete Institute, 66(12), pp 974-985.
- 108 Ang, A.H.-S. (1973). 'Structural Risk Analysis and Reliability-Based Design', Journal of Structural Division of American Society of Civil Engineers, 99(ST9), pp 1891-1910.
- 109 Ang, A.H.S. and Cornell, C.A., 'Reliability Bases of Structural Safety and Design', Journal of Structural Division, American Society of Civil Engineers, Vol. 94, No. EM2, April, 1974, pp. 1755-1769.
- 110 Hasofer, A.M. and Lind, N., 'An Exact and Invariant First-Order Reliability Format', Journal of Engineering Mechanics, American Society of Civil Engineers, Vol. 100, No. EM1, February, 1974, pp. 111-121.
- 111 Rackwitz, R. and Fiessler, B. (1976). 'Note on Discrete Safety Checking When Using Non-Normal Stochastic Models for Basic Variables', Loads Project Working Session, Cambridge, Massachusetts, Massachusetts Institute of Technology.
- 112 Chen, X. and Lind, N.C. (1982). 'A New Method for Fast Probability Integration', Paper No. 171, Waterloo, Canada, University of Waterloo.
- 113 Freudenthal, A.M., Garrelt, J.M. and Shinozuka, M., 'The Analysis of Structural Safety', Journal of Structural Division, American Society of Civil Engineers, Vol. 92, No. ST1, February, 1966.

- 114 Freudenthal, A.M. (1956). 'Safety and the Probability of Structural Failure', ASCE Transactions, Vol. 121, pp. 1337-1397.
- 115 Shinozuka, M., 'Basic Analysis of Structural Safety', Journal of Structural Division, American Society of Civil Engineers, Vol. 198, No. 3, March, 1983.
- 116 Paloheimo, E. and Hannus, H. (1974). 'Structural Design Based on Weighted Fractiles', Journal of Structural Division, American Society of Civil Engineers, Vol. 100, No. ST7, pp. 1367-1378.
- 117 Rackwitz, R. (1976). 'Practical Probability Approach to Design', Bulletin No. 112, Paris France, Comité Européen du Béton.
- 118 Rosenblatt, M., 'Remarks on A Multivariate Transformation', Annals of Math. Stat., Vol. 23, No. 3, September, 1952, pp. 470-472.
- 119 Fiessler, B., Neumann, H.J. and Rackwitz, R. (1979). 'Quadratic Limit States in Structural Reliability', Journal of Engineering Mechanics Division of American Society of Civil Engineers, Vol. 105, No. 4, pp. 661-676.
- 120 Hasofer, A.M. and Lind, N.C. (1974). 'Exact and Invariant Second-Moment Code Format', Journal of Engineering Mechanics Division of American Society of Civil Engineers, Vol. 100, No. EM1, pp. 111-121.
- 121 Chau, K.W. (1995). 'The Validity of the Triangular Distribution Assumption in Monte Carlo Simulation of Construction Costs', Construction Management and Economics, 13 (1), pp. 15-21.
- 122 Harris, F. and McCaffer, R. (1982). 'Construction Plant Management and Investment Decisions', Granada Publishing Limited, London, pp. 101.
- 123 Raftery, J.(1991). 'Principles of Building Economics: An Introduction', BSP Professional Books, Oxford.

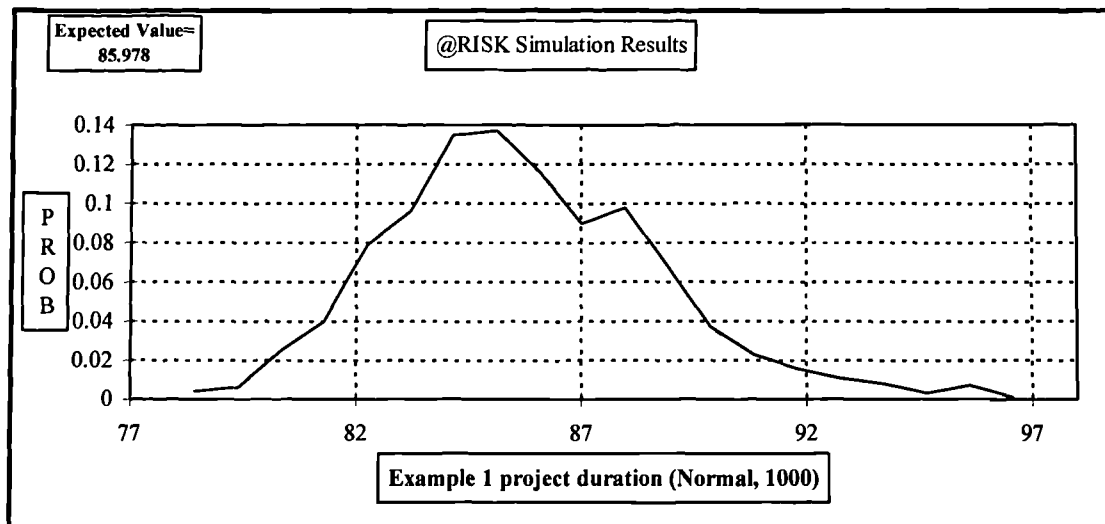
# **APPENDIX A** **RESULTS OF MCS FOR EXAMPLE 3.1**



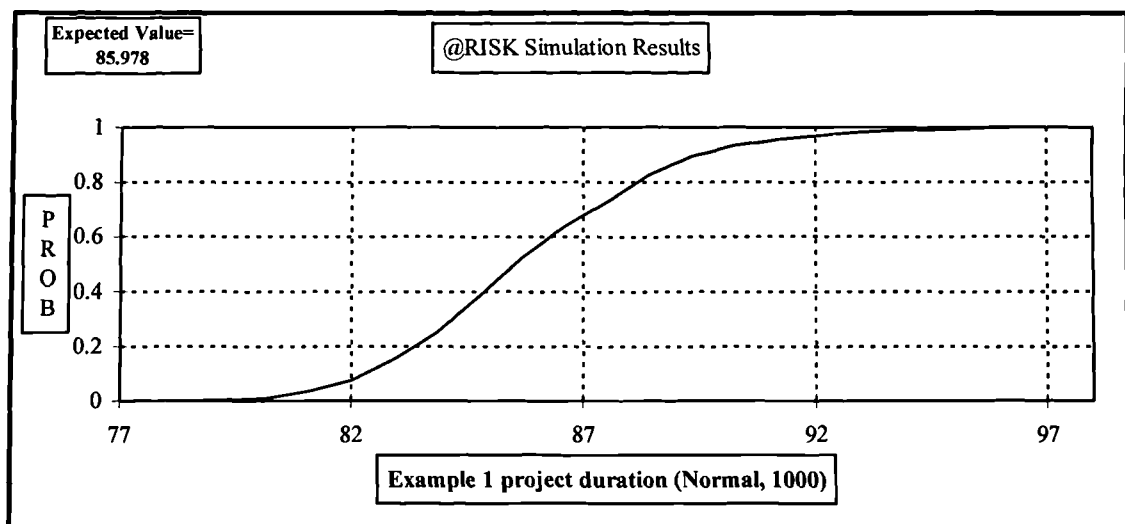
**Figure A.1 Total project duration of Example 3.1. Histogram in curve.**  
**Normal distribution, 100 iterations**



**Figure A.1a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Normal distribution, 100 iterations**

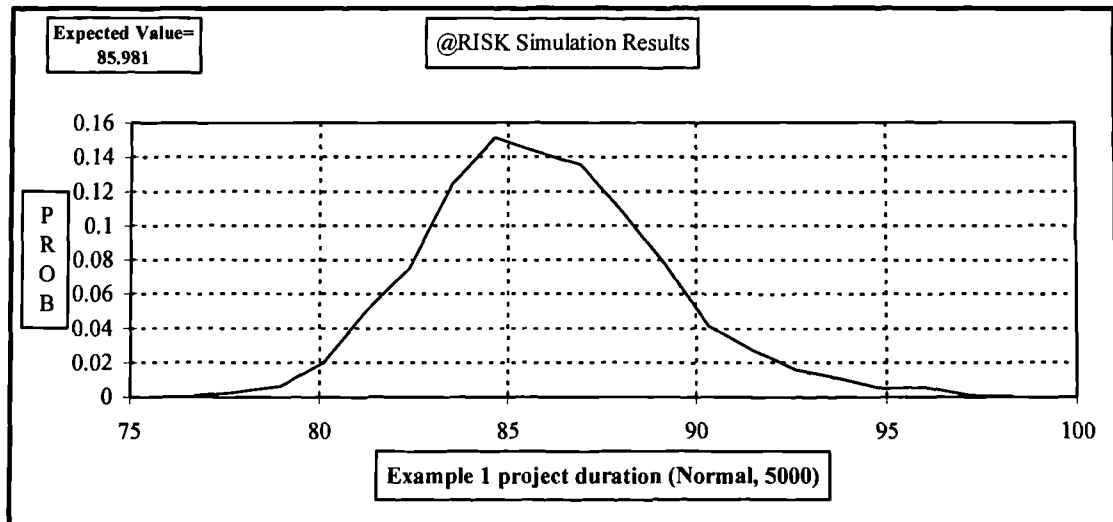


**Figure A.2 Total project duration of Example 3.1. Histogram in curve.**  
**Normal distribution, 1000 iterations**

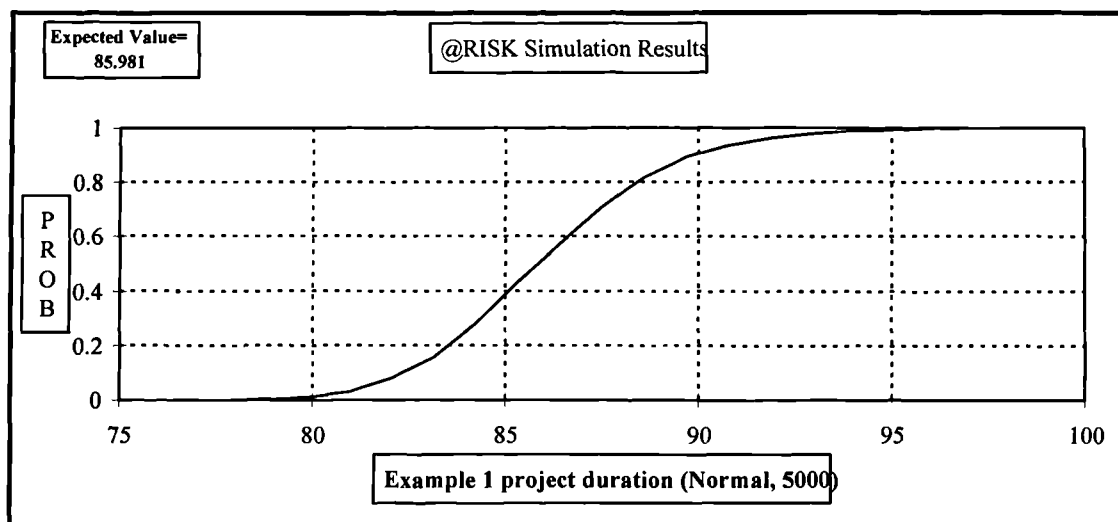


**Figure A.2a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Normal distribution, 1000 iterations**

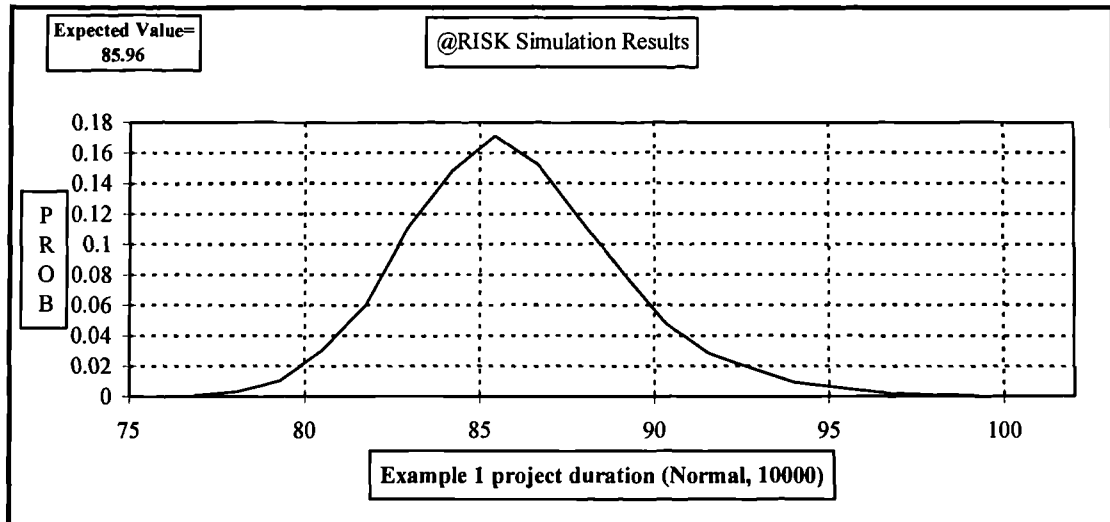




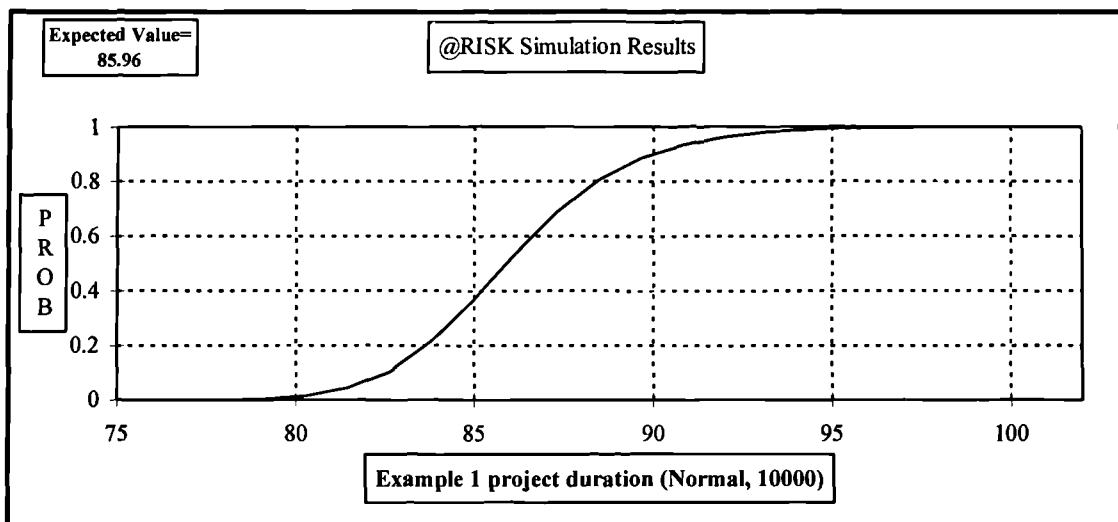
**Figure A.3 Total project duration of Example 3.1. Histogram in curve.**  
**Normal distribution, 5000 iterations**



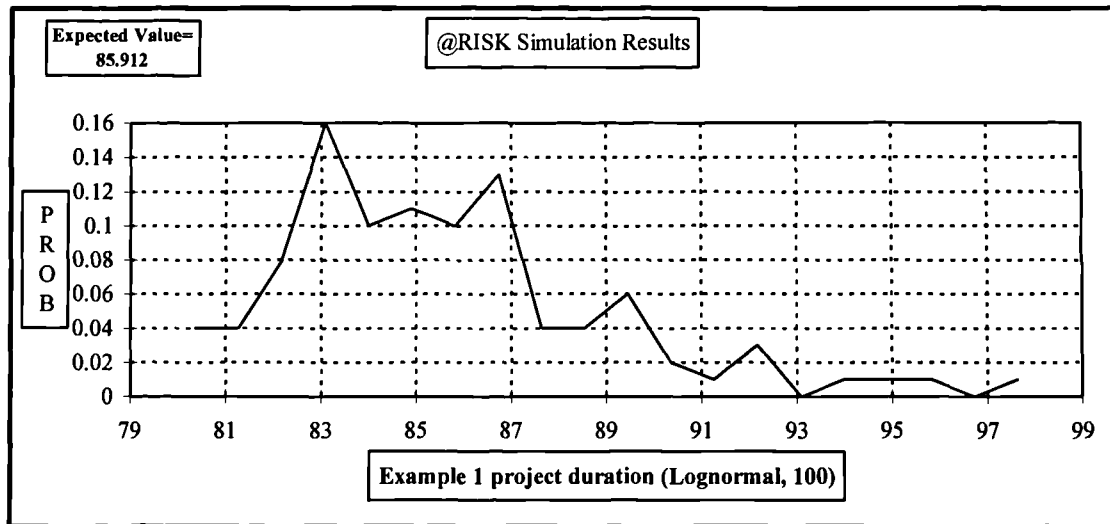
**Figure A.3a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Normal distribution, 5000 iterations**



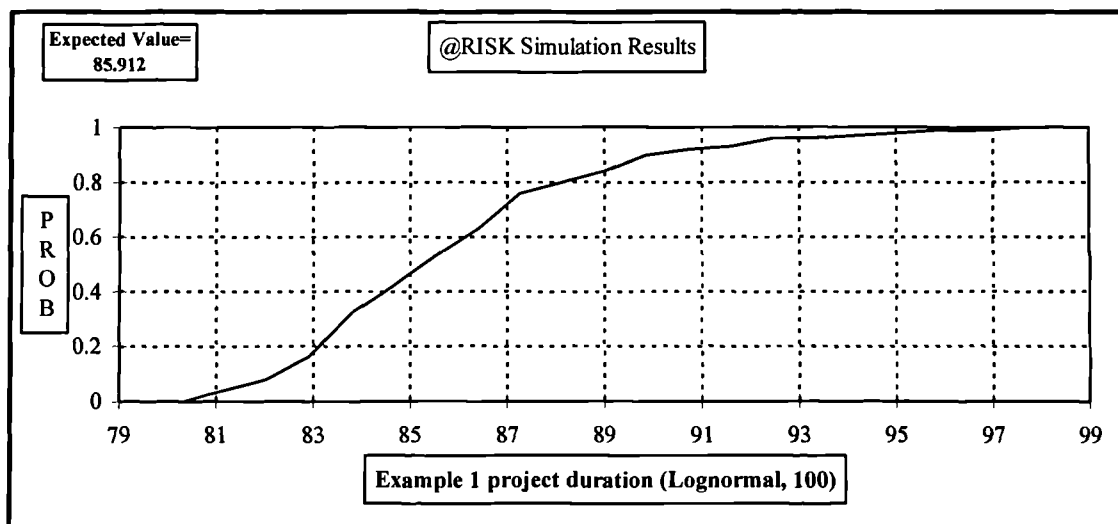
**Figure A.4 Total project duration of Example 3.1. Histogram in curve.**  
**Normal distribution, 10000 iterations**



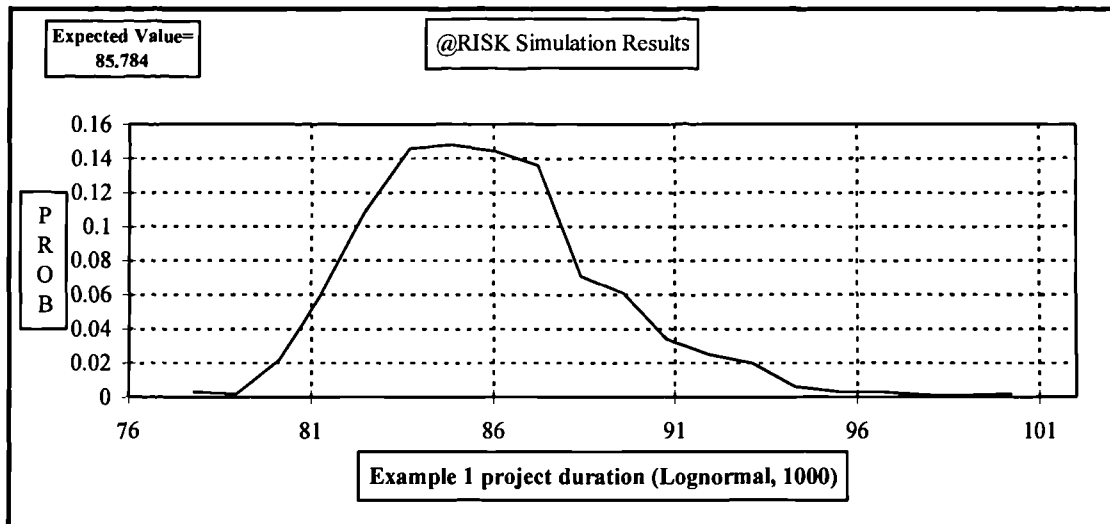
**Figure A.4a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Normal distribution, 10000 iterations**



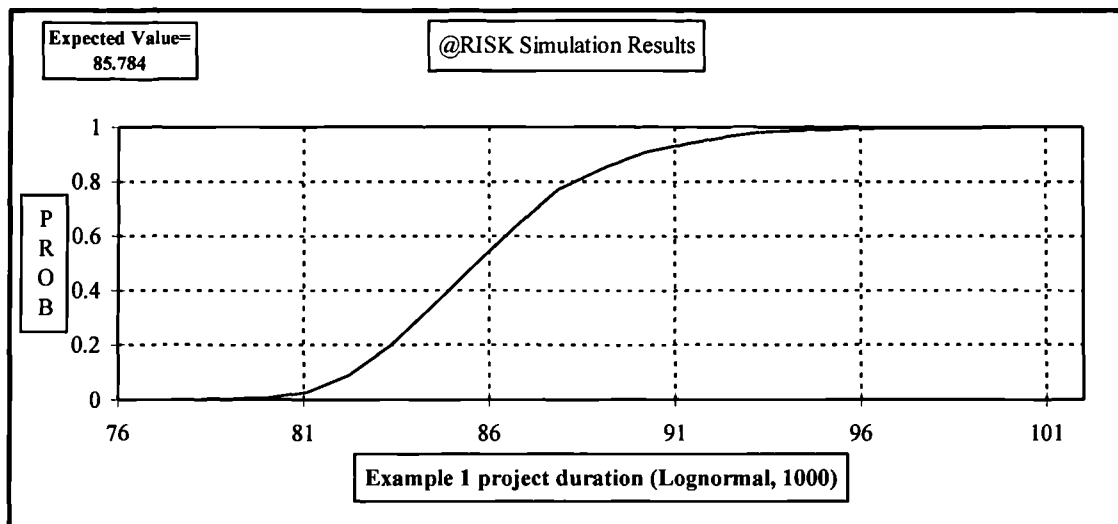
**Figure A.5 Total project duration of Example 3.1. Histogram in curve.**  
**Log-Normal distribution, 100 iterations**



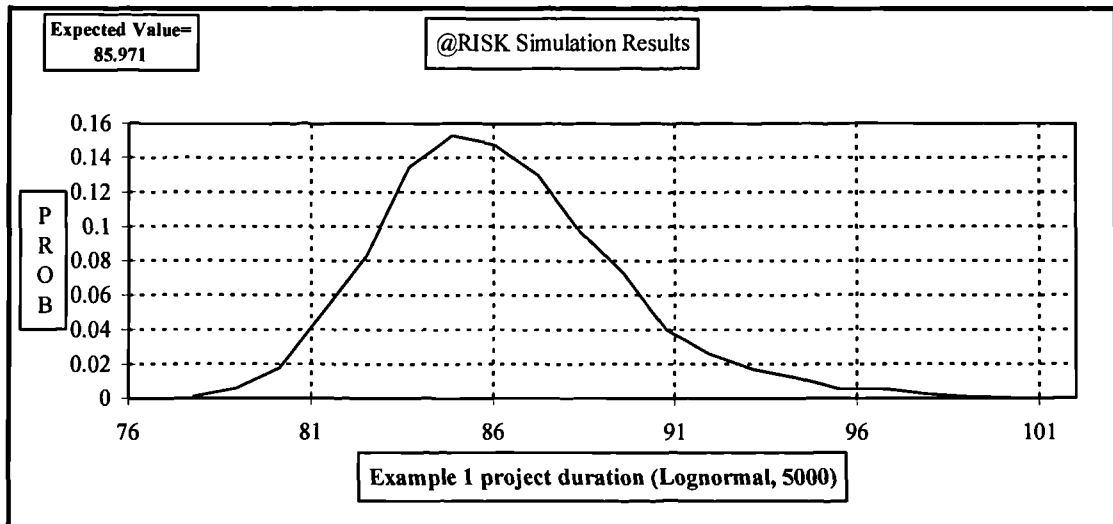
**Figure A.5a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Log-Normal distribution, 100 iterations**



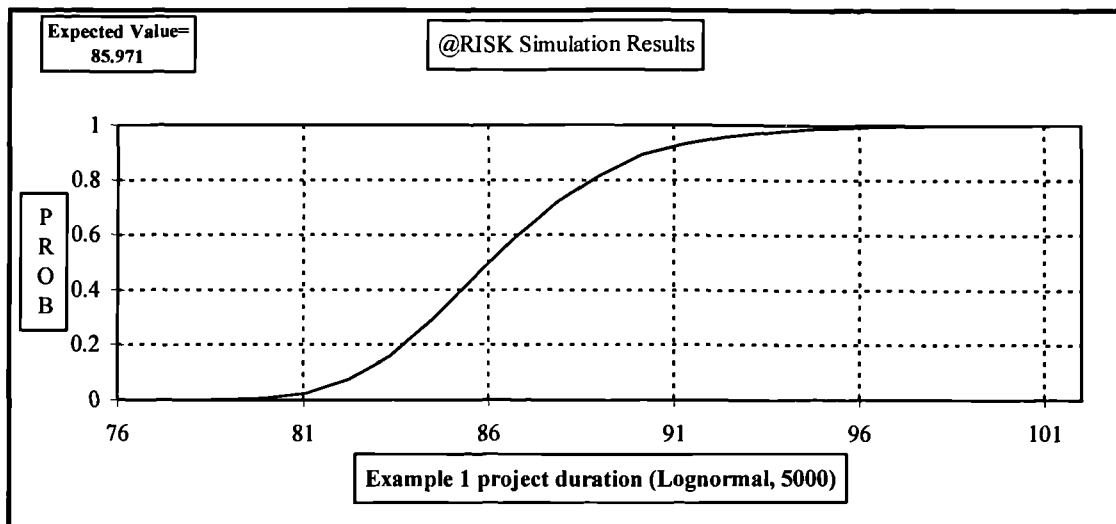
**Figure A.6 Total project duration of Example 3.1. Histogram in curve.  
Log-Normal distribution, 1000 iterations**



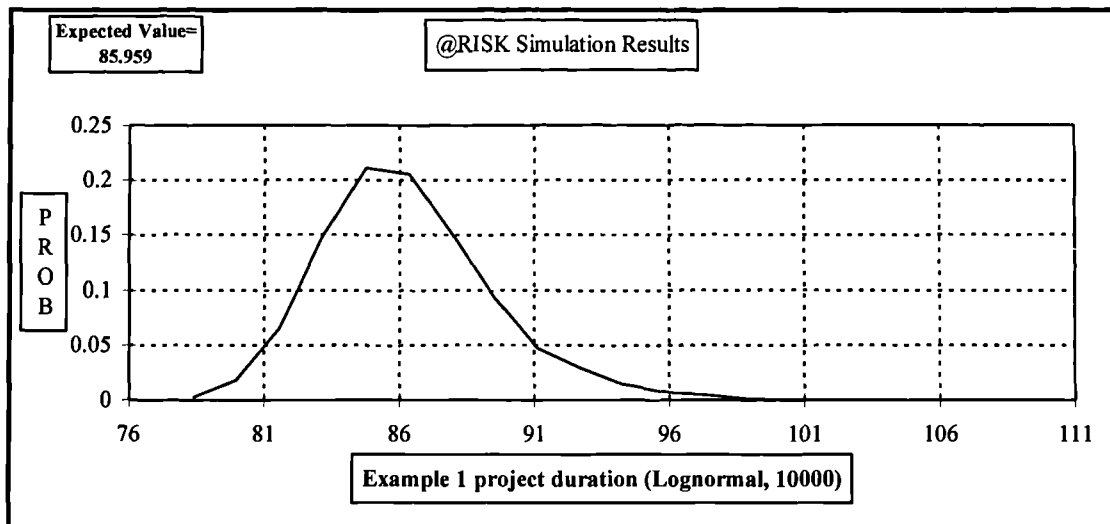
**Figure A.6a Total project duration of Example 3.1. Ascending cumulative curve.  
Log-Normal distribution, 1000 iterations**



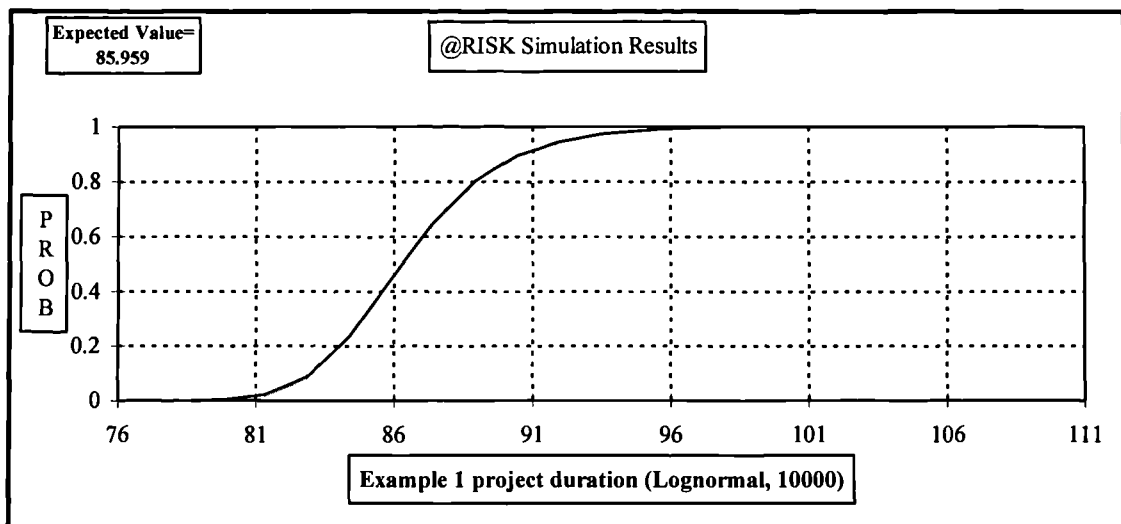
**Figure A.7 Total project duration of Example 3.1. Histogram in curve.  
Log-Normal distribution, 5000 iterations**



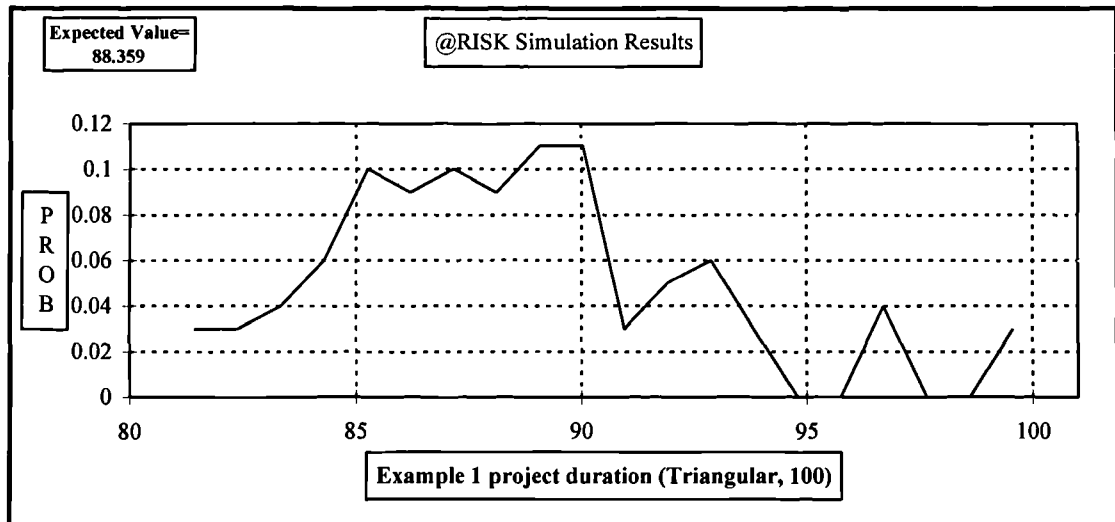
**Figure A.7a Total project duration of Example 3.1. Ascending cumulative curve.  
Log-Normal distribution, 5000 iterations**



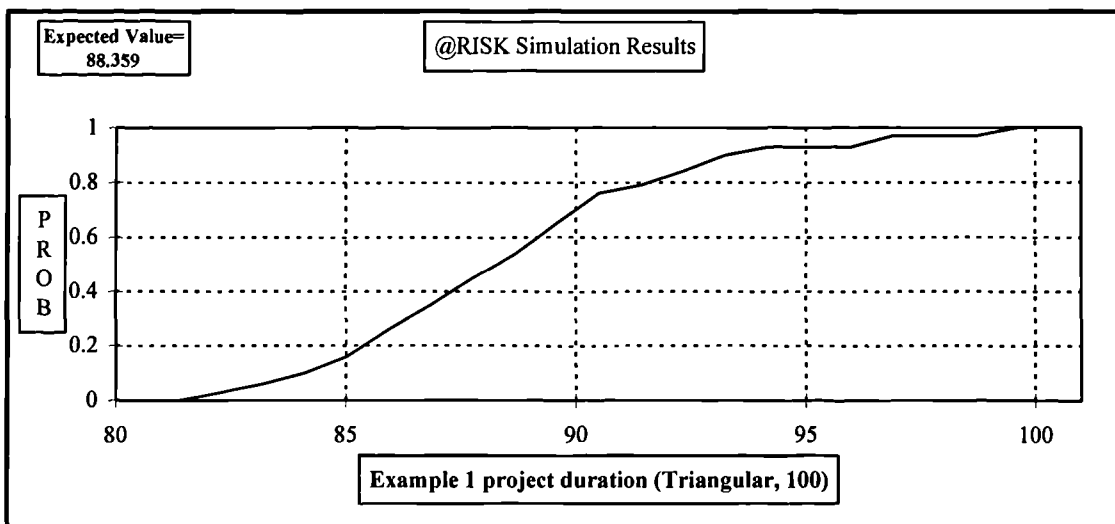
**Figure A.8 Total project duration of Example 3.1. Histogram in curve.  
Log-Normal distribution, 10000 iterations**



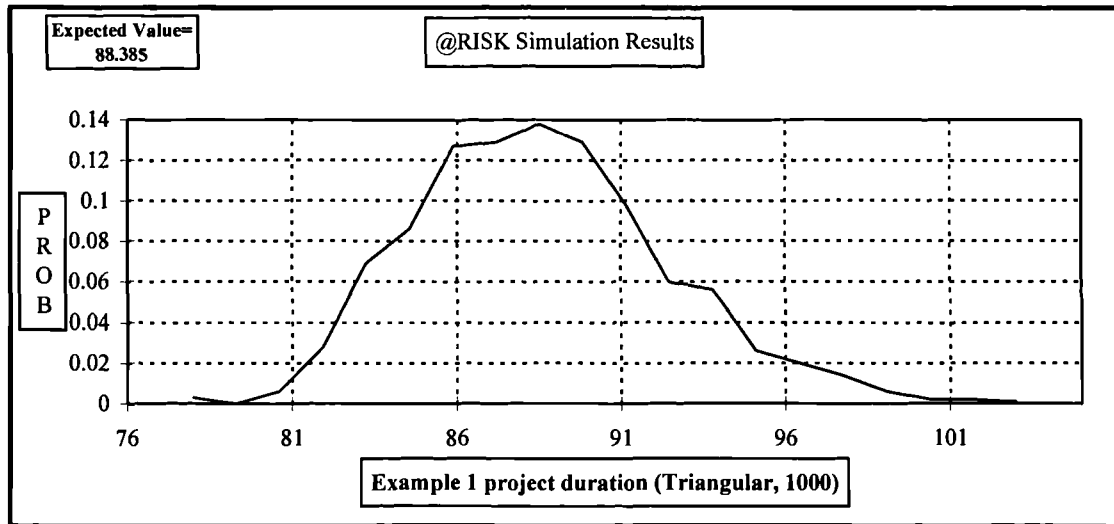
**Figure A.8a Total project duration of Example 3.1. Ascending cumulative curve.  
Log-Normal distribution, 10000 iterations**



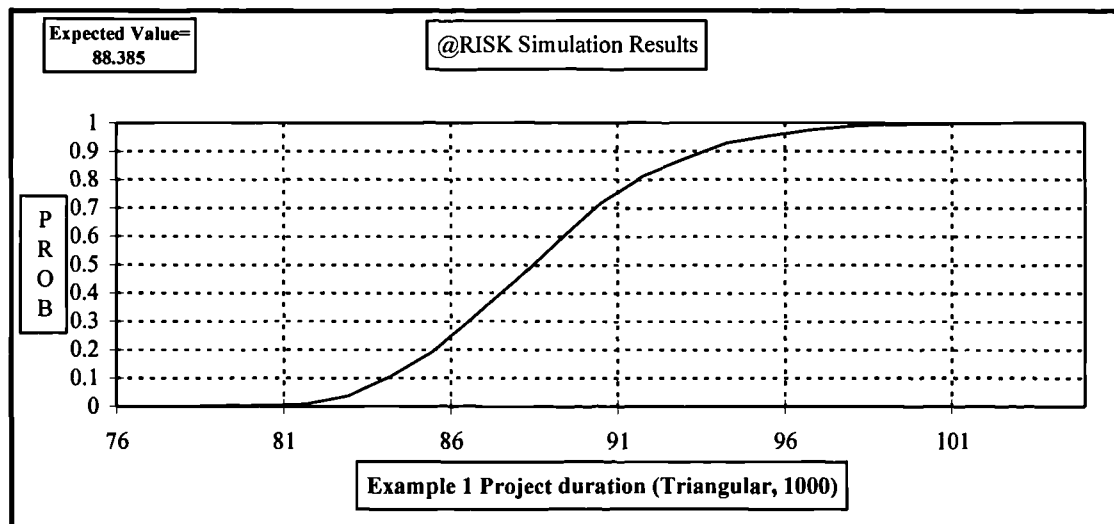
**Figure A.9 Total project duration of Example 3.1. Histogram in curve.  
Triangular distribution, 100 iterations**



**Figure A.9a Total project duration of Example 3.1. Ascending cumulative curve.  
Triangular distribution, 100 iterations**

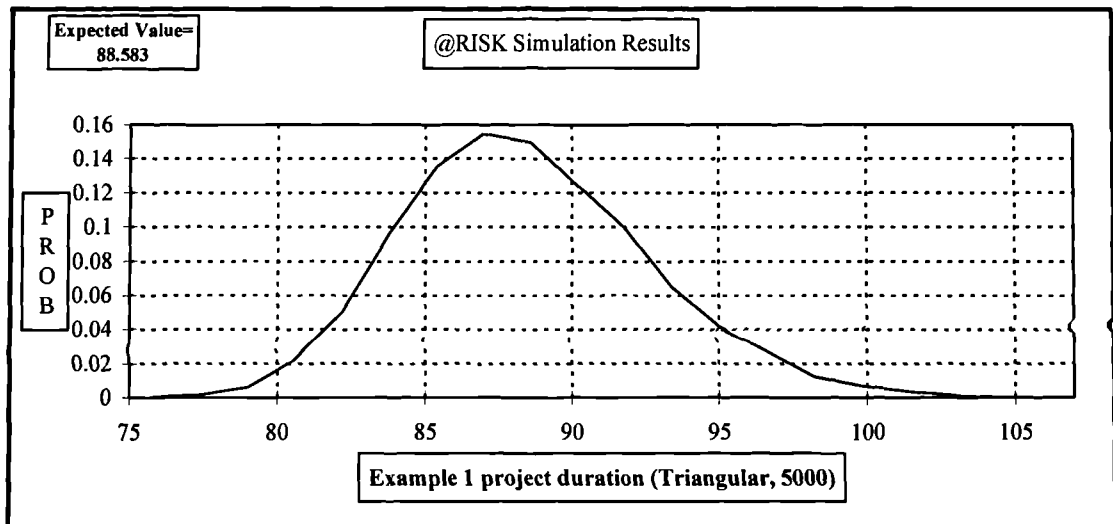


**Figure A.10 Total project duration of Example 3.1. Histogram in curve.**  
**Triangular distribution, 1000 iterations**

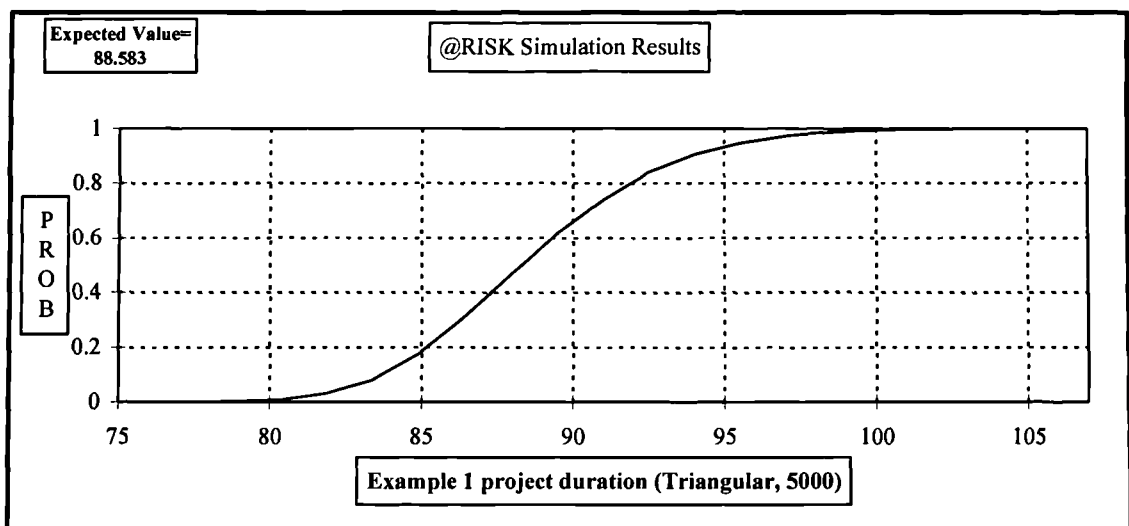


**Figure A.10a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Triangular distribution, 1000 iterations**

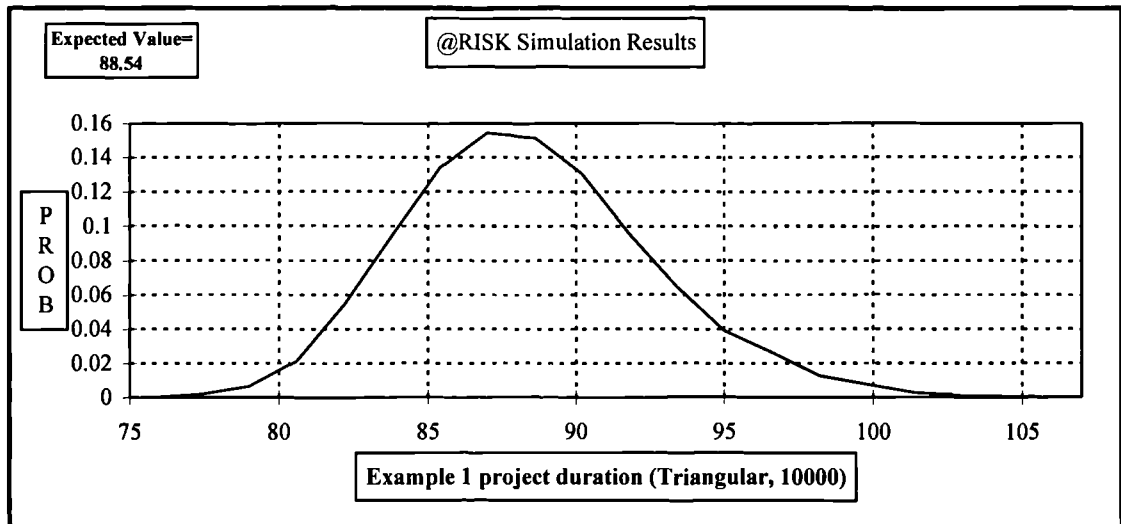




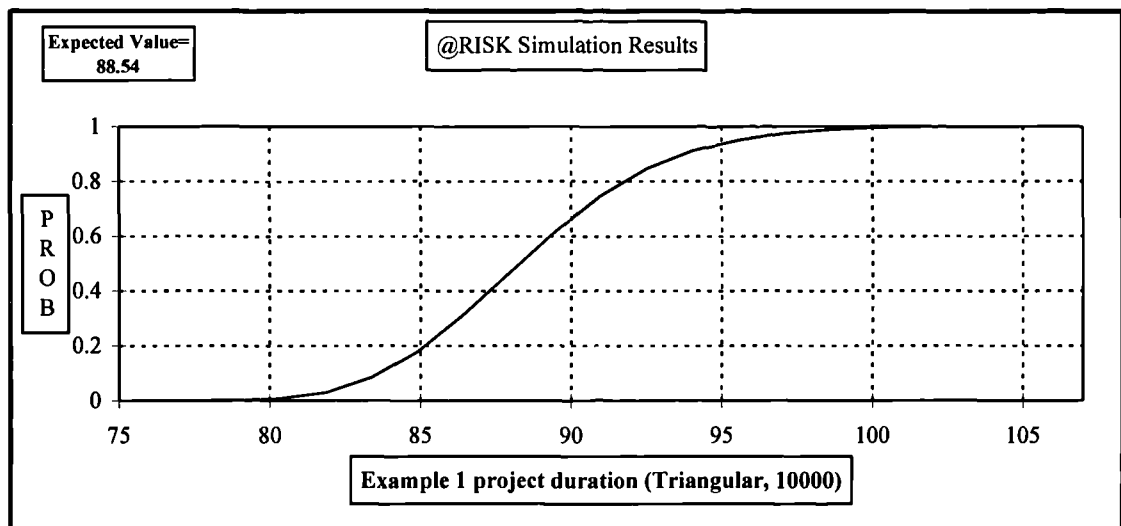
**Figure A.11 Total project duration of Example 3.1. Histogram in curve.  
Triangular distribution, 5000 iterations**



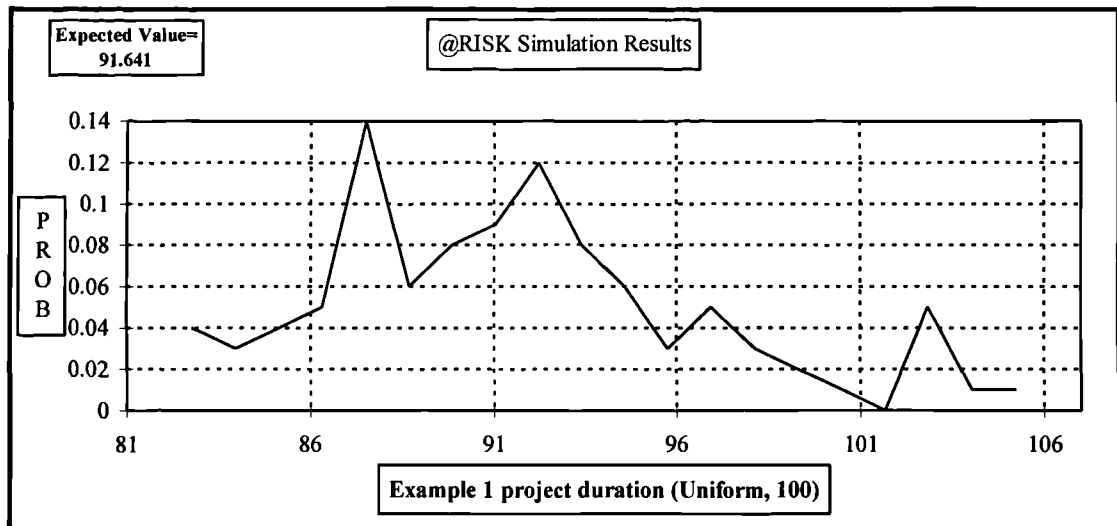
**Figure A.11a Total project duration of Example 3.1. Ascending cumulative curve.  
Triangular distribution, 5000 iterations**



**Figure A.12 Total project duration of Example 3.1. Histogram in curve.**  
**Triangular distribution, 10000 iterations**

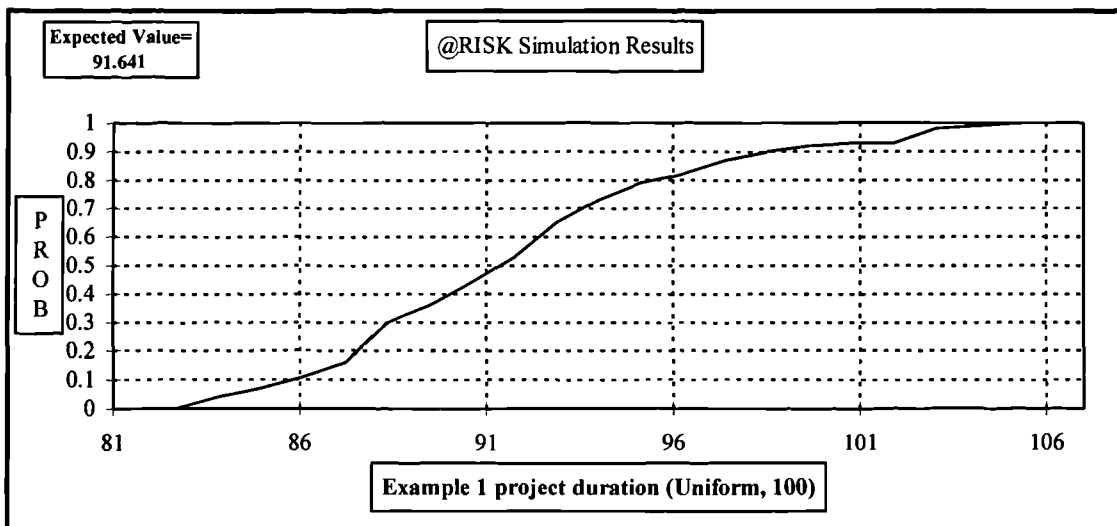


**Figure A.12a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Triangular distribution, 10000 iterations**



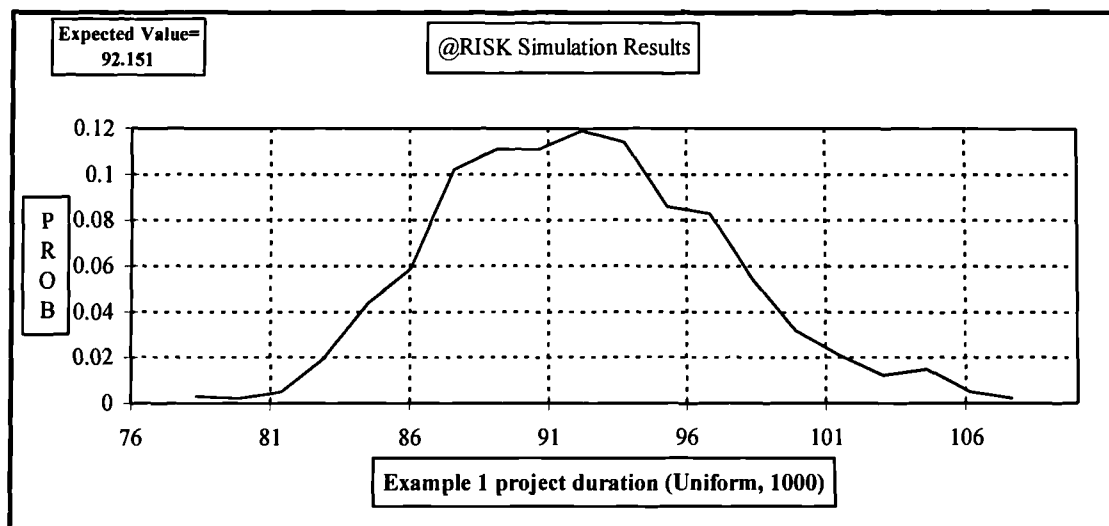
**Figure A.13 Total project duration of Example 3.1. Histogram in curve.**

**Uniform distribution, 100 iterations**

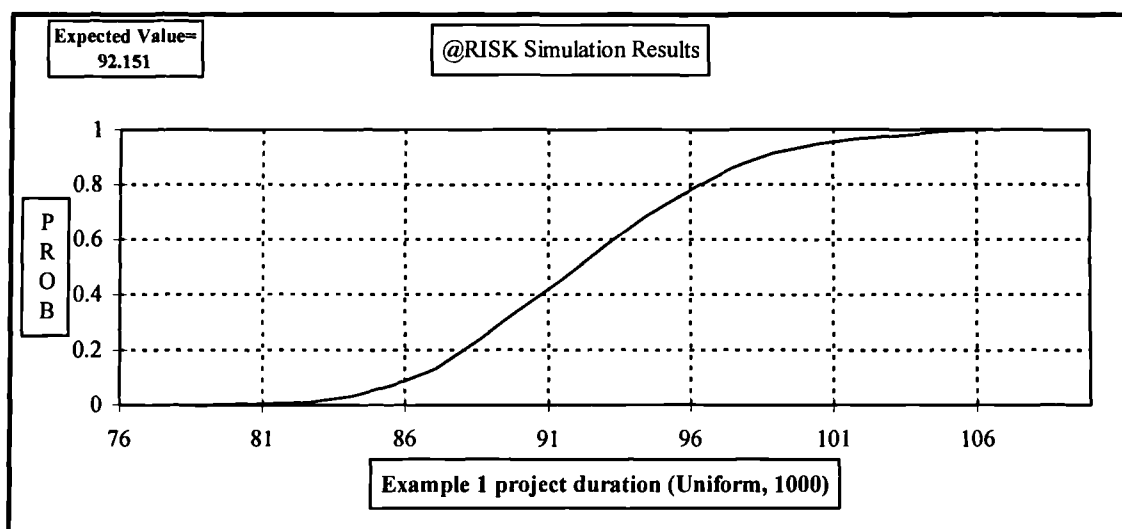


**Figure A.13a Total project duration of Example 3.1. Ascending cumulative curve.**

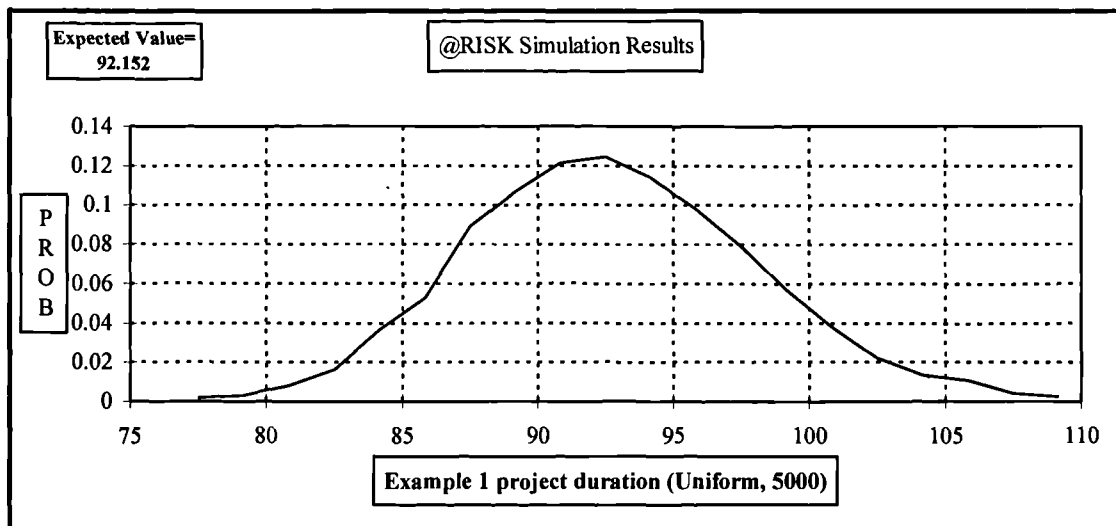
**Uniform distribution, 100 iterations**



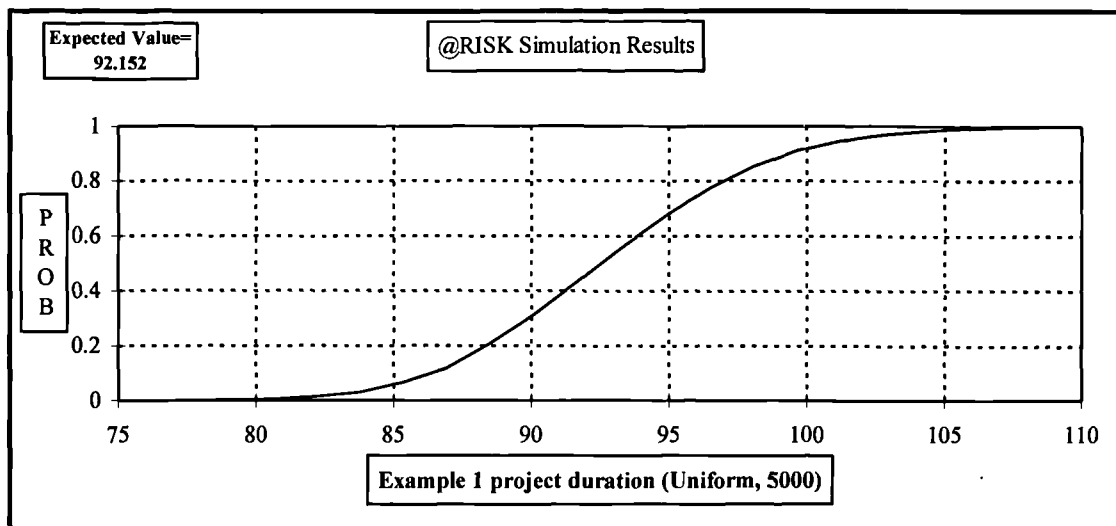
**Figure A.14 Total project duration of Example 3.1. Histogram in curve.**  
**Uniform distribution, 1000 iterations**



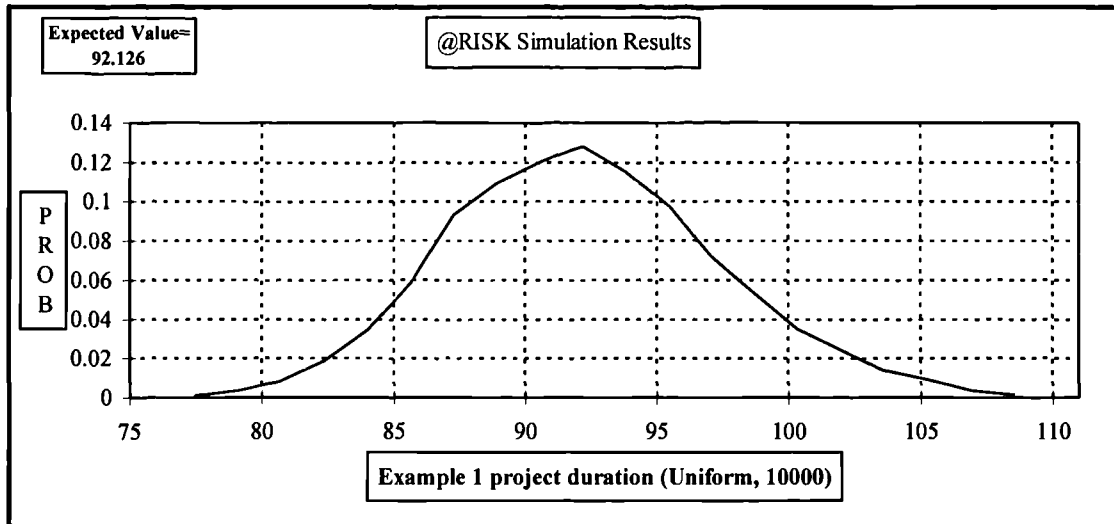
**Figure A.14a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Uniform distribution, 1000 iterations**



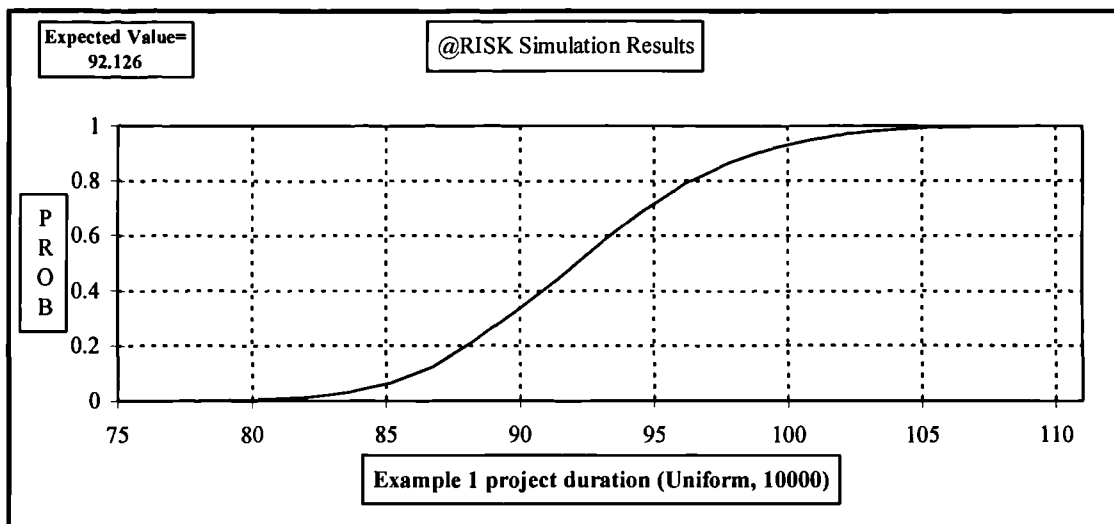
**Figure A.15 Total project duration of Example 3.1. Histogram in curve.**  
**Uniform distribution, 5000 iterations**



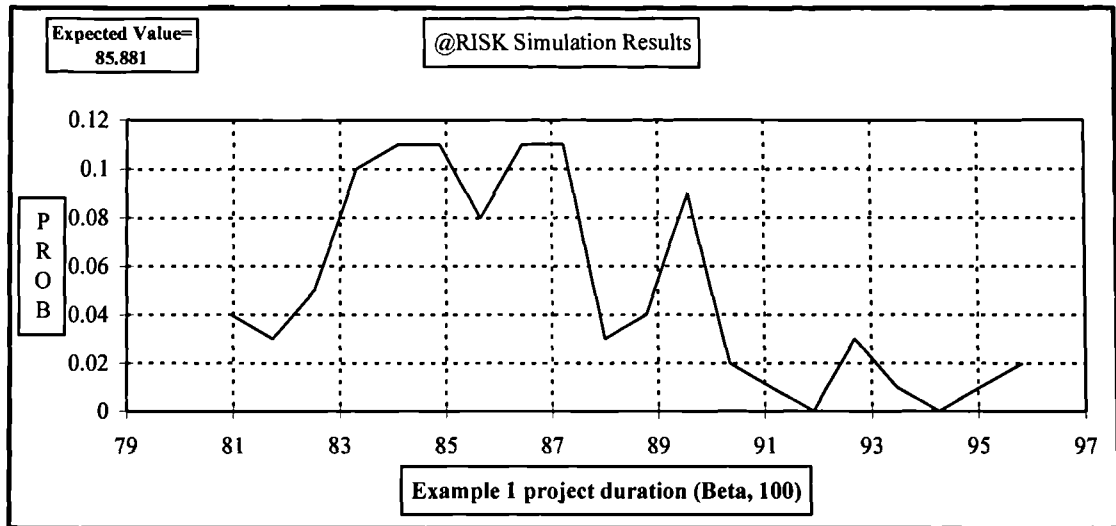
**Figure A.15a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Uniform distribution, 5000 iterations**



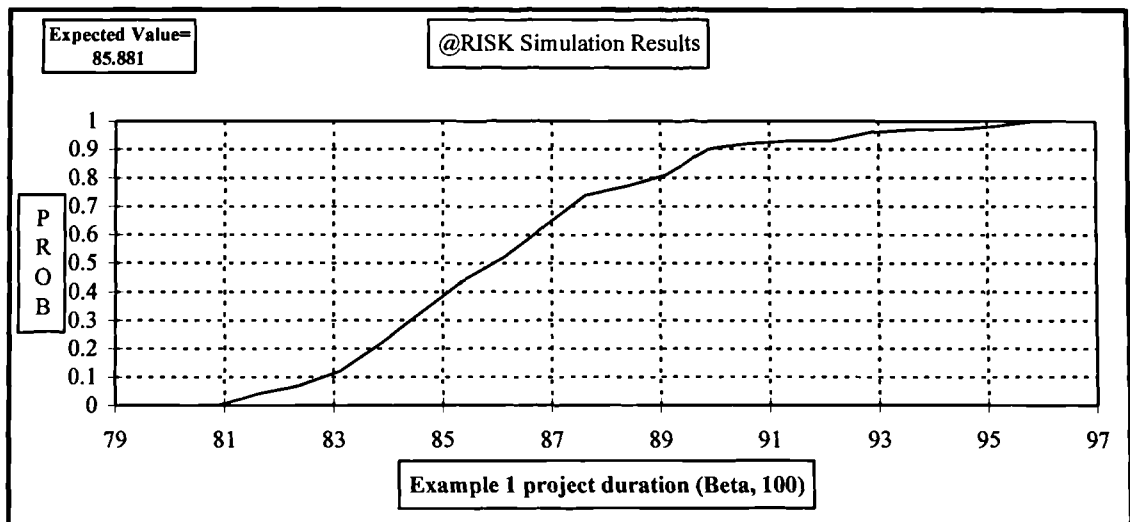
**Figure A.16 Total project duration of Example 3.1. Histogram in curve.**  
**Uniform distribution, 10000 iterations**



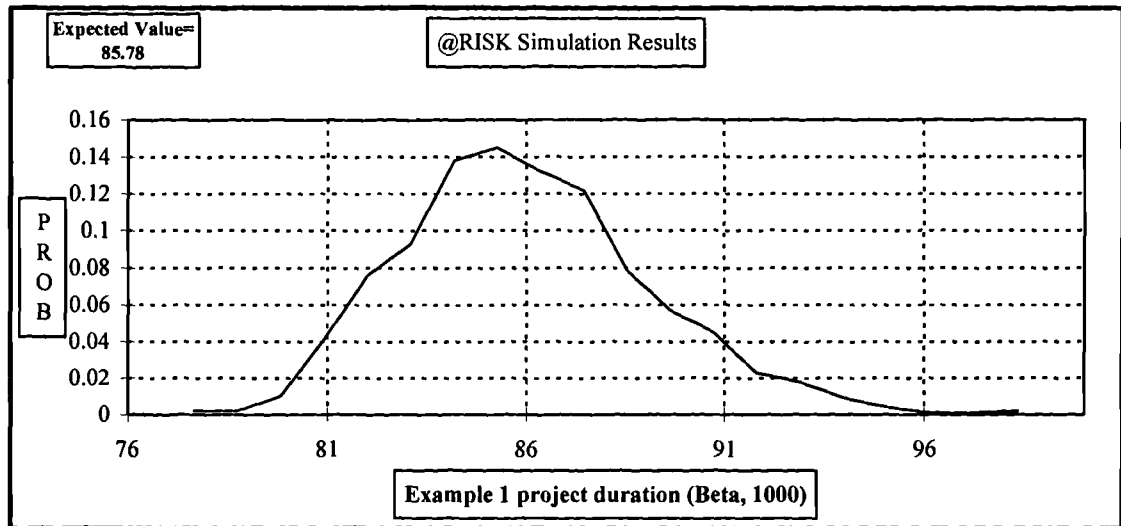
**Figure A.16a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Uniform distribution, 10000 iterations**



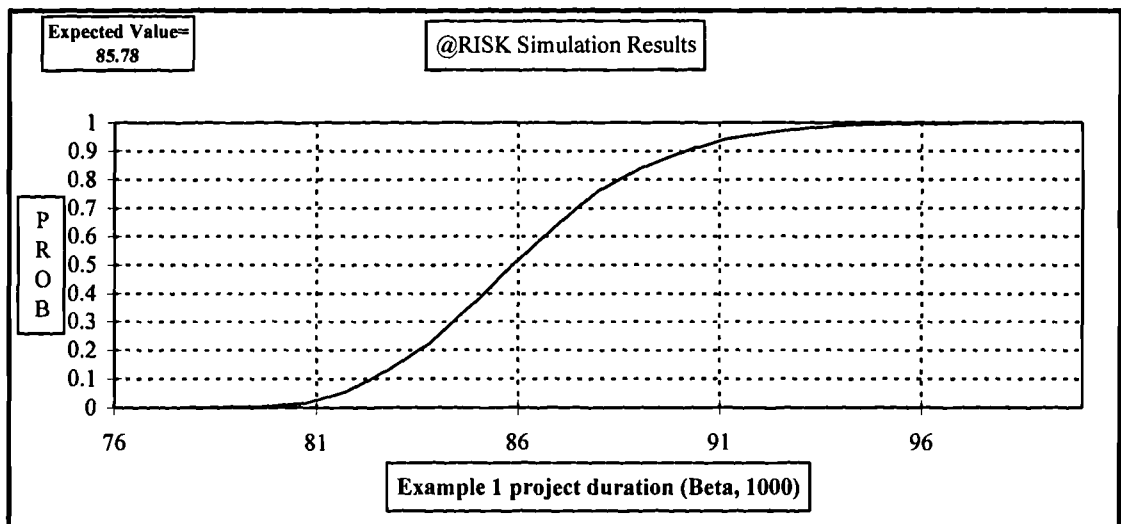
**Figure A.17 Total project duration of Example 3.1. Histogram in curve.  
Beta distribution, 100 iterations**



**Figure A.17a Total project duration of Example 3.1. Ascending cumulative curve.  
Beta distribution, 100 iterations**

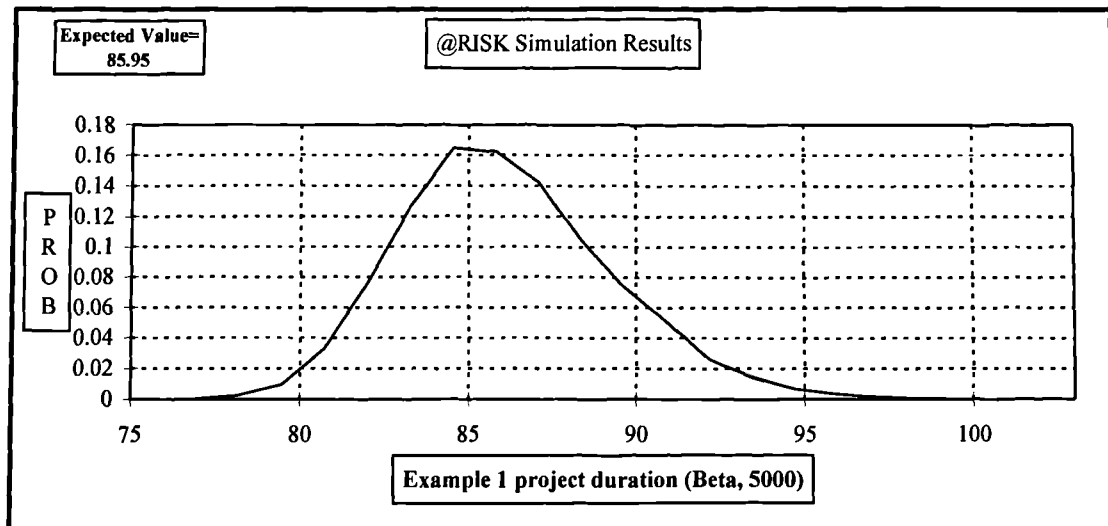


**Figure A.18 Total project duration of Example 3.1. Histogram in curve.  
Beta distribution, 1000 iterations**

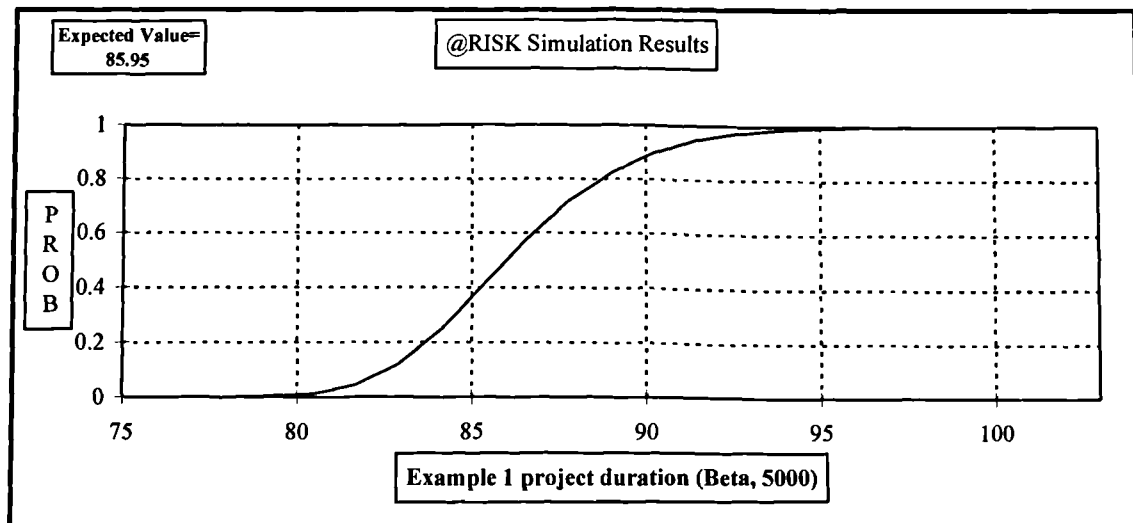


**Figure A.18a Total project duration of Example 3.1. Ascending cumulative curve.  
Beta distribution, 1000 iterations**

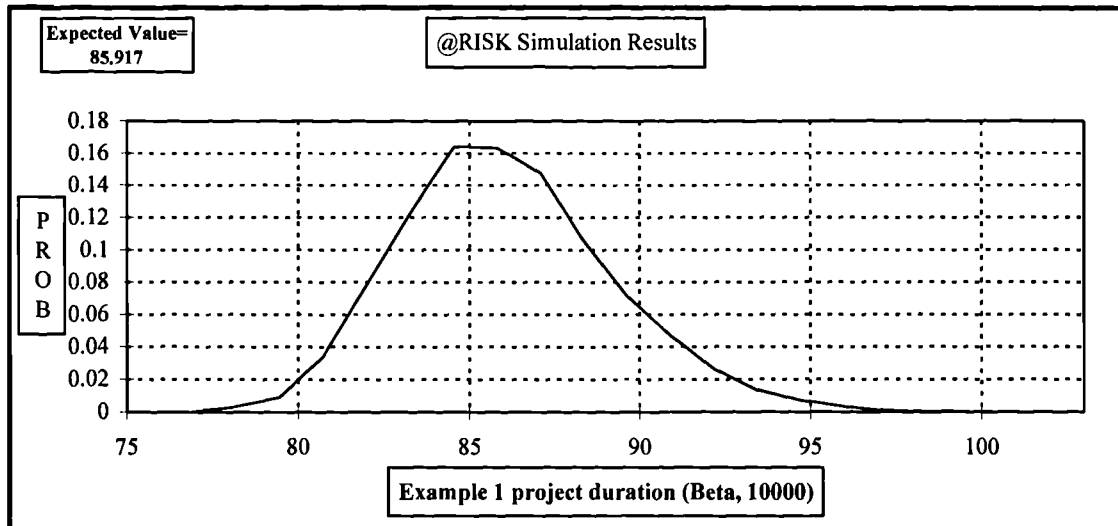




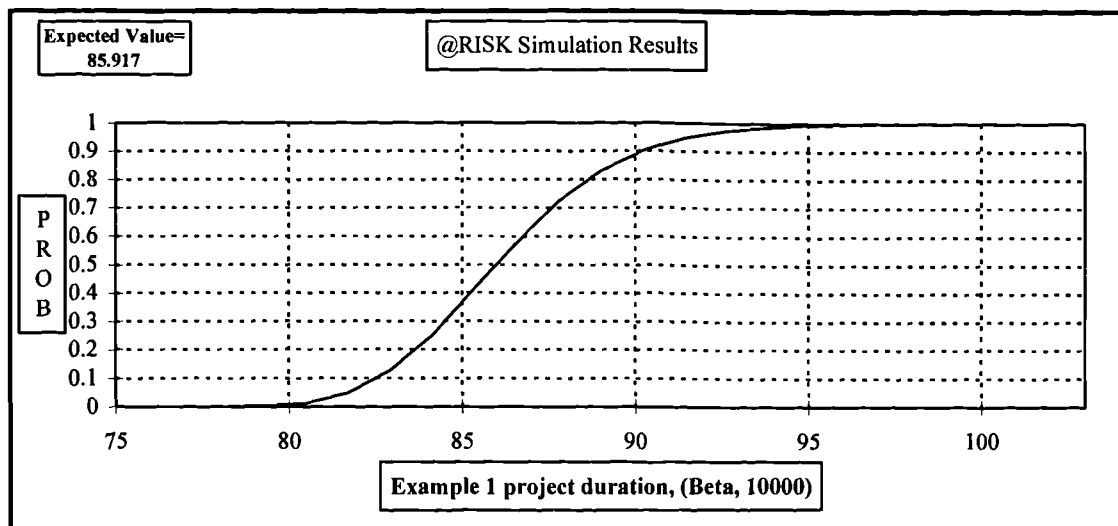
**Figure A.19 Total project duration of Example 3.1. Histogram in curve.  
Beta distribution, 5000 iterations**



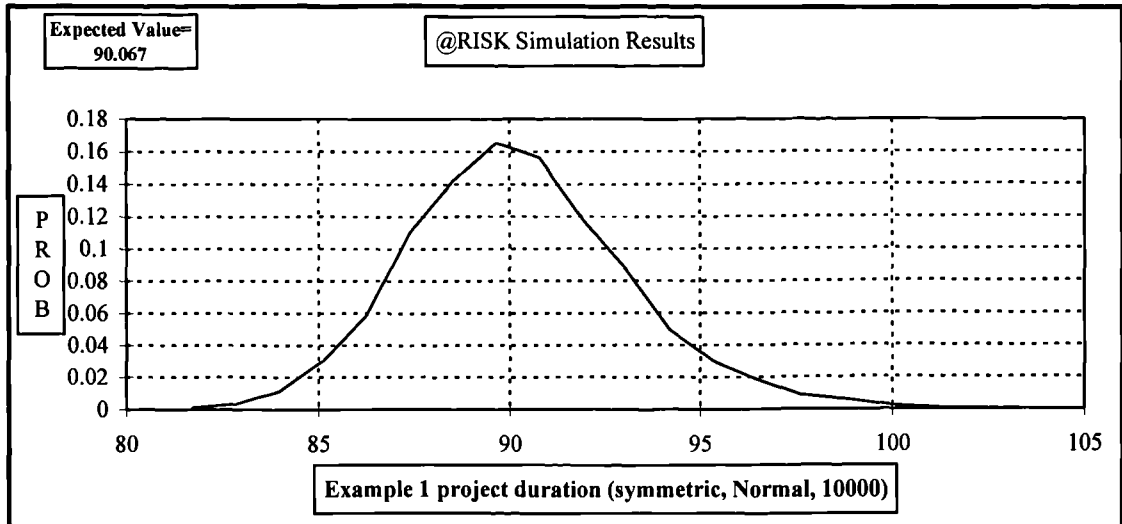
**Figure A.19a Total project duration of Example 3.1. Ascending cumulative curve.  
Beta distribution, 5000 iterations**



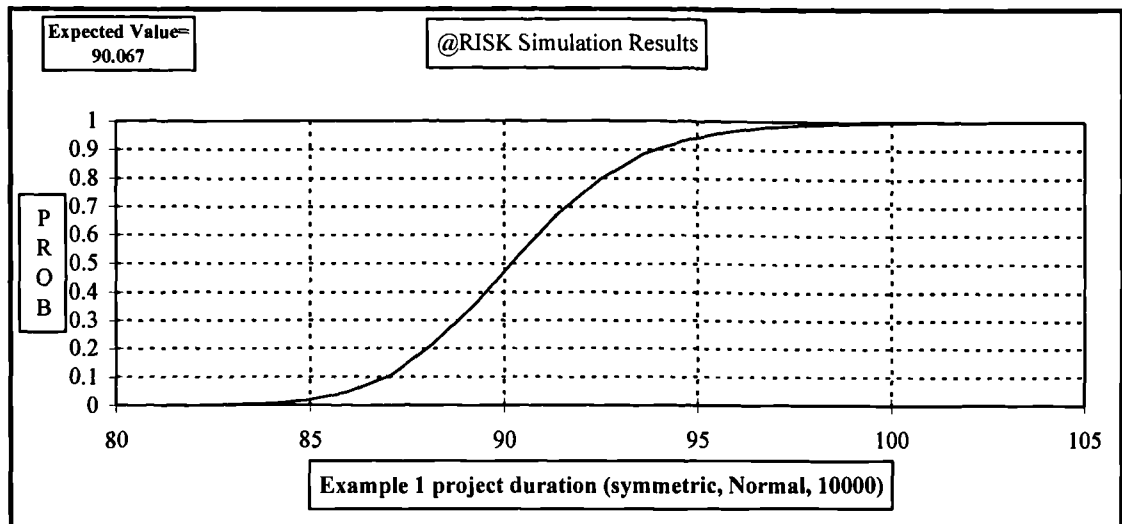
**Figure A.20 Total project duration of Example 3.1. Histogram in curve.  
Beta distribution, 10000 iterations**



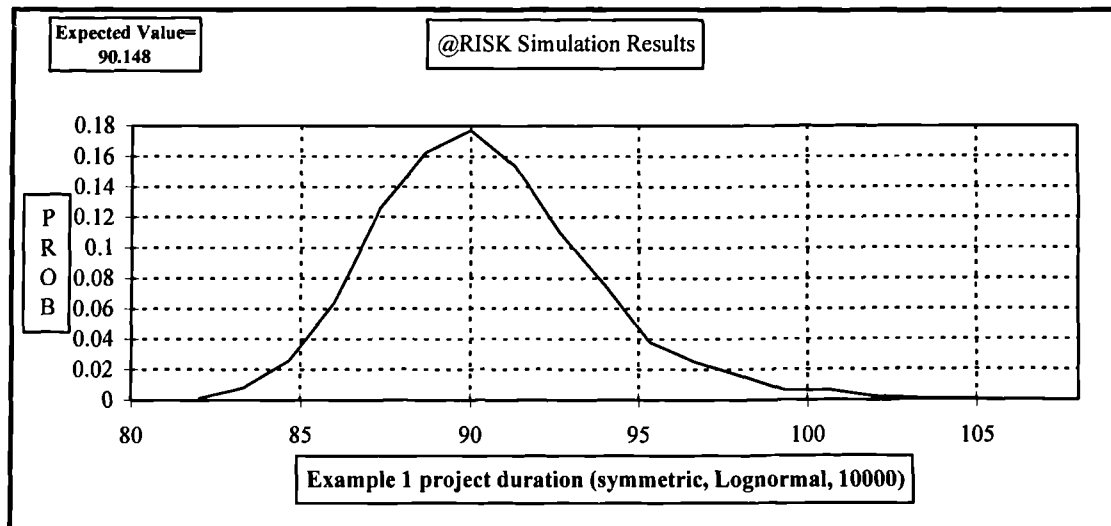
**Figure A.20a Total project duration of Example 3.1. Ascending cumulative curve.  
Beta distribution, 10000 iterations**



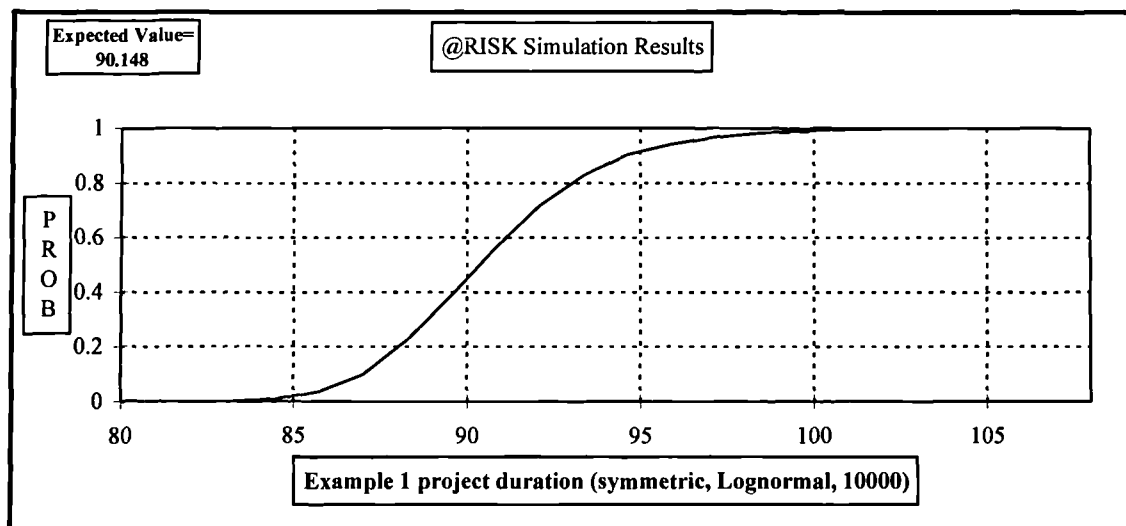
**Figure A.21 Total project duration of Example 3.1. Histogram in curve.**  
**Symmetric Normal distribution, 10000 iterations**



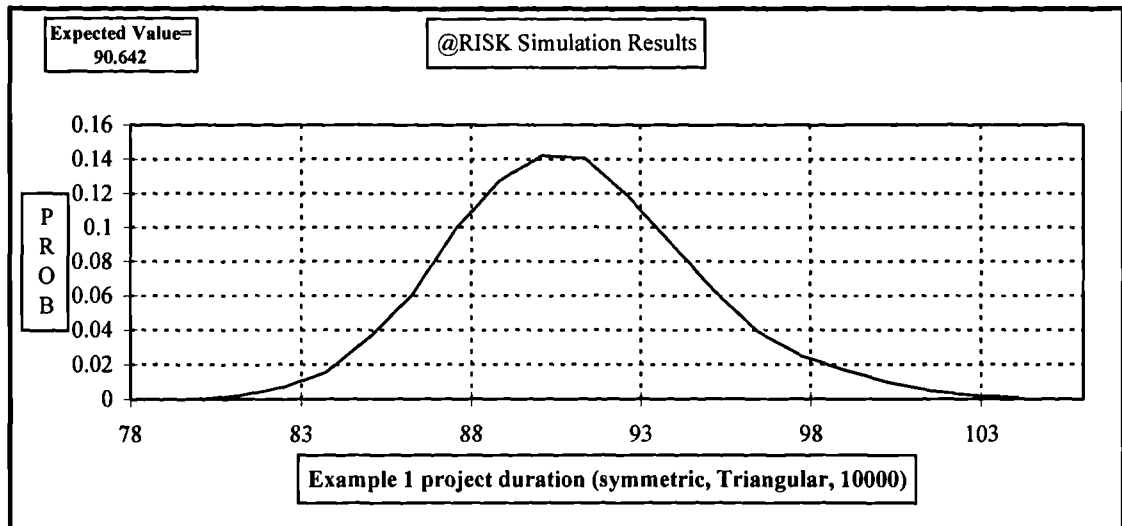
**Figure A.21a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Symmetric Normal distribution, 10000 iterations**



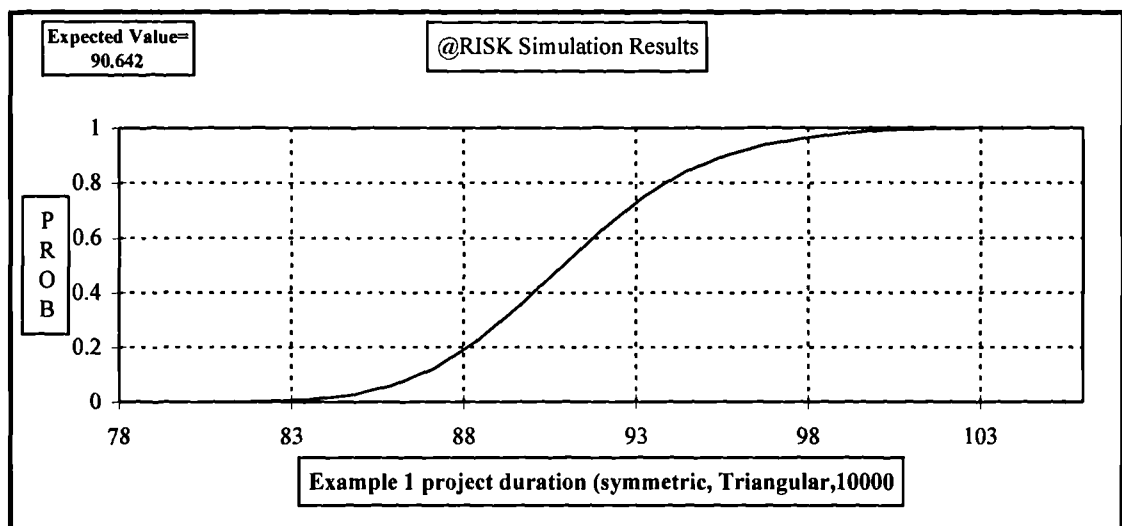
**Figure A.22 Total project duration of Example 3.1. Histogram in curve.  
Symmetric Log-Normal distribution, 10000 iterations**



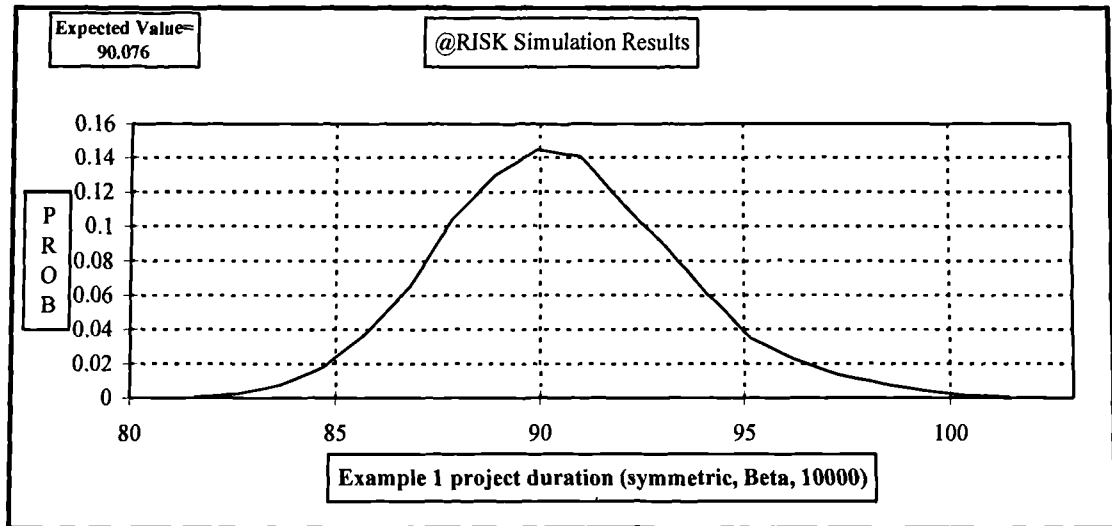
**Figure A.22a Total project duration of Example 3.1. Ascending cumulative curve.  
Symmetric Log-Normal distribution, 10000 iterations**



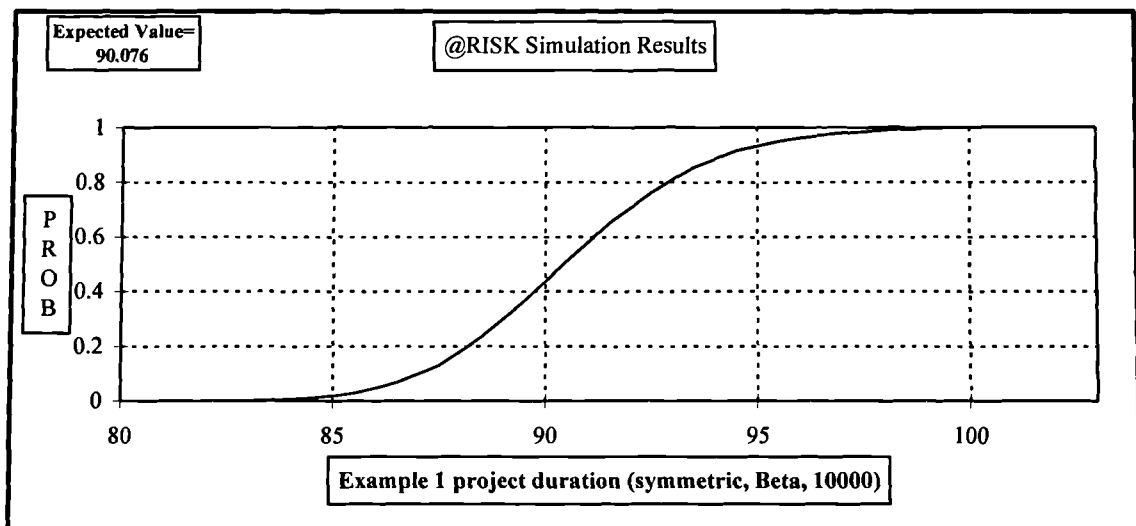
**Figure A.23 Total project duration of Example 3.1. Histogram in curve.  
Symmetric Triangular distribution, 10000 iterations**



**Figure A.23a Total project duration of Example 3.1. Ascending cumulative curve.  
Symmetric Triangular distribution, 10000 iterations**



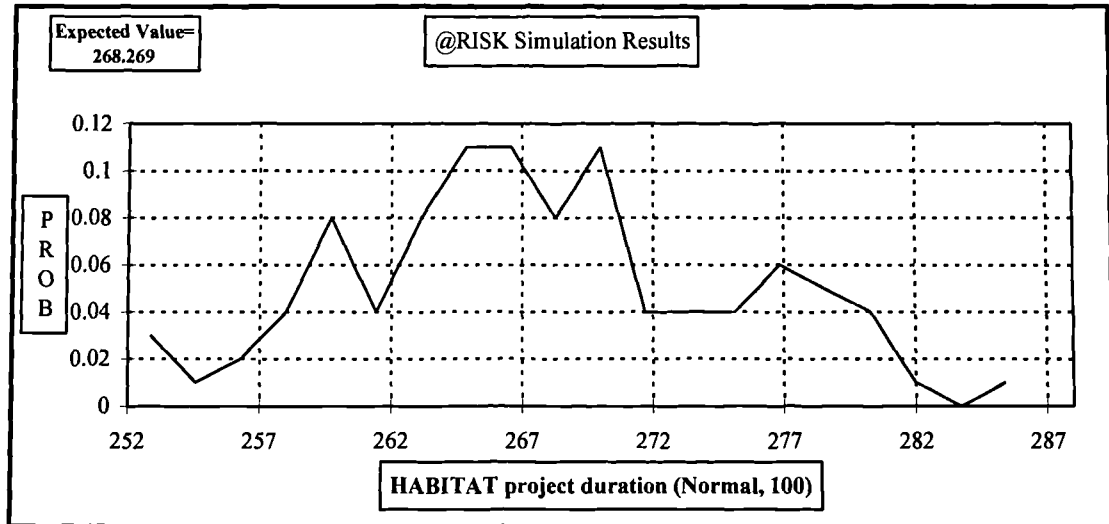
**Figure A.24 Total project duration of Example 3.1. Histogram in curve.**  
**Symmetric Beta distribution, 10000 iterations**



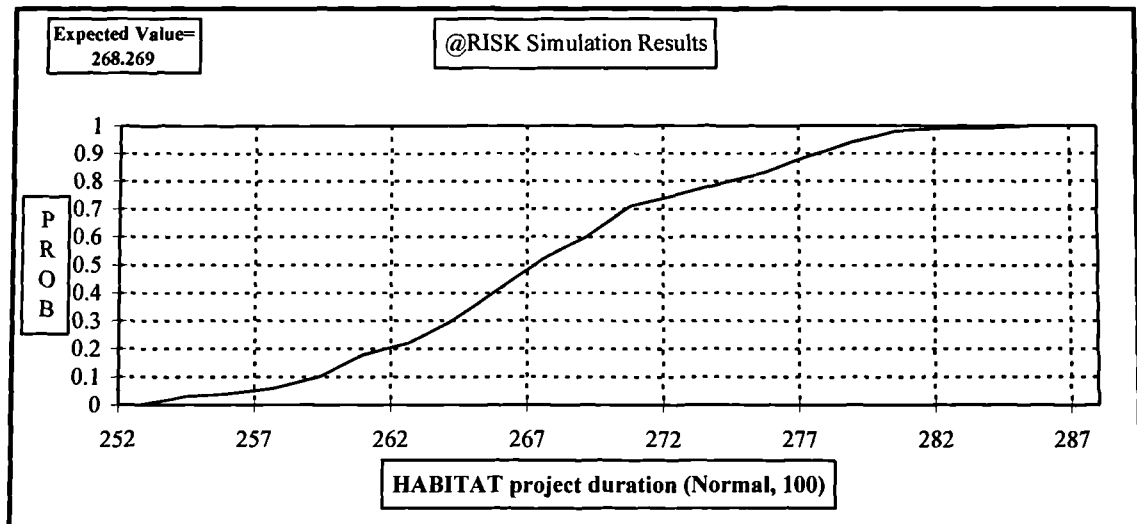
**Figure A.24a Total project duration of Example 3.1. Ascending cumulative curve.**  
**Symmetric Beta distribution, 10000 iterations**

## APPENDIX B

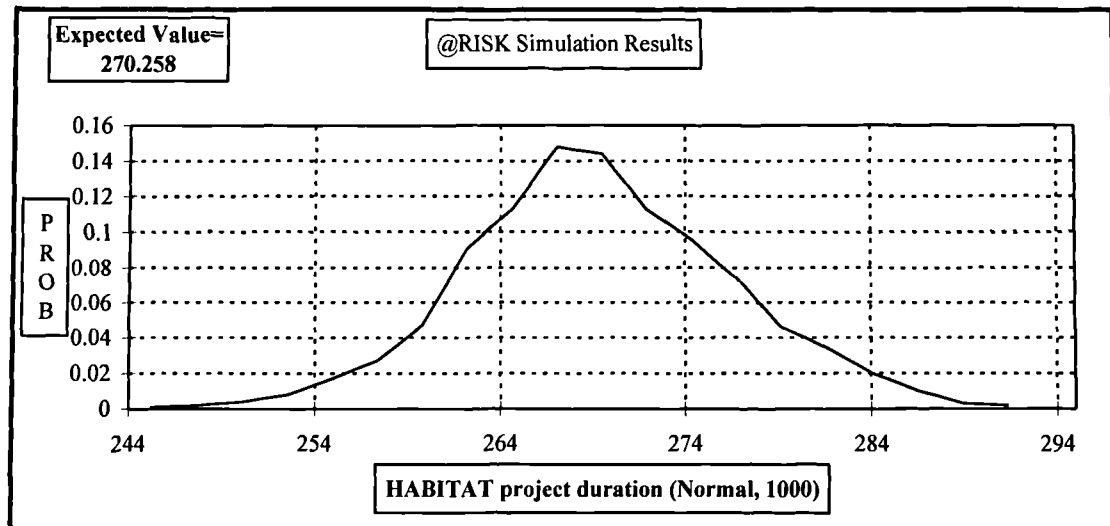
### RESULTS OF MCS FOR EXAMPLE 3.2



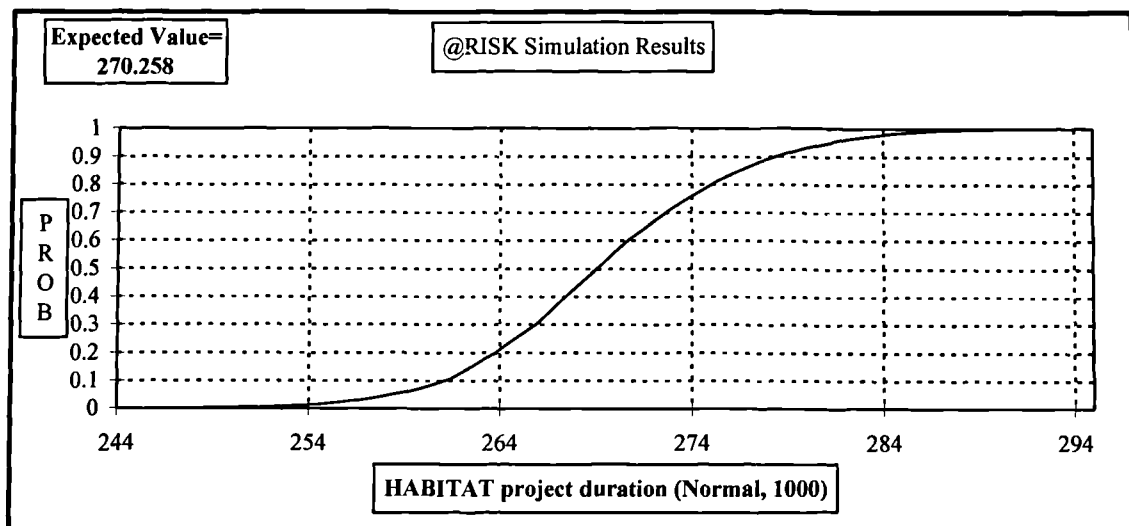
**Figure B.1 Total project duration of Example 3.2. Histogram in curve.**  
**Normal distribution, 100 iterations**



**Figure B.1a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Normal distribution, 100 iterations**

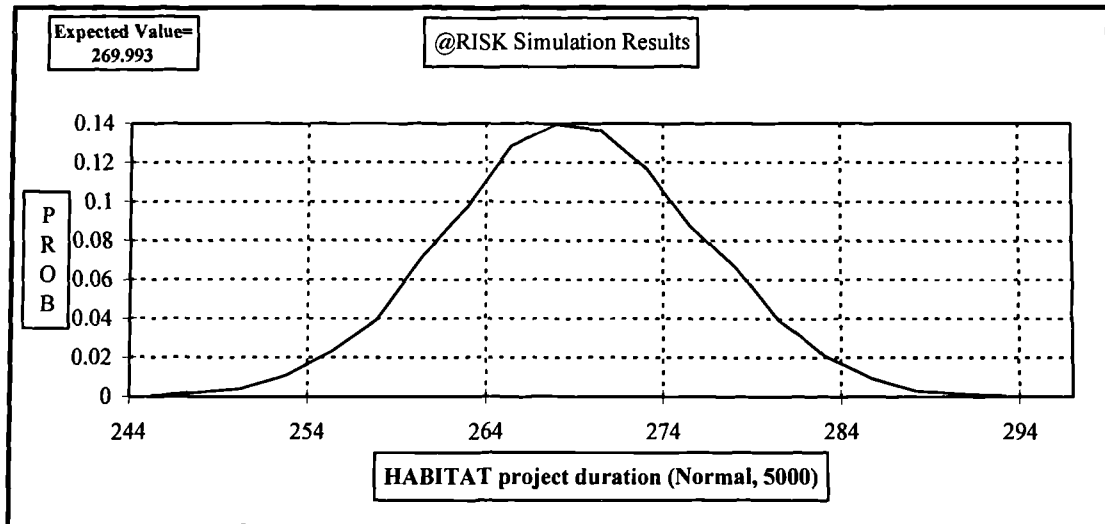


**Figure B.2 Total project duration of Example 3.2. Histogram in curve.**  
**Normal distribution, 1000 iterations**

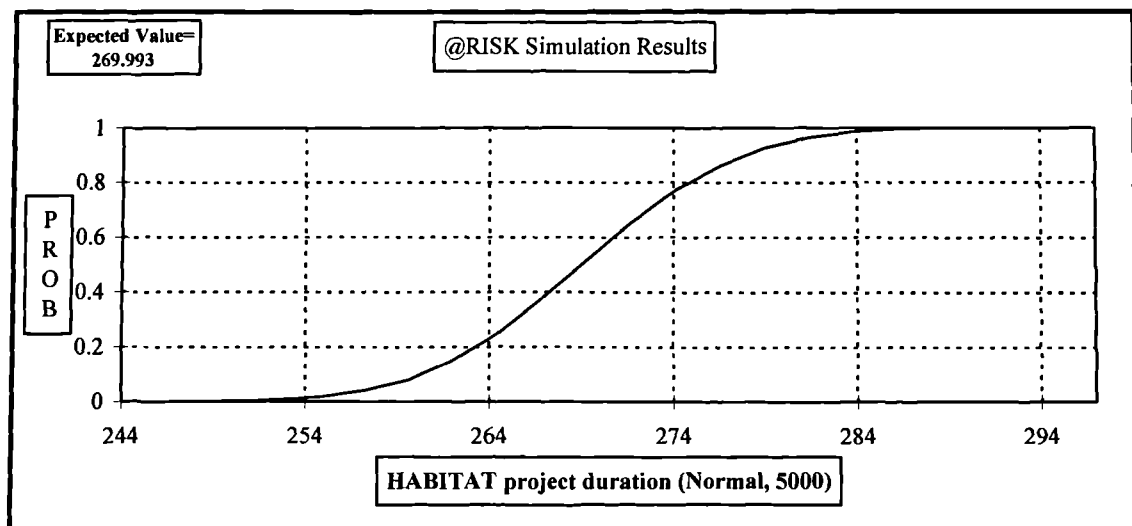


**Figure B.2a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Normal distribution, 1000 iterations**

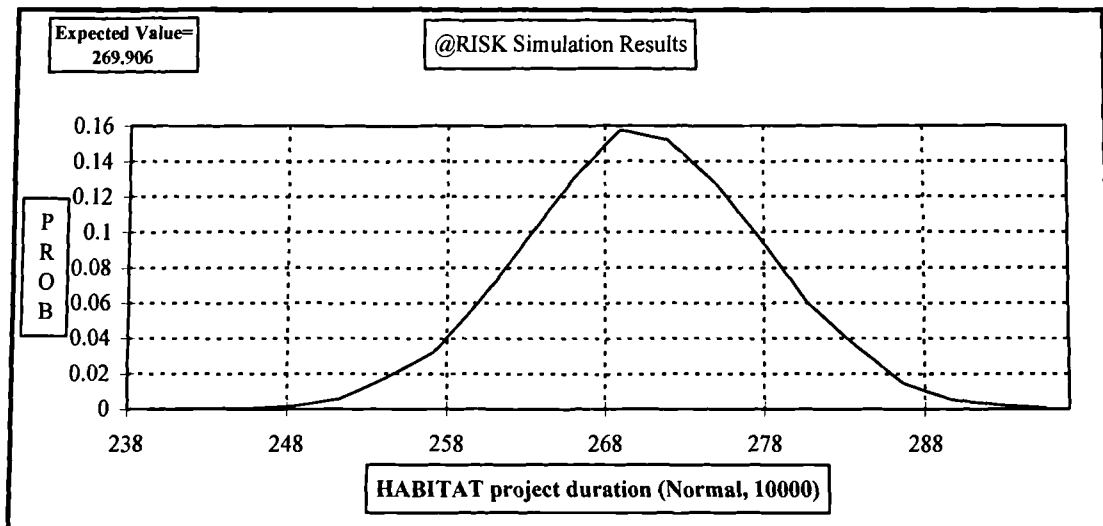




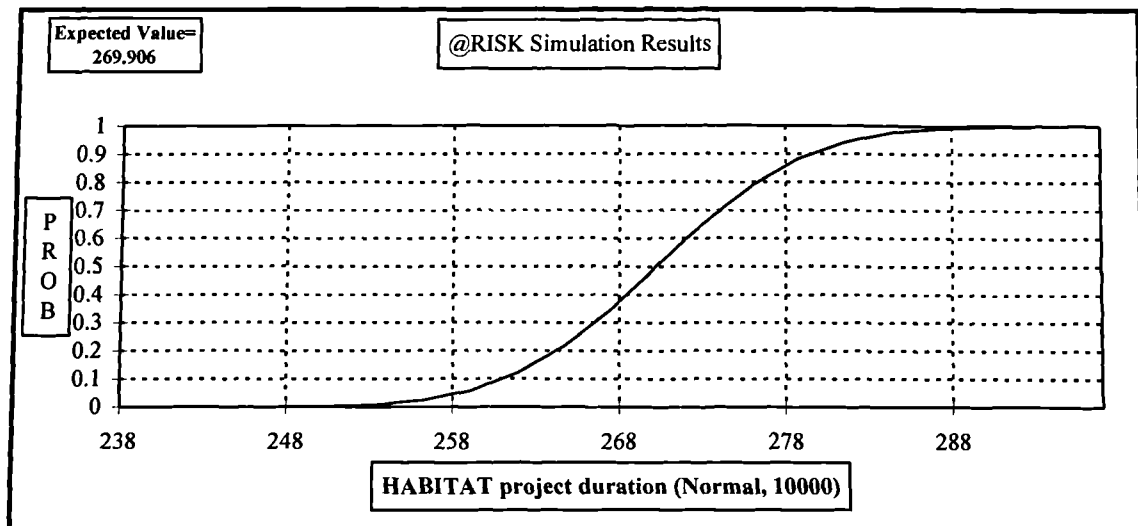
**Figure B.3 Total project duration of Example 3.2. Histogram in curve.**  
**Normal distribution, 5000 iterations**



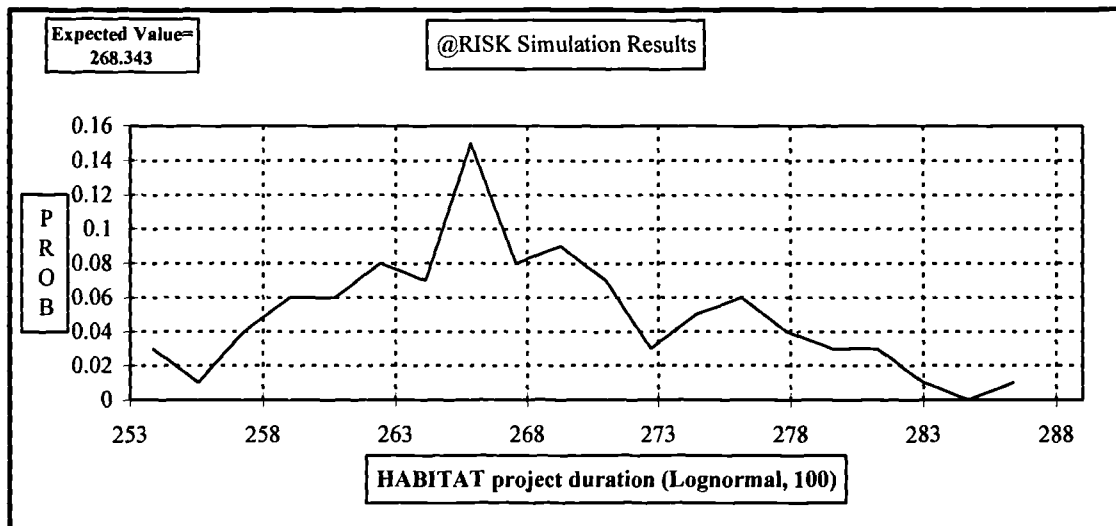
**Figure B.3a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Normal distribution, 5000 iterations**



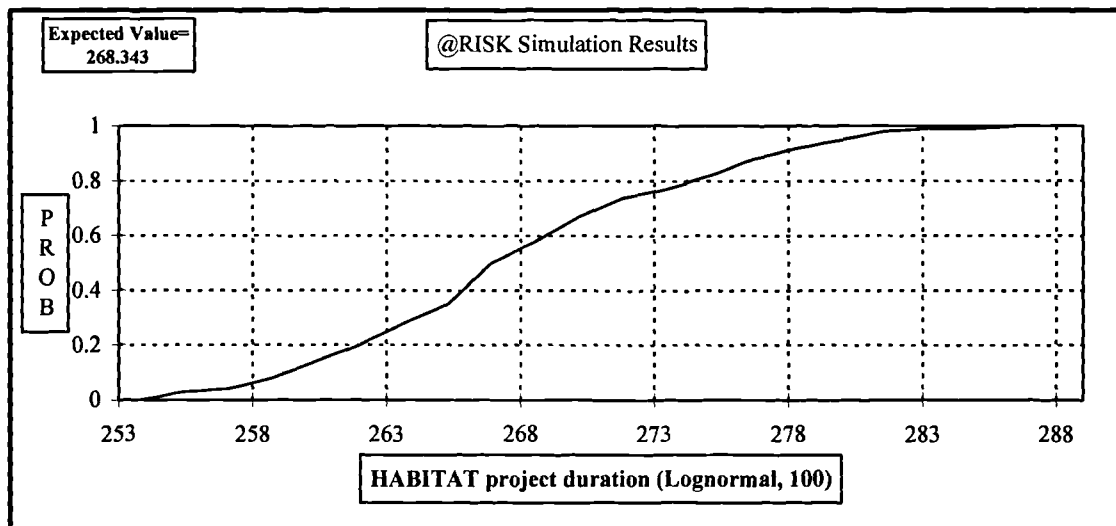
**Figure B.4 Total project duration of Example 3.2. Histogram in curve.**  
**Normal distribution, 10000 iterations**



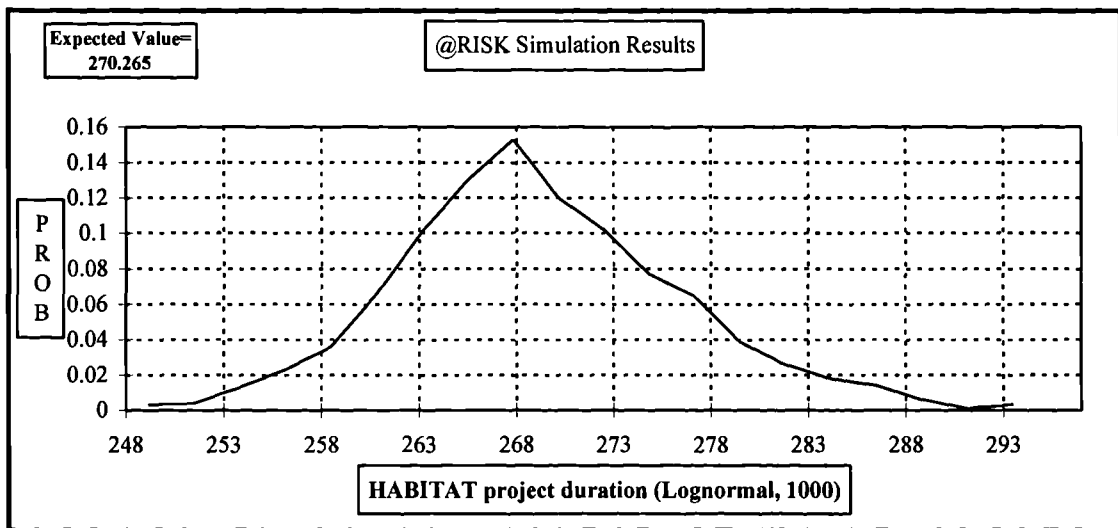
**Figure B.4a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Normal distribution, 10000 iterations**



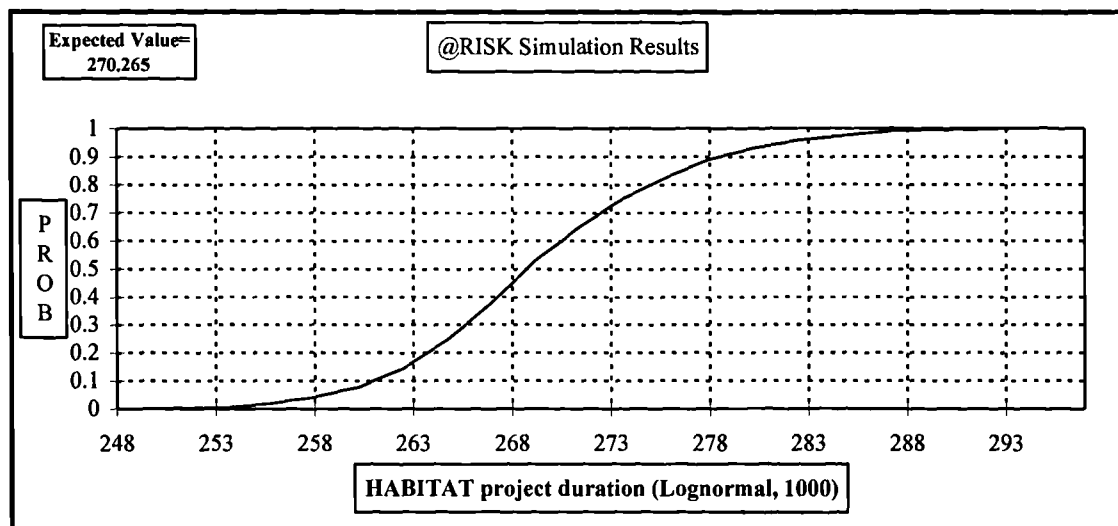
**Figure B.5 Total project duration of Example 3.2. Histogram in curve.**  
**Log-Normal distribution, 100 iterations**



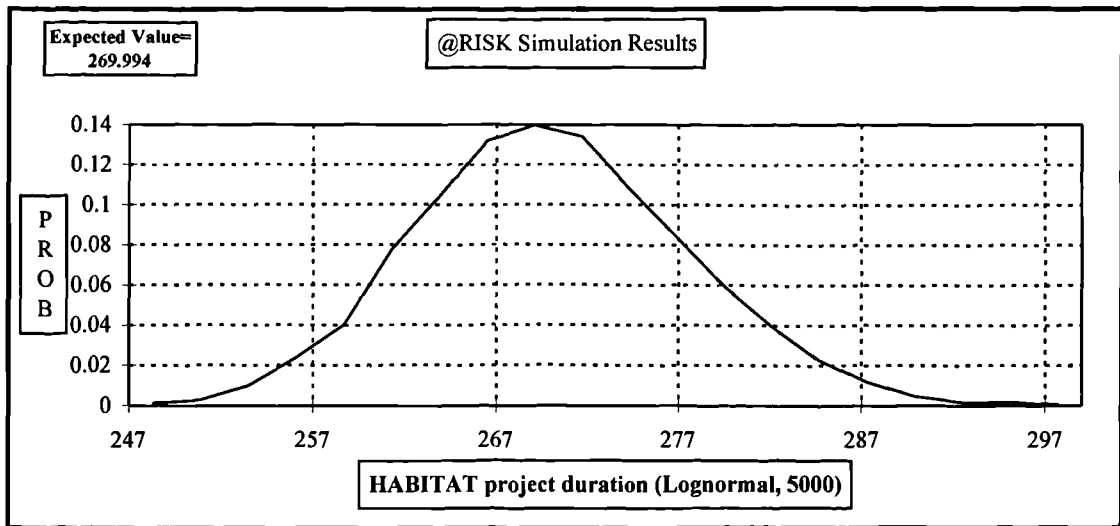
**Figure B.5a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Log-Normal distribution, 100 iterations**



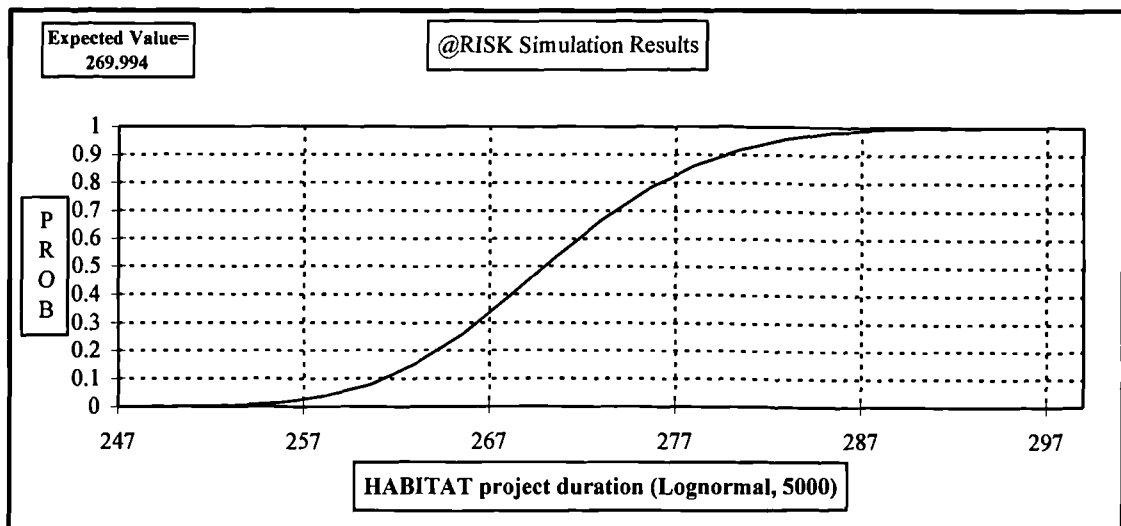
**Figure B.6 Total project duration of Example 3.2. Histogram in curve.  
Log-Normal distribution, 1000 iterations**



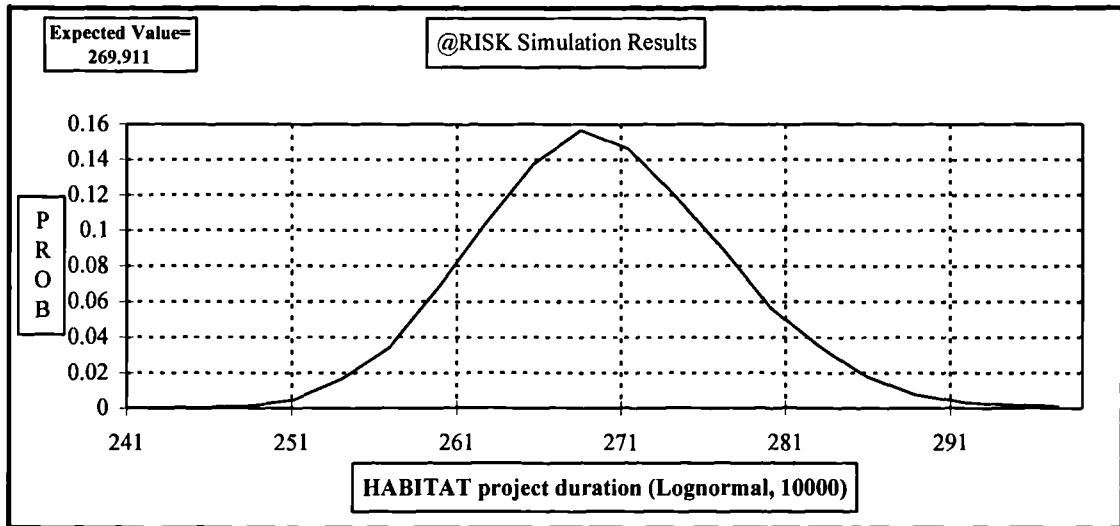
**Figure B.6a Total project duration of Example 3.2. Ascending cumulative curve.  
Log-Normal distribution, 1000 iterations**



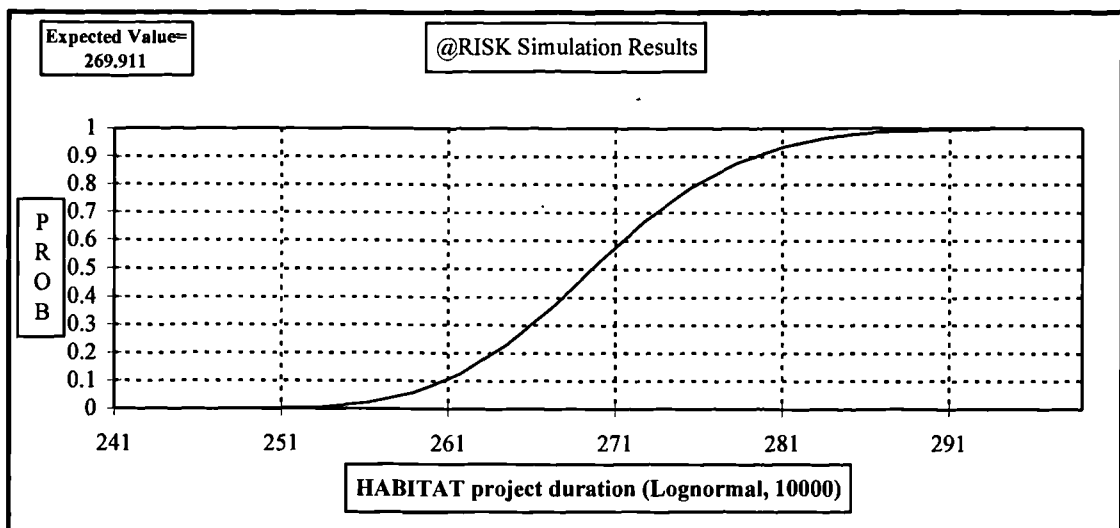
**Figure B.7 Total project duration of Example 3.2. Histogram in curve.  
Log-Normal distribution, 5000 iterations**



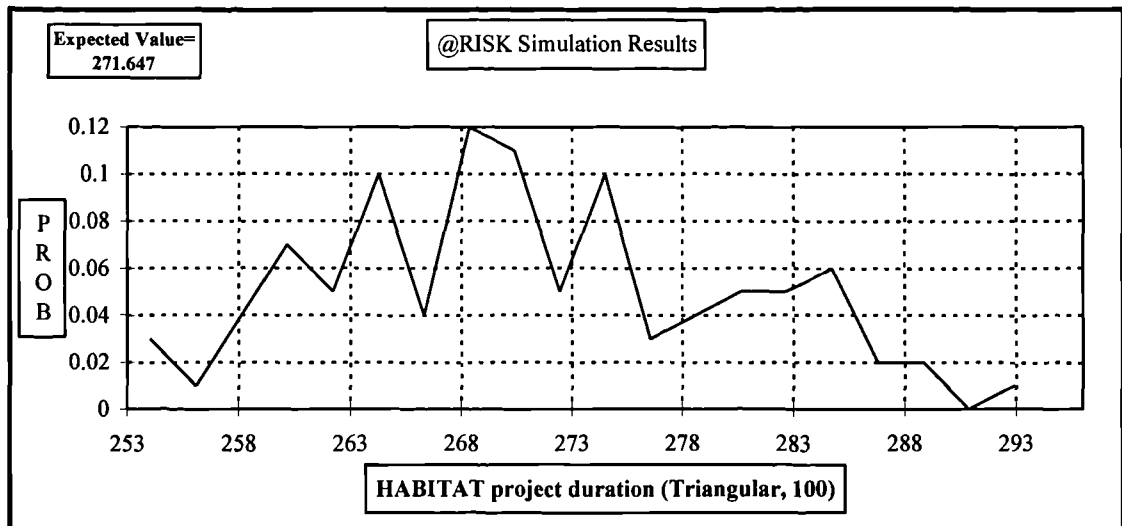
**Figure B.7a Total project duration of Example 3.2. Ascending cumulative curve.  
Log-Normal distribution, 5000 iterations**



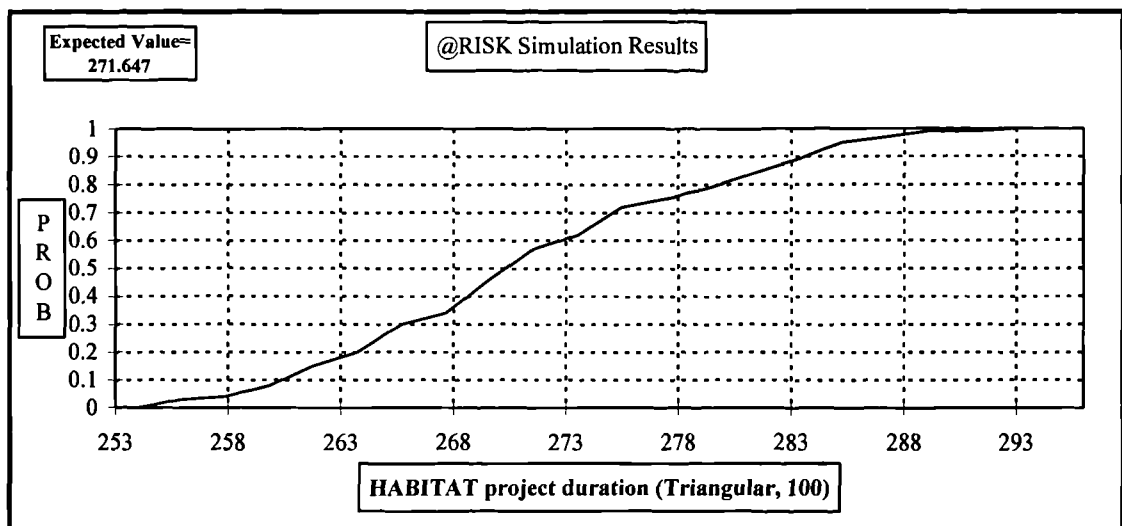
**Figure B.8 Total project duration of Example 3.2. Histogram in curve.  
Log-Normal distribution, 10000 iterations**



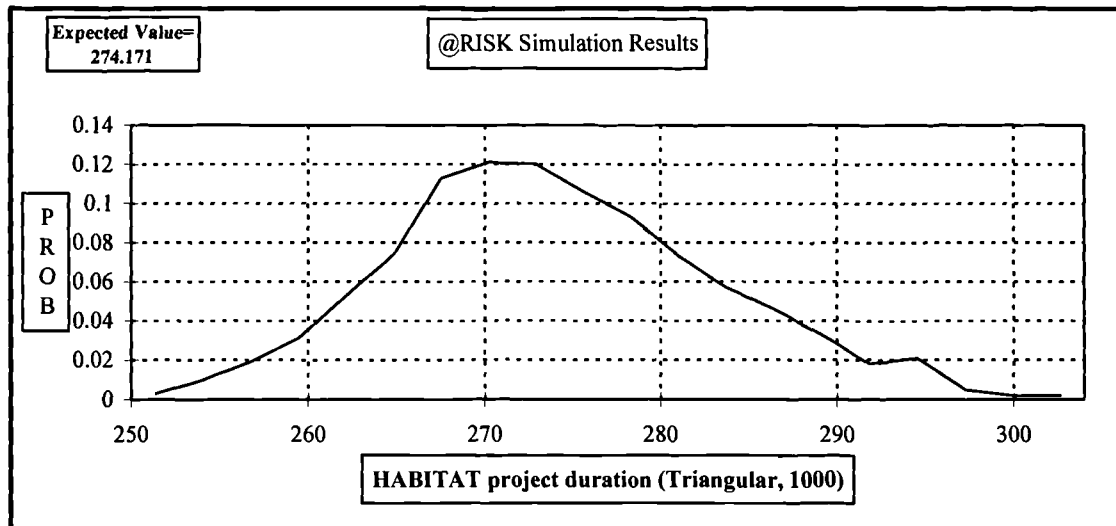
**Figure B.8a Total project duration of Example 3.2. Ascending cumulative curve.  
Log-Normal distribution, 10000 iterations**



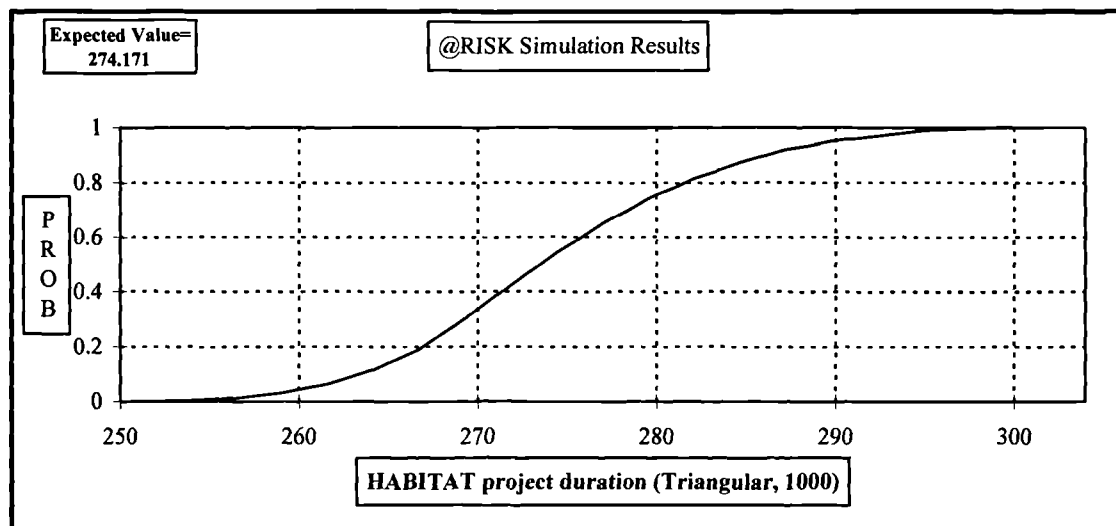
**Figure B.9 Total project duration of Example 3.2. Histogram in curve.**  
**Triangular distribution, 100 iterations**



**Figure B.9a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Triangular distribution, 100 iterations**

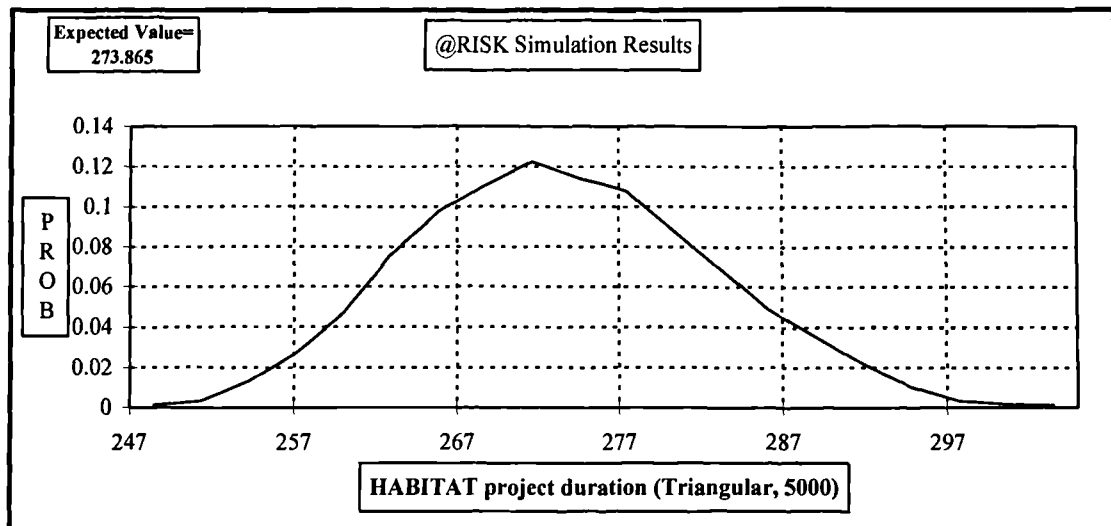


**Figure B.10 Total project duration of Example 3.2. Histogram in curve.**  
**Triangular distribution, 1000 iterations**

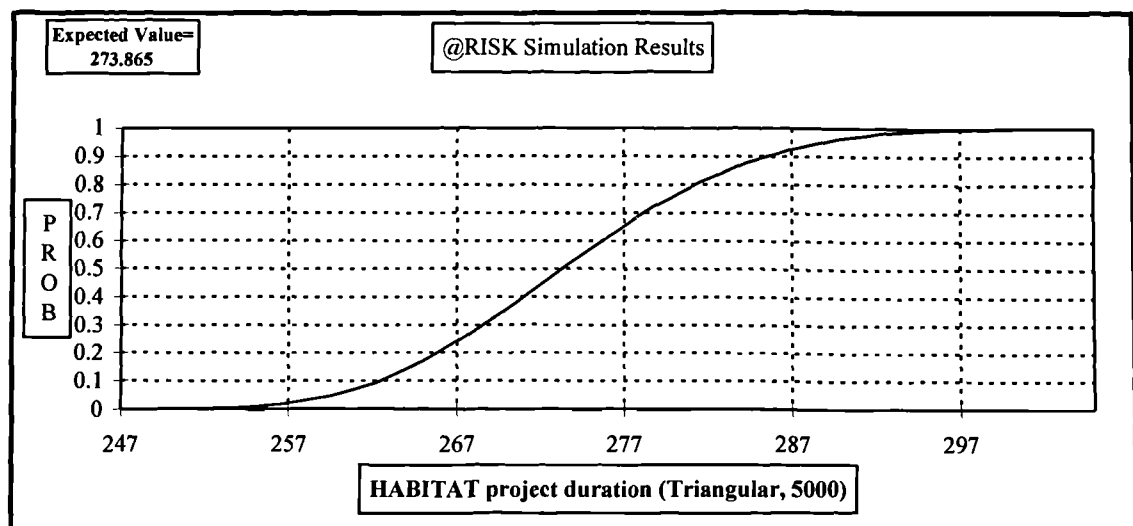


**Figure B.10a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Triangular distribution, 1000 iterations**

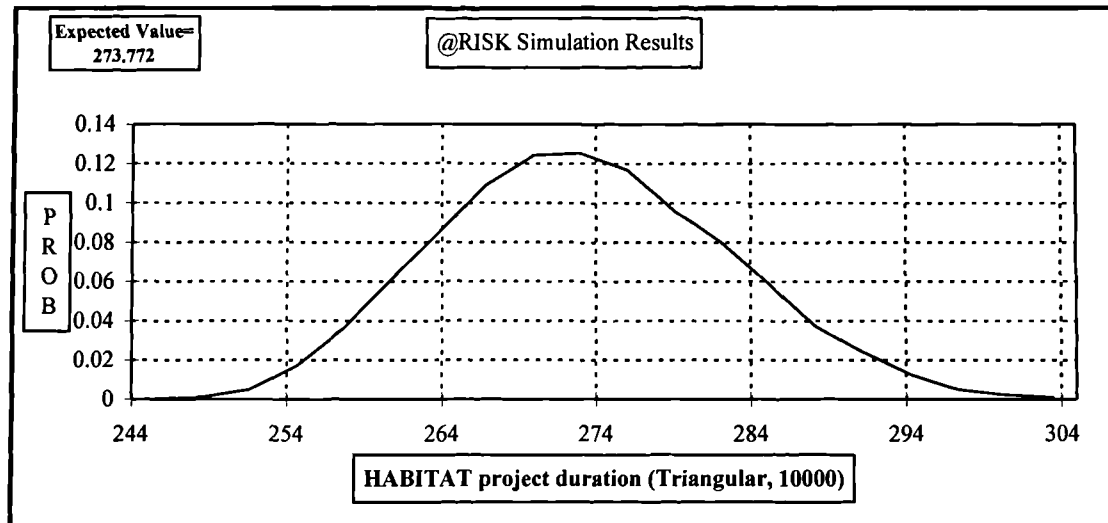




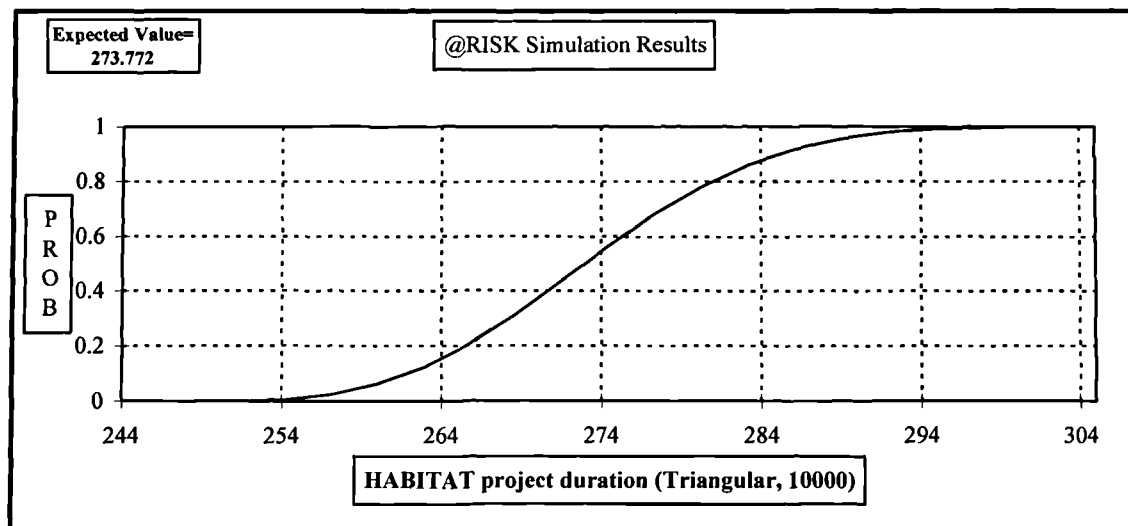
**Figure B.11 Total project duration of Example 3.2. Histogram in curve.  
Triangular distribution, 5000 iterations**



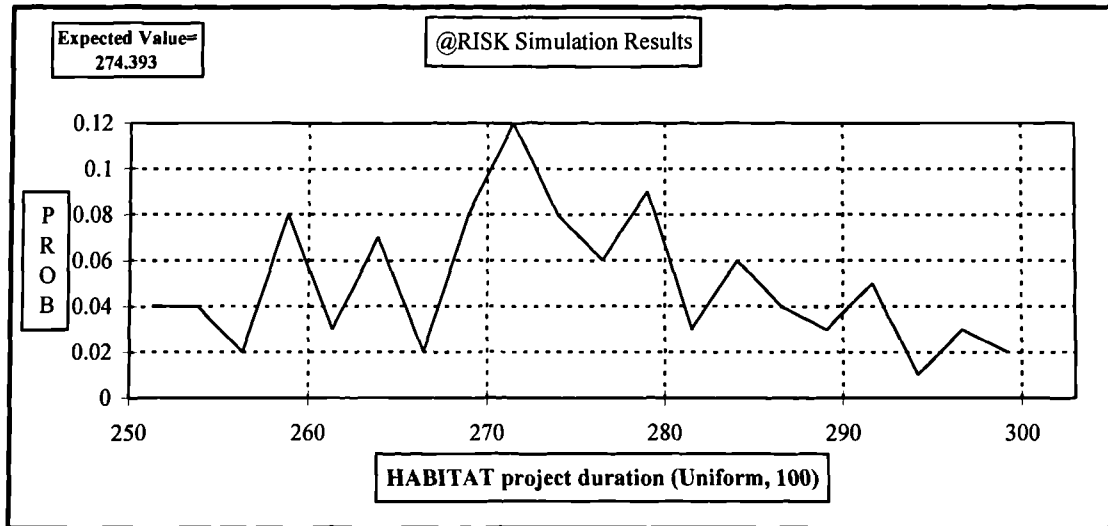
**Figure B.11a Total project duration of Example 3.2. Ascending cumulative curve.  
Triangular distribution, 5000 iterations**



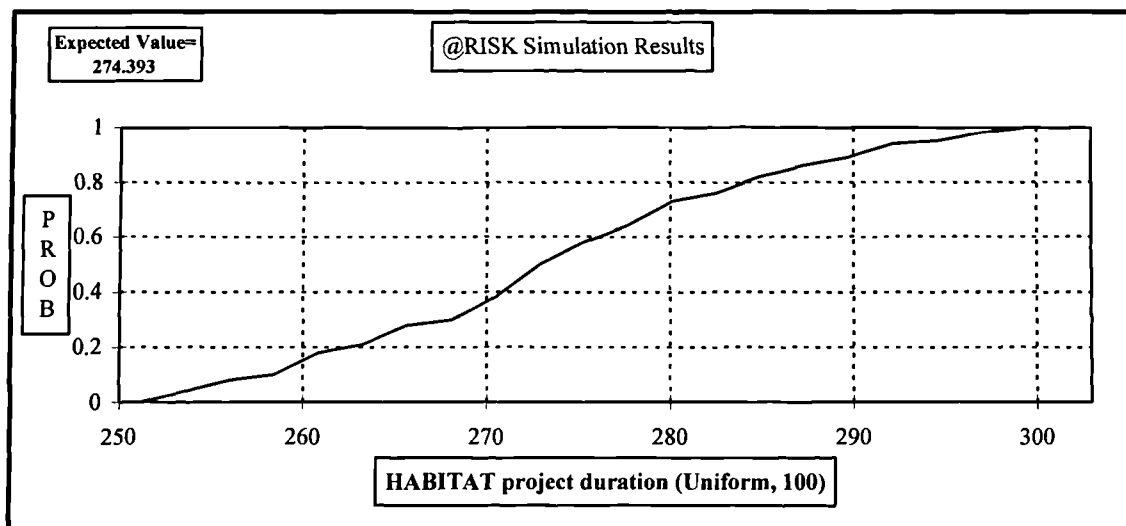
**Figure B.12 Total project duration of Example 3.2. Histogram in curve.**  
**Triangular distribution, 10000 iterations**



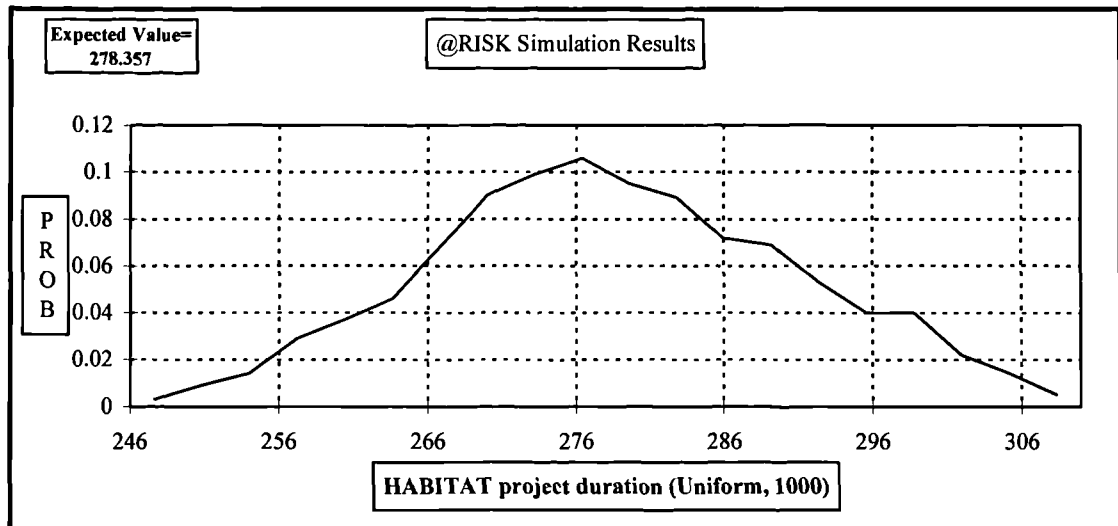
**Figure B.12a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Triangular distribution, 10000 iterations**



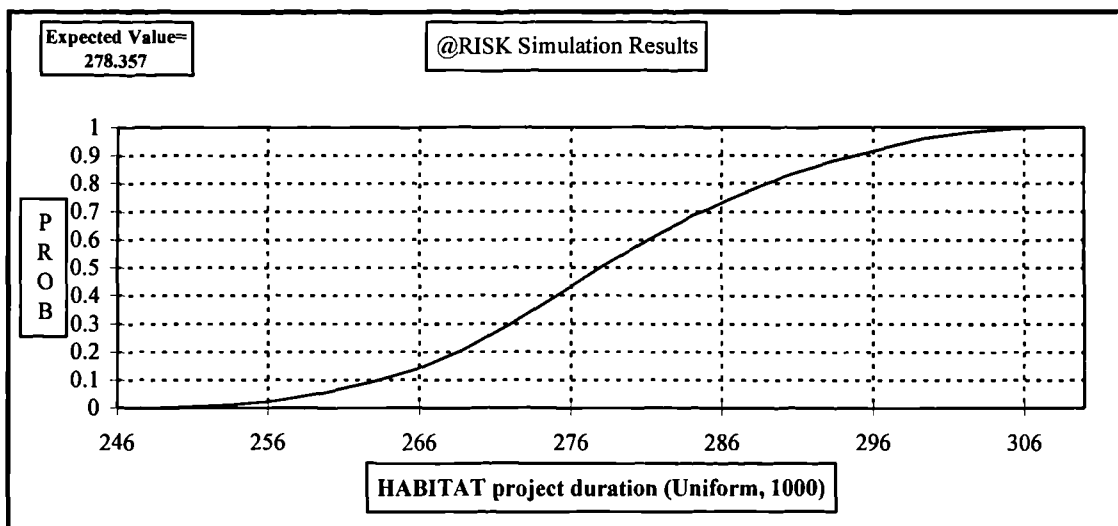
**Figure B.13 Total project duration of Example 3.2. Histogram in curve.**  
**Uniform distribution, 100 iterations**



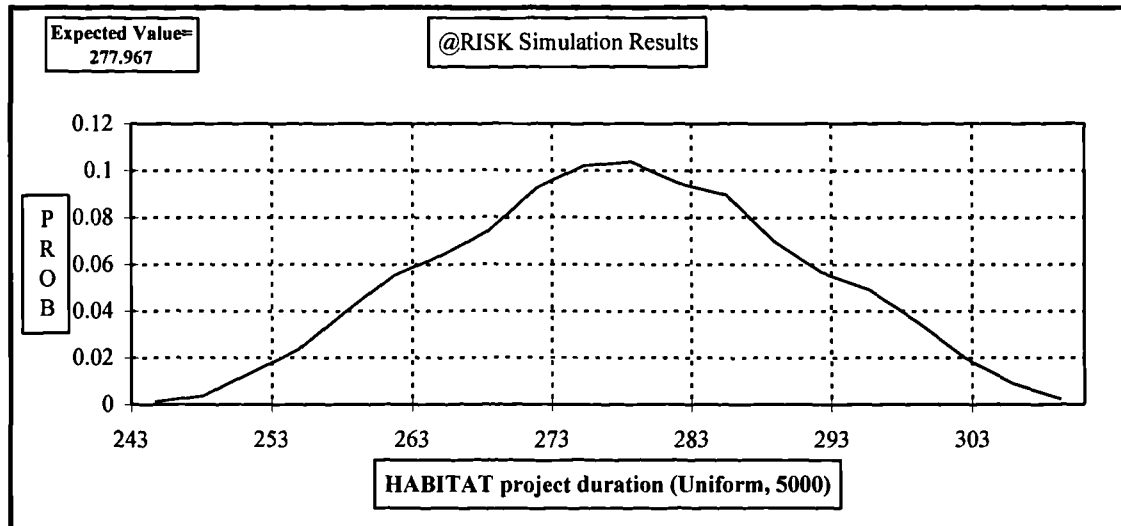
**Figure B.13a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Uniform distribution, 100 iterations**



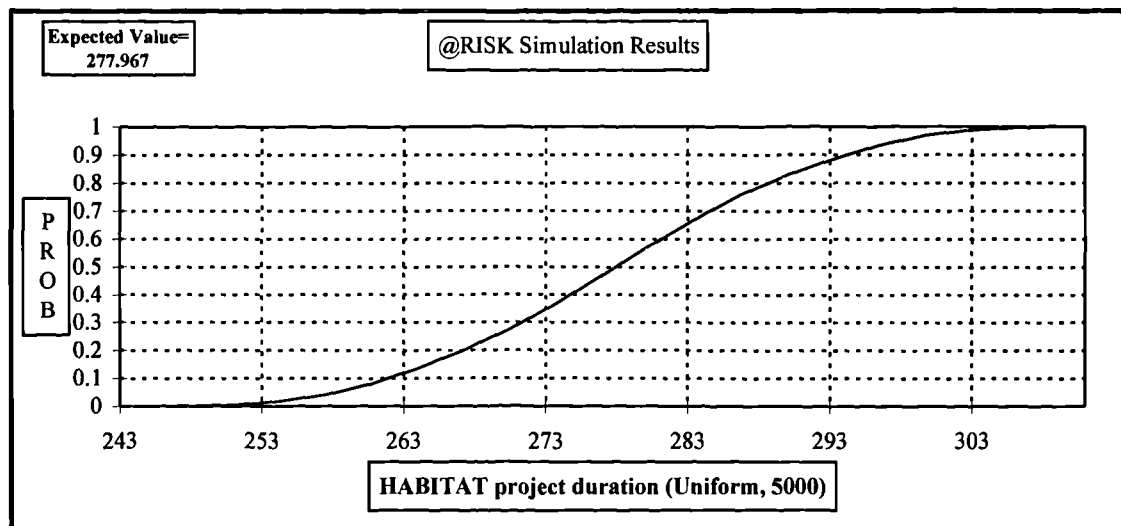
**Figure B.14 Total project duration of Example 3.2. Histogram in curve.**  
**Uniform distribution, 1000 iterations**



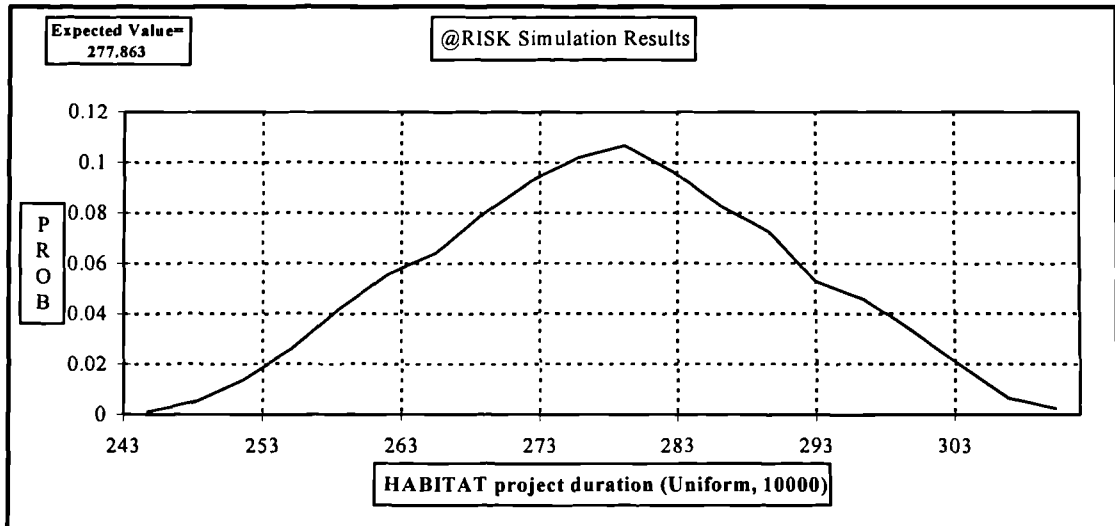
**Figure B.14a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Uniform distribution, 1000 iterations**



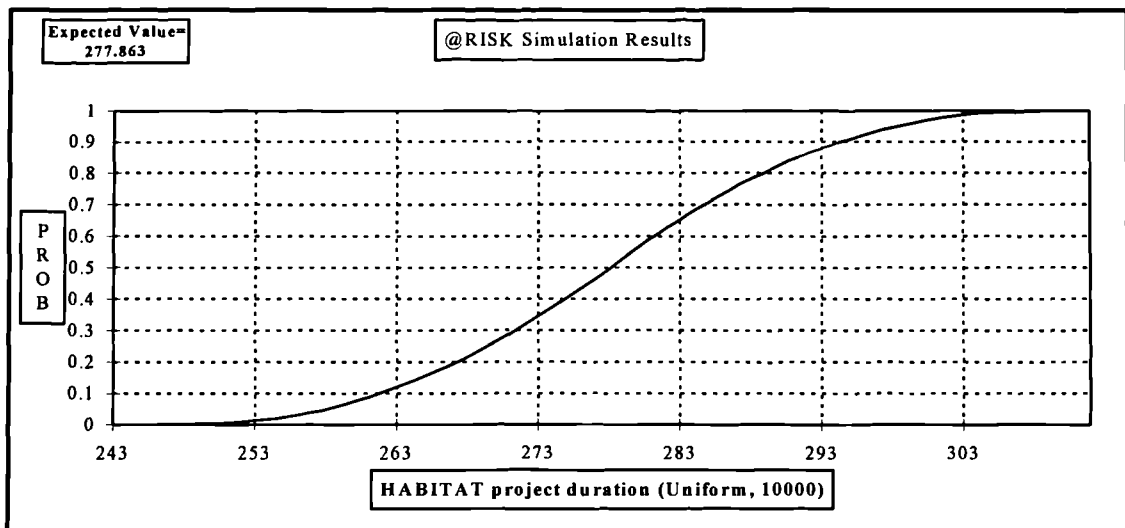
**Figure B.15 Total project duration of Example 3.2. Histogram in curve.**  
**Uniform distribution, 5000 iterations**



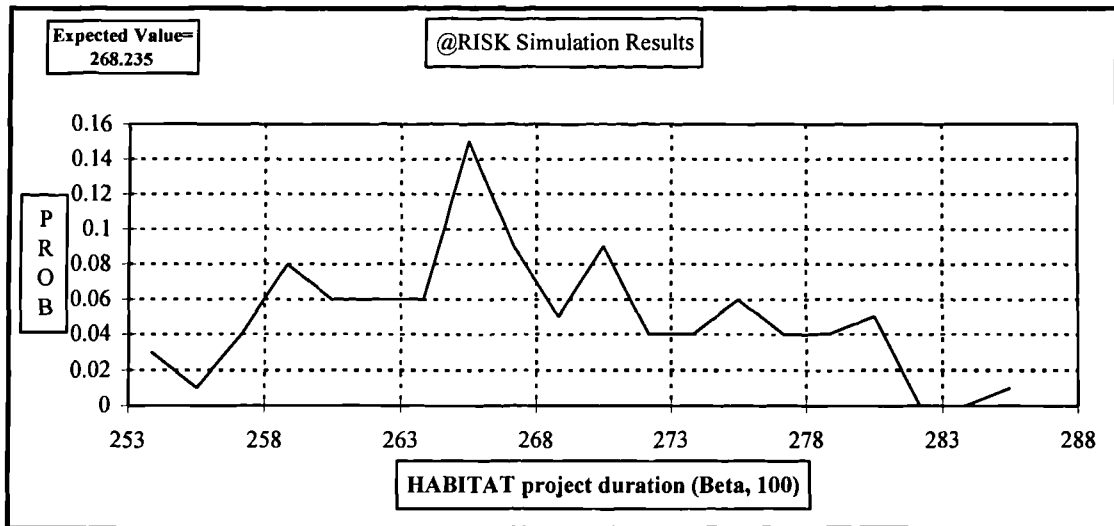
**Figure B.15a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Uniform distribution, 5000 iterations**



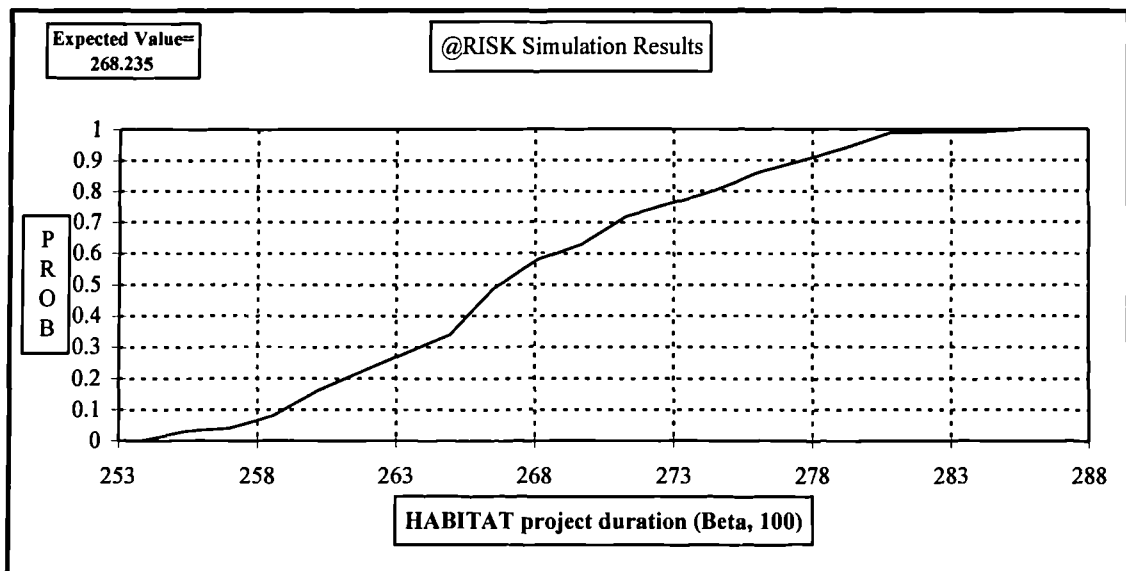
**Figure B.16 Total project duration of Example 3.2. Histogram in curve.**  
**Uniform distribution, 10000 iterations**



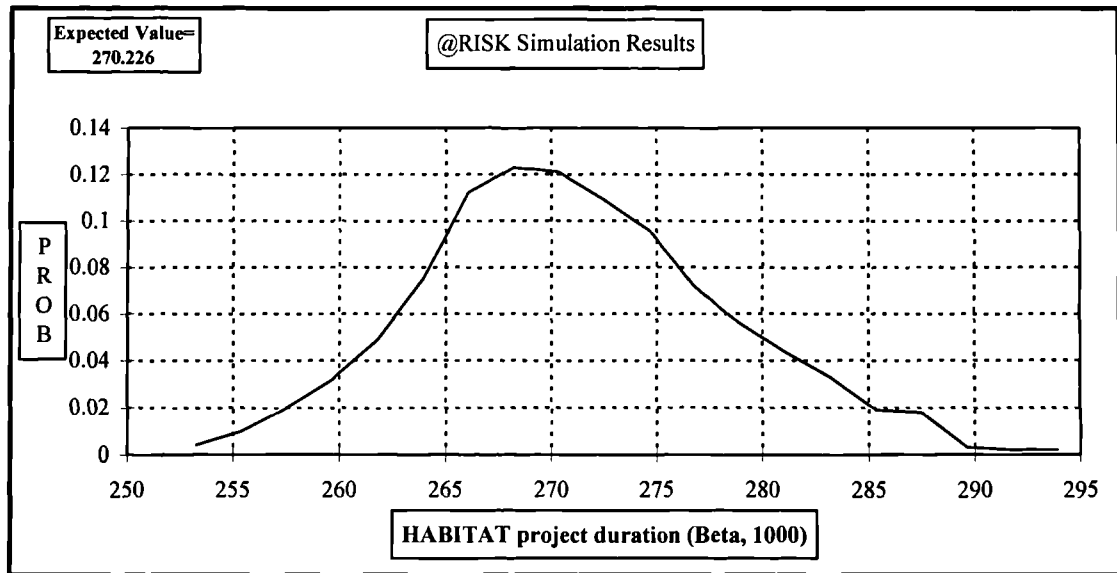
**Figure B.16a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Uniform distribution, 10000 iterations**



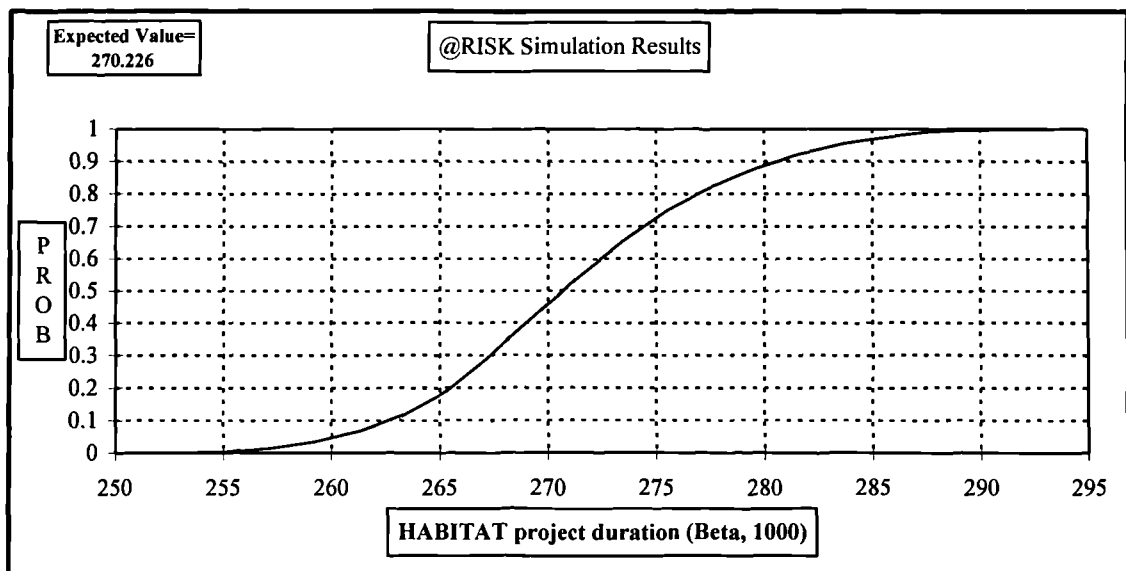
**Figure B.17 Total project duration of Example 3.2. Histogram in curve.**  
**Beta distribution, 100 iterations**



**Figure B.17a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Beta distribution, 100 iterations**

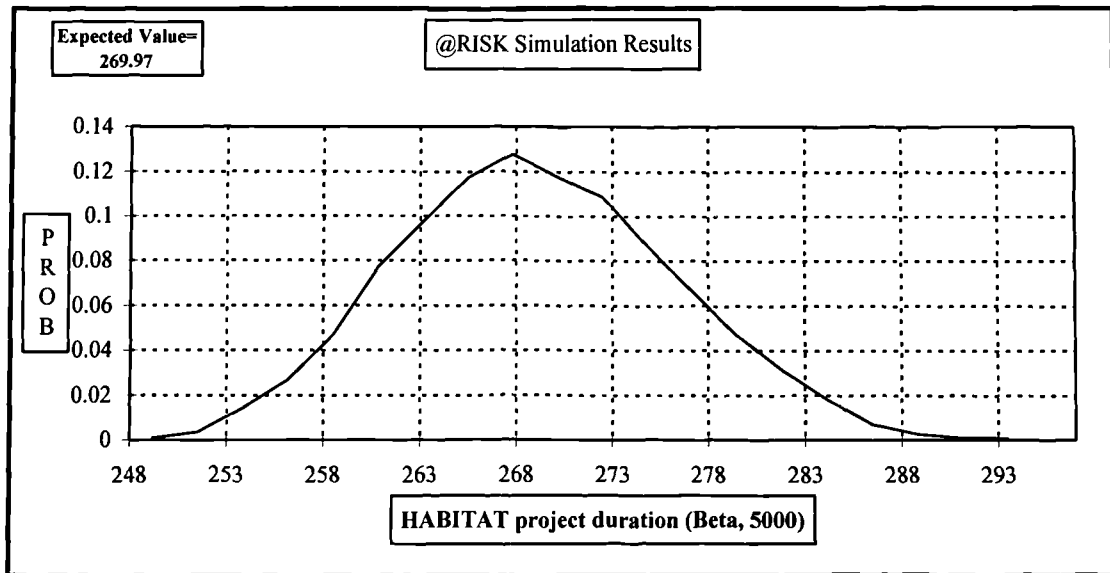


**Figure B.18 Total project duration of Example 3.2. Histogram in curve.  
Beta distribution, 1000 iterations**

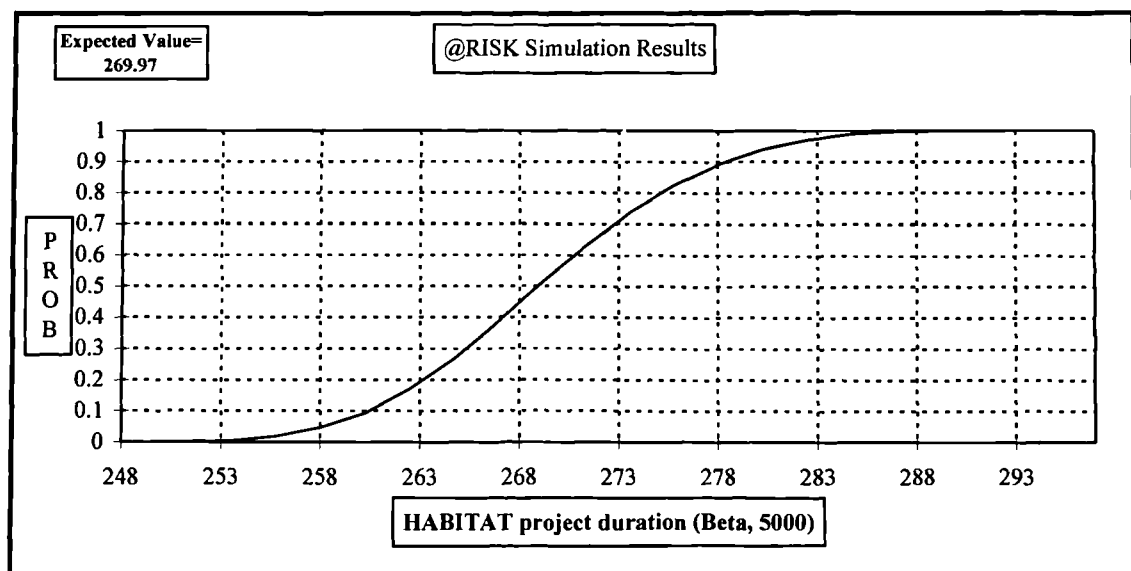


**Figure B.18a Total project duration of Example 3.2. Ascending cumulative curve.  
Beta distribution, 1000 iterations**

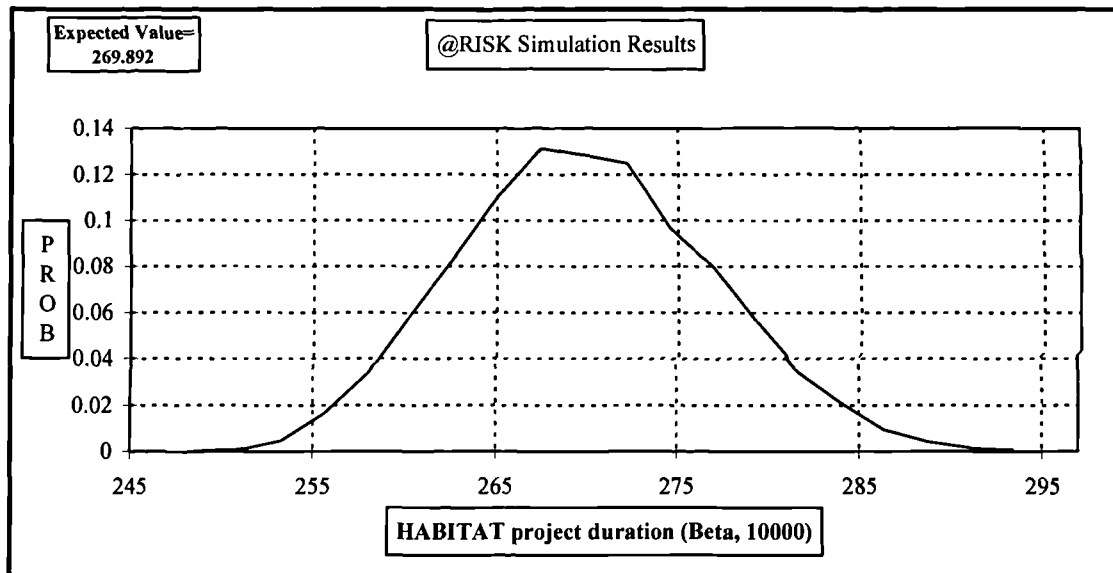




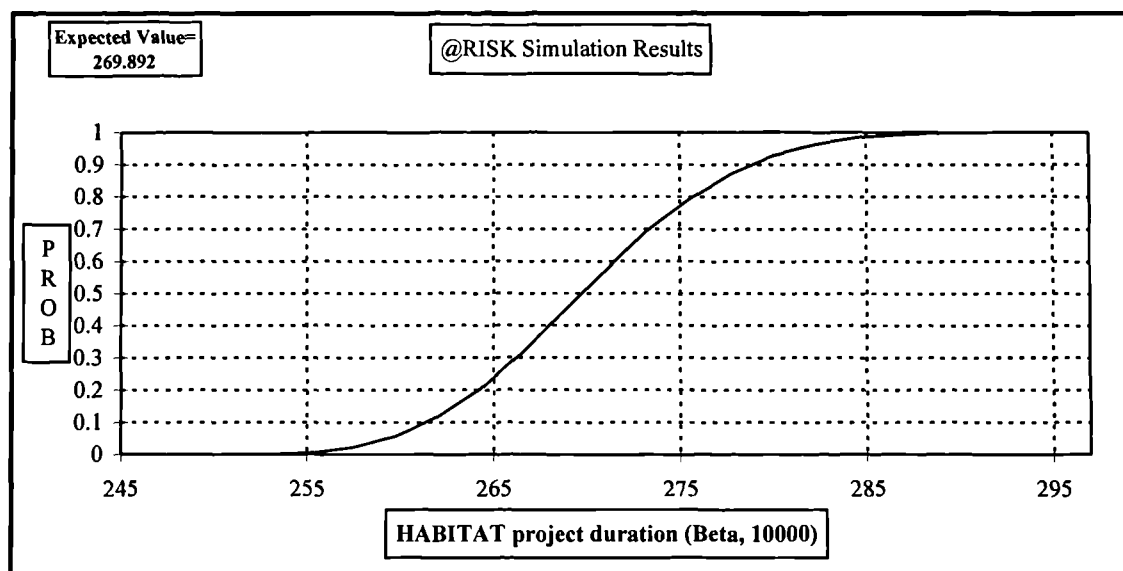
**Figure B.19 Total project duration of Example 3.2. Histogram in curve.**  
**Beta distribution, 5000 iterations**



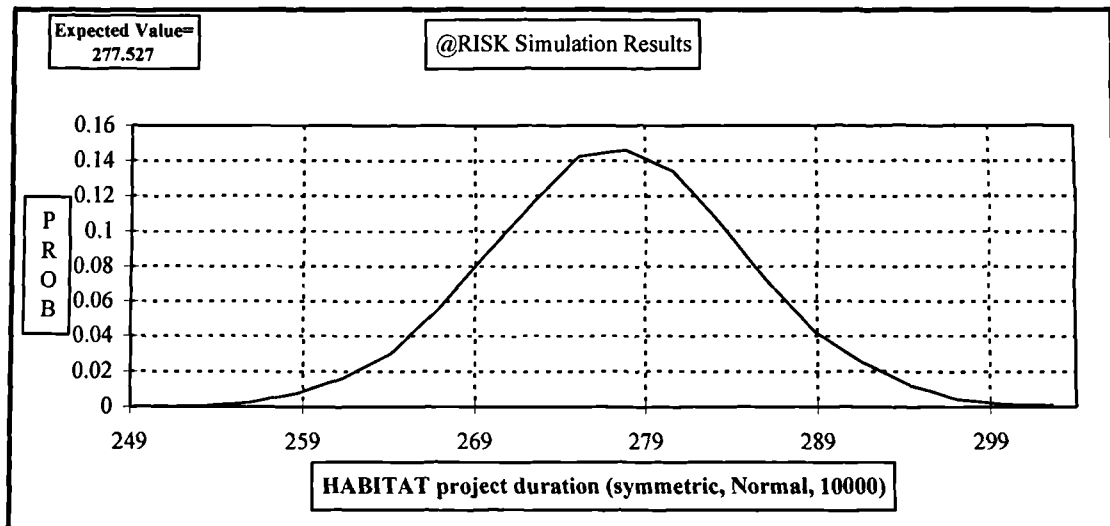
**Figure B.19a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Beta distribution, 5000 iterations**



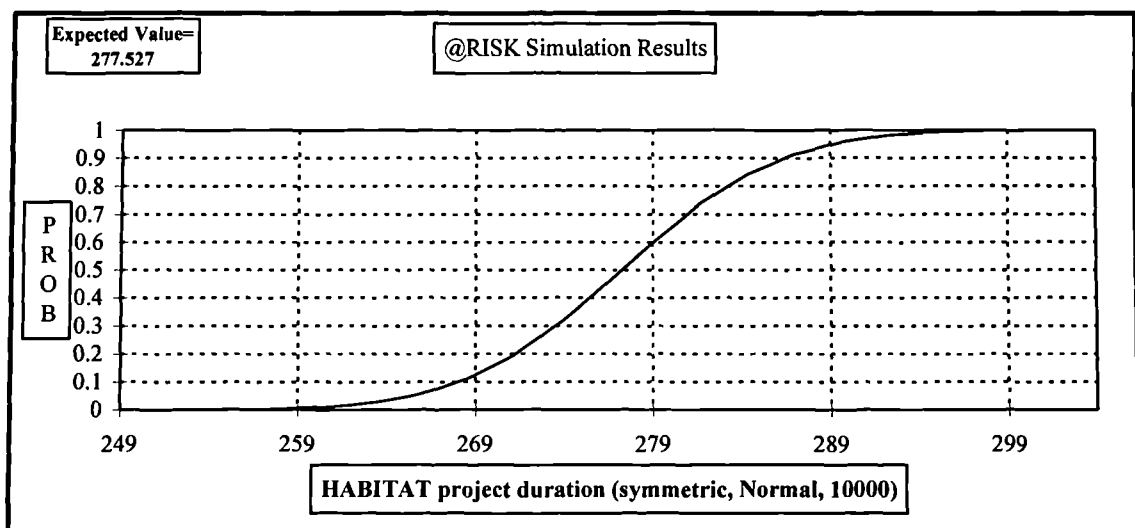
**Figure B.20 Total project duration of Example 3.2. Histogram in curve.  
Beta distribution, 10000 iterations**



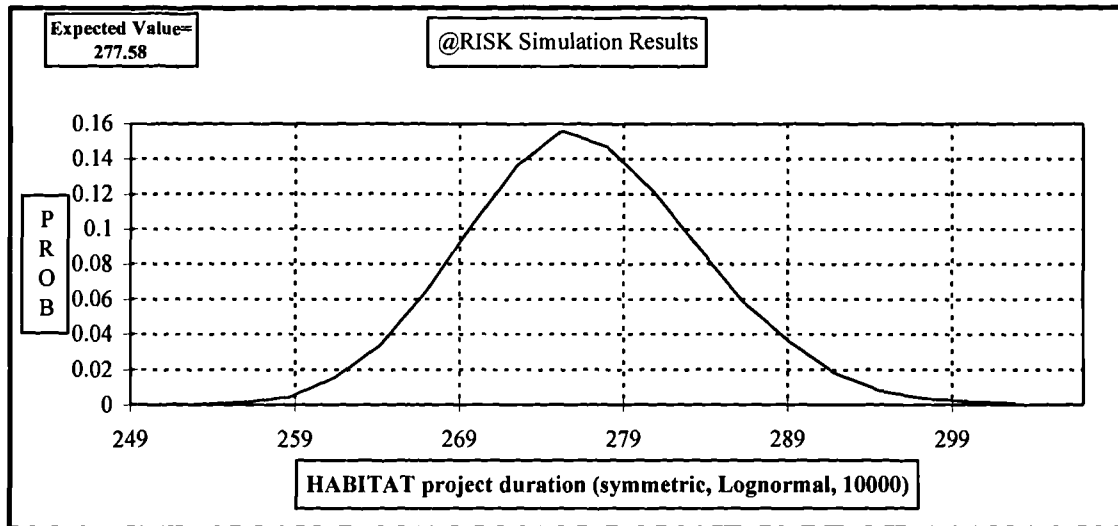
**Figure B.20a Total project duration of Example 3.2. Ascending cumulative curve.  
Beta distribution, 10000 iterations**



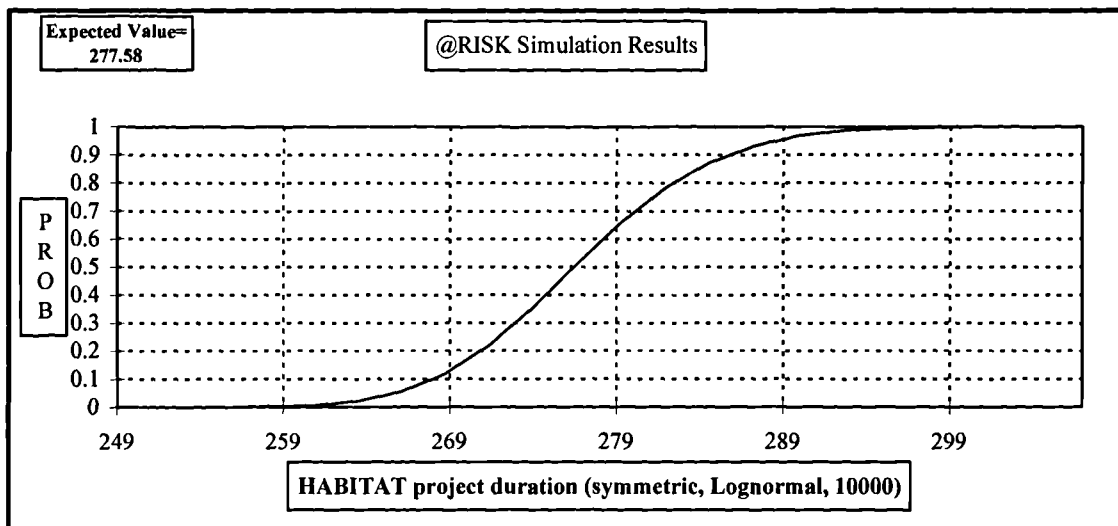
**Figure B.21 Total project duration of Example 3.2. Histogram in curve.**  
**Symmetric Normal distribution, 10000 iterations**



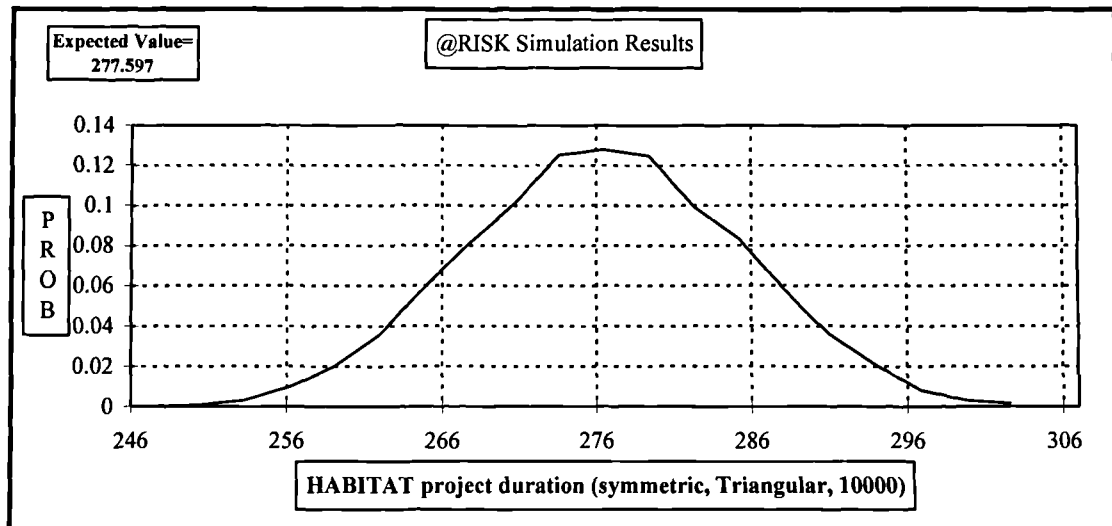
**Figure B.21a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Symmetric Normal distribution, 10000 iterations**



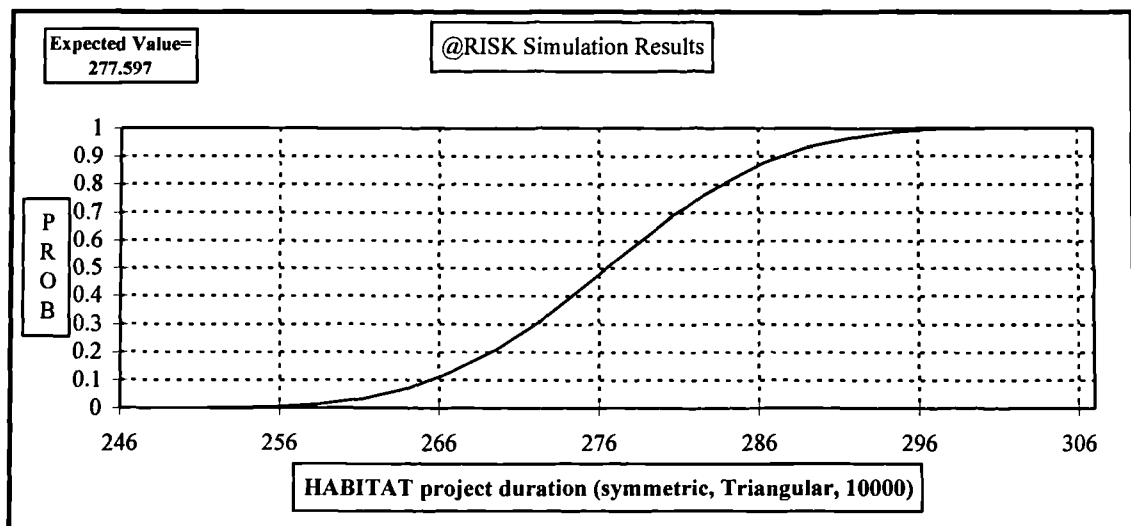
**Figure B.22 Total project duration of Example 3.2. Histogram in curve.  
Symmetric Log-Normal distribution, 10000 iterations**



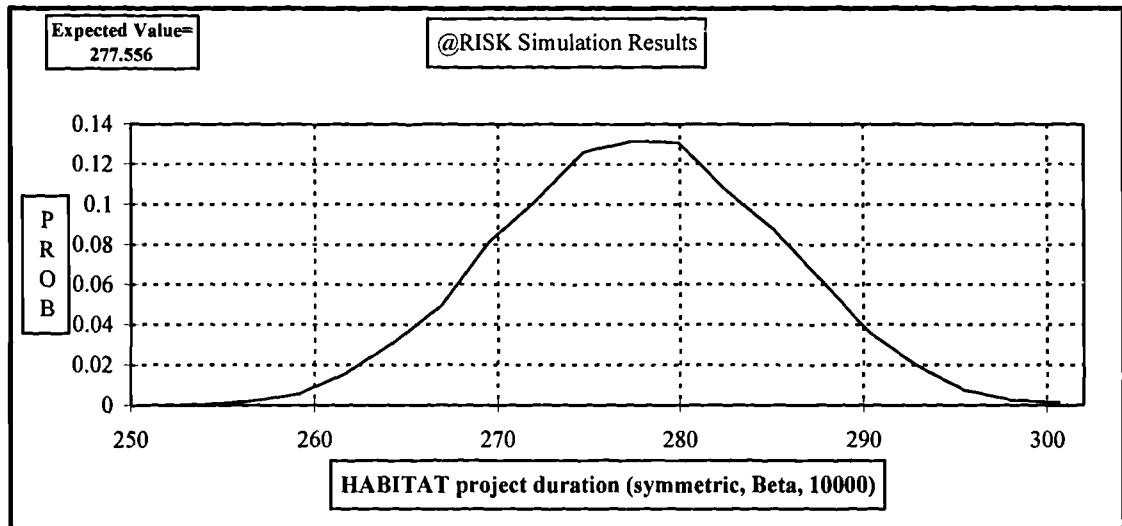
**Figure B.22a Total project duration of Example 3.2. Ascending cumulative curve.  
Symmetric Log-Normal distribution, 10000 iterations**



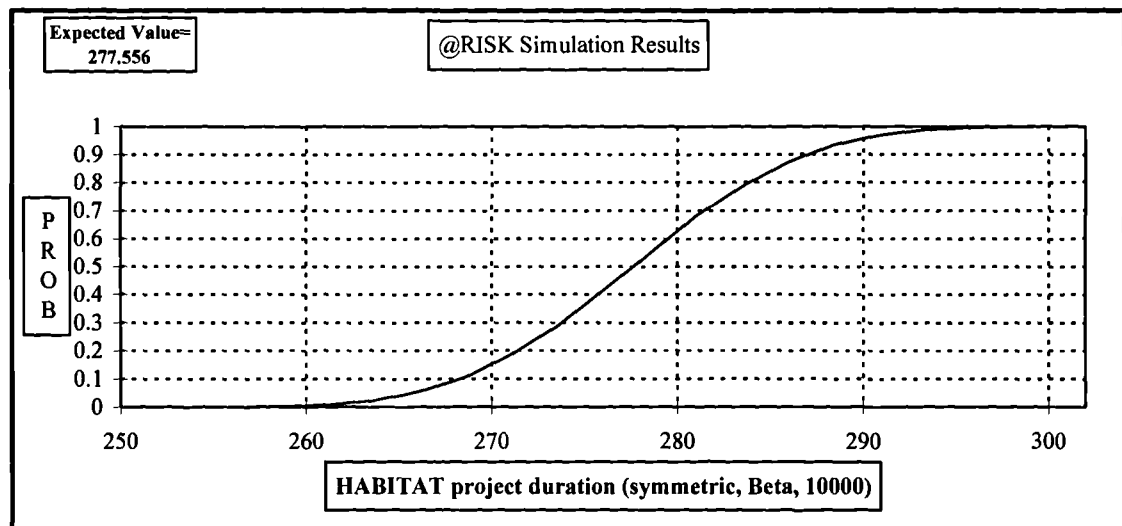
**Figure B.23 Total project duration of Example 3.2. Histogram in curve.**  
**Symmetric Triangular distribution, 10000 iterations**



**Figure B.23a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Symmetric Triangular distribution, 10000 iterations**



**Figure B.24 Total project duration of Example 3.2. Histogram in curve.**  
**Symmetric Beta distribution, 10000 iterations**



**Figure B.24a Total project duration of Example 3.2. Ascending cumulative curve.**  
**Symmetric Beta distribution, 10000 iterations**

# APPENDIX C

## DATA INPUTS AND OUTPUTS OF MSAM

### FOR EXAMPLES 4.1-4.5

#### Appendix C.1 Example 4.1

##### Data input for MSAM

100	0	0	0.00	0		
101	12	0	1.33	1	100	
102	20	0	3.00	1	100	
103	14	0	2.00	1	100	
104	16	0	4.00	1	103	
105	28	0	6.33	1	101	
106	15	0	2.00	2	102	104
107	36	0	4.00	2	102	104
108	22	0	2.67	1	103	
109	18	0	1.67	2	105	106
110	24	0	3.33	1	108	
111	0	0	0.00	3	107	109 110

##### Data output from MSAM

Mean:	67.8802762953
Variance:	22.3480833290
Standard Deviation:	4.72737594539
Run Time:	0.2197802

## Appendix C.2 Example 4.2

### Data input for MSAM

100	0	0	0	0				
101	3	4	5	1	100			
102	10	15	23	1	100			
103	3	5	9	1	101			
104	0	0	0	1	103			
105	7	8	10	1	103			
106	18	26	40	2	102	104		
107	13	14	16	1	105			
108	6	8	10	1	107			
109	1	2	10	1	108			
110	1	2	15	1	107			
111	0	0	0	1	107			
112	0	0	0	1	108			
113	8	10	11	2	106	109		
114	4	8	10	2	111	113		
115	0	0	0	1	114			
116	1	2	3	2	111	113		
117	2	6	10	1	107			
118	0	0	0	1	117			
119	6	7	8	2	116	118		
120	0	0	0	2	111	113		
121	8	10	12	2	112	120		
122	4	6	10	1	121			
123	0	0	0	1	122			
124	0	0	0	1	110			
125	2	3	20	2	123	124		
126	13	15	18	1	122			
127	0	0	0	1	110			
128	10	12	15	2	119	127		
129	3	5	7	1	128			
130	0	0	0	4	115	125	126	129

### Data output from MSAM

Mean: 85.8927999511  
 Variance: 9.67770709467  
 Standard Deviation: 3.11090133155  
 Run Time: 0.1098901



### Appendix C.3 Example 4.3

#### Data input for MSAM

100	0	0	0	0					
101	9	10	11	1	100				
102	14	15	21	1	101				
103	59	60	80	1	101				
104	50	60	90	1	101				
105	40	50	80	1	101				
106	20	30	40	1	101				
107	10	15	25	1	101				
108	10	15	30	1	101				
109	20	30	35	1	101				
110	40	50	60	1	101				
111	70	80	90	1	101				
112	14	15	16	1	102				
113	14	15	16	1	102				
114	50	60	80	5	103	143	128	135	122
115	50	60	61	5	103	143	128	135	122
116	14	15	16	1	104				
117	14	15	16	1	104				
118	29	30	31	1	104				
119	39	40	41	1	105				
120	59	60	61	3	106	129	133		
121	89	90	91	2	107	145			
122	0	0	0	2	108	130			
123	9	10	11	2	108	130			
124	0	0	0	2	109	137			
125	19	20	21	2	109	137			
126	14	15	16	1	112				
127	9	10	11	1	114				
128	0	0	0	1	117				
129	0	0	0	1	117				
130	0	0	0	1	117				
131	29	30	31	1	117				
132	0	0	0	0	0				
133	0	0	0	1	119				
134	29	30	31	1	119				
135	0	0	0	2	120	124			
136	25	27	27	2	120	124			
137	0	0	0	1	121				
138	4	5	6	1	123				
139	9	10	11	1	125				
140	19	20	21	1	110				
141	29	30	31	1	111				
142	29	30	31	1	113				
143	0	0	0	1	126				

144	39	40	41	1	126					
145	30	41	60	1	126					
146	19	20	21	6	131	134	136	138	139	140
147	4	5	6	4	141	142	144	146		
148	4	5	6	3	115	127	147			
149	4	5	6	1	148					
150	0	0	0	1	116					
151	0	0	0	1	118					
152	0	0	0	3	149	150	151			

### Data output from MSAM

Mean: 269.838918321  
 Variance: 51.9581049811  
 Standard Deviation: 7.20819706869  
 Run Time: 0.7142856

## Appendix C.4 Example 4.4

### Data input for MSAM

100	0	0	0.0	0		
101	0	0	0	2	103	117
102	2	0	0.5	1	100	
103	5	0	1.0	1	100	
104	6	0	1.5	1	100	
105	3	0	0.5	1	100	
106	7	0	4.0	1	100	
107	10	0	2.0	1	104	
108	3	0	1.0	1	105	
109	7	0	1.5	1	105	
110	5	0	2.0	1	106	
111	3	0	1.5	1	106	
112	9	0	2.0	1	107	
113	5	0	1.5	2	108	112
114	3	0	0.5	1	109	
115	9	0	4.5	1	110	
116	6	0	2.0	1	111	
117	2	0	0.5	1	102	
118	7	0	1.73	1	113	
119	5	0	2.0	3	114	115 116
120	10	0	2.0	2	118	128
121	7	0	3.31	1	119	
122	6	0	1.5	1	120	
123	10	0	4.5	2	101	121
124	6	0	1.5	1	123	
125	3	0	1.0	1	122	
126	3	0	1.0	1	124	
127	5	0	1.5	2	125	126
128	0	0	0.0	2	103	117
129	0	0	0.0	1	127	

### Data output from MSAM

Mean:	63.3899700621
Variance:	27.9750679538
Standard Deviation:	5.28914624053
Run Time:	0.1098901

## Appendix C.5 Example 4.5

### Data input for MSAM

100	0	0	0.0	0		
101	32	0	3.2	1	100	
102	2	0	0.5	1	101	
103	2	0	0.5	1	101	
104	0	0	0.0	1	103	
105	2	0	0.5	2	102	104
106	1	0	0.5	1	103	
107	2	0	1.0	2	105	106
108	0	0	0.0	1	107	
109	40	0	12	1	100	
110	2	0	0.5	2	108	109
111	2	0	0.5	1	110	
112	4	0	0.8	1	111	
113	1	0	0.1	1	112	
114	0	0	0.0	1	112	
115	0	0	0.0	1	113	
116	1	0	0.5	1	107	
117	60	0	12	1	100	
118	5	0	1.0	2	114	117
119	30	0	6.0	1	100	
120	3	0	0.9	2	115	116
121	1	0	0.3	1	120	
122	4	0	0.4	1	121	
123	1	0	0.1	1	122	
124	2	0	0.5	1	123	
125	2	0	0.2	1	124	
126	2	0	0.5	1	125	
127	2	0	0.5	1	126	
128	0	0	0.0	1	127	
129	2	0	0.2	1	124	
130	1	0	0.2	1	129	
131	0	0	0.0	1	127	
132	5	0	1.5	2	118	119
133	6	0	1.2	1	132	
134	0	0	0.0	1	132	
135	4	0	1.2	1	134	
136	2	0	0.5	3	131	133 135
137	3	0	0.3	2	128	130
138	0	0	0.0	2	136	137

### Data output from MSAM

Mean: 83.2956197052  
 Variance: 66.6662400250  
 Standard Deviation: 8.16493968287  
 Run Time: 0.0549451

## APPENDIX D

### RESULTS OF MCS FOR EXAMPLES 4.1-4.5

**Table D.1 Results of MCS for Example 4.1**

<b>Simulation Statistics</b>	
Date: 7/31/99 at 15:09	
Iterations: 10000	
Simulations: 1	
Worksheet: Project1	
Output Range: DURATION_	
p	
Cell: Duration 1	
Minimum=	53.98958206
Maximum=	88.31874847
Mean=	67.85377102
Std Deviation=	4.793780295
Variance=	22.98032951
Skewness=	0.323868499
Kurtosis=	3.075345021

**Percentile Values**

5Perc=	60.48125076
10Perc=	61.83124924
15Perc=	62.87083435
20Perc=	63.76041794
25Perc=	64.4604187
30Perc=	65.13124847
35Perc=	65.81041718
40Perc=	66.40208435
45Perc=	67
50Perc=	67.55000305
55Perc=	68.16874695
60Perc=	68.76875305
65Perc=	69.44999695
70Perc=	70.18125153
75Perc=	70.94999695
80Perc=	71.87916565
85Perc=	72.86042023
90Perc=	74.12916565
95Perc=	76.18958282

**Target Values:**

Value#1=	54
Prob#1=	0.01%
Value#2=	56
Prob#2=	0.24%
Value#3=	58
Prob#3=	1.04%
Value#4=	60
Prob#4=	3.85%
Value#5=	62
Prob#5=	10.69%
Value#6=	64
Prob#6=	21.63%
Value#7=	66
Prob#7=	36.68%
Value#8=	68
Prob#8=	53.70%
Value#9=	70
Prob#9=	68.80%
Value#10=	72
Prob#10=	80.64%

<b>Simulation Statistics</b>	
Date: 7/31/99 at 15:16	
Iterations: 10000	
Simulations: 1	
Worksheet: Project1	
Output Range: DURATION_	
p	
Cell: Duration 1	
Minimum=	53.98958206
Maximum=	88.31874847
Mean=	67.85377102
Std Deviation=	4.793780295
Variance=	22.98032951
Skewness=	0.323868499
Kurtosis=	3.075345021

**Percentile Values**

5Perc=	60.48125076
10Perc=	61.83124924
15Perc=	62.87083435
20Perc=	63.76041794
25Perc=	64.4604187
30Perc=	65.13124847
35Perc=	65.81041718
40Perc=	66.40208435
45Perc=	67
50Perc=	67.55000305
55Perc=	68.16874695
60Perc=	68.76875305
65Perc=	69.44999695
70Perc=	70.18125153
75Perc=	70.94999695
80Perc=	71.87916565
85Perc=	72.86042023
90Perc=	74.12916565
95Perc=	76.18958282

**Target Values:**

Value#1=	74
Prob#1=	89.60%
Value#2=	76
Prob#2=	94.60%
Value#3=	78
Prob#3=	97.55%
Value#4=	80
Prob#4=	99.08%
Value#5=	82
Prob#5=	99.58%
Value#6=	84
Prob#6=	99.88%
Value#7=	86
Prob#7=	99.96%
Value#8=	88
Prob#8=	99.99%
Value#9=	90
Prob#9=	100.00%
Value#10=	92
Prob#10=	100.00%

**Table D.2 Results of MCS for Example 4.2**

<b>Simulation Statistics</b>	
Date: 8/2/99 at 14:41	
Iterations: 10000	
Simulations: 1	
Worksheet: EXAM2.MPP	
Output Range: duration_p	
Cell: Duration 1	
Minimum=	75.40208435
Maximum=	105.8104172
Mean=	85.90435374
Std Deviation=	3.104058027
Variance=	9.635176236
Skewness=	0.551237836
Kurtosis=	3.858188682

<b>Percentile Values</b>	
5Perc=	81.26875305
10Perc=	82.20207977
15Perc=	82.84999847
20Perc=	83.34791565
25Perc=	83.76875305
30Perc=	84.17708588
35Perc=	84.56874847
40Perc=	84.9375
45Perc=	85.2895813
50Perc=	85.68125153
55Perc=	86.03749847
60Perc=	86.42082977
65Perc=	86.78333282
70Perc=	87.21875
75Perc=	87.70207977
80Perc=	88.2895813
85Perc=	89.03125
90Perc=	89.9291687
95Perc=	91.40208435

<b>Target Values:</b>	
Value#1=	75
Prob#1=	0.00%
Value#2=	77
Prob#2=	0.05%
Value#3=	79
Prob#3=	0.52%
Value#4=	81
Prob#4=	4.05%
Value#5=	83
Prob#5=	16.58%
Value#6=	85
Prob#6=	41.06%
Value#7=	87
Prob#7=	67.43%
Value#8=	89
Prob#8=	84.87%
Value#9=	91
Prob#9=	93.96%
Value#10=	93
Prob#10=	97.68%

<b>Simulation Statistics</b>	
Date: 8/2/99 at 14:45	
Iterations: 10000	
Simulations: 1	
Worksheet: EXAM2.MPP	
Output Range: duration_p	
Cell: Duration 1	
Minimum=	75.40208435
Maximum=	105.8104172
Mean=	85.90435374
Std Deviation=	3.104058027
Variance=	9.635176236
Skewness=	0.551237836
Kurtosis=	3.858188682

<b>Percentile Values</b>	
5Perc=	81.26875305
10Perc=	82.20207977
15Perc=	82.84999847
20Perc=	83.34791565
25Perc=	83.76875305
30Perc=	84.17708588
35Perc=	84.56874847
40Perc=	84.9375
45Perc=	85.2895813
50Perc=	85.68125153
55Perc=	86.03749847
60Perc=	86.42082977
65Perc=	86.78333282
70Perc=	87.21875
75Perc=	87.70207977
80Perc=	88.2895813
85Perc=	89.03125
90Perc=	89.9291687
95Perc=	91.40208435

<b>Target Values:</b>	
Value#1=	95
Prob#1=	99.33%
Value#2=	97
Prob#2=	99.75%
Value#3=	99
Prob#3=	99.91%
Value#4=	101
Prob#4=	99.97%
Value#5=	103
Prob#5=	99.99%
Value#6=	105
Prob#6=	99.99%
Value#7=	107
Prob#7=	100.00%
Value#8=	109
Prob#8=	100.00%
Value#9=	111
Prob#9=	100.00%
Value#10=	113
Prob#10=	100.00%

**Table D.3 Results of MCS for Example 4.3**

<b>Simulation Statistics</b>	
Date: 8/8/00 at 19:32	
Iterations: 10000	
Simulations: 1	
Worksheet: E2NOR4.MPP	
Output Range: duration_p	
Cell: Duration 1	
Minimum=	238.5500031
Maximum=	295.6104126
Mean=	269.9060544
Std Deviation=	7.229252349
Variance=	52.26208952
Skewness=	0.017100126
Kurtosis=	2.980746525
<b>Percentile Values</b>	
5Perc=	258.0187378
10Perc=	260.6708374
15Perc=	262.3916626
20Perc=	263.7895813
25Perc=	265.0520935
30Perc=	266.1104126
35Perc=	267.1291809
40Perc=	268.0125122
45Perc=	268.9729309
50Perc=	269.8687439
55Perc=	270.8020935
60Perc=	271.7000122
65Perc=	272.6708374
70Perc=	273.6020813
75Perc=	274.75
80Perc=	276.0604248
85Perc=	277.4708252
90Perc=	279.1979065
95Perc=	281.8979187
<b>Target Values:</b>	
Value#1=	238
Prob#1=	0.00%
Value#2=	241
Prob#2=	0.01%
Value#3=	244
Prob#3=	0.02%
Value#4=	247
Prob#4=	0.06%
Value#5=	250
Prob#5=	0.24%
Value#6=	253
Prob#6=	0.87%
Value#7=	256
Prob#7=	2.76%
Value#8=	259
Prob#8=	6.47%
Value#9=	262
Prob#9=	13.77%
Value#10=	265
Prob#10=	24.79%

<b>Simulation Statistics</b>	
Date: 8/8/00 at 20:03	
Iterations: 10000	
Simulations: 1	
Worksheet: E2NOR4.MPP	
Output Range: duration_p	
Cell: Duration 1	
Minimum=	238.5500031
Maximum=	295.6104126
Mean=	269.9060544
Std Deviation=	7.229252349
Variance=	52.26208952
Skewness=	0.017100126
Kurtosis=	2.980746525
<b>Percentile Values</b>	
5Perc=	258.0187378
10Perc=	260.6708374
15Perc=	262.3916626
20Perc=	263.7895813
25Perc=	265.0520935
30Perc=	266.1104126
35Perc=	267.1291809
40Perc=	268.0125122
45Perc=	268.9729309
50Perc=	269.8687439
55Perc=	270.8020935
60Perc=	271.7000122
65Perc=	272.6708374
70Perc=	273.6020813
75Perc=	274.75
80Perc=	276.0604248
85Perc=	277.4708252
90Perc=	279.1979065
95Perc=	281.8979187
<b>Target Values:</b>	
Value#1=	268
Prob#1=	39.89%
Value#2=	271
Prob#2=	55.98%
Value#3=	274
Prob#3=	71.64%
Value#4=	277
Prob#4=	83.38%
Value#5=	280
Prob#5=	91.90%
Value#6=	283
Prob#6=	96.47%
Value#7=	286
Prob#7=	98.74%
Value#8=	289
Prob#8=	99.58%
Value#9=	292
Prob#9=	99.86%
Value#10=	295
Prob#10=	99.97%

**Table D.4 Results of MCS for Example 4.4**

<b>Simulation Statistics</b>	
Date: 7/31/99 at 14:45	
Iterations: 10000	
Simulations: 1	
Worksheet: EXAM4.MPP	
Output Range: duration_p	
Cell: Duration 1	
Minimum=	42.25
Maximum=	92.56874847
Mean=	63.62653417
Std Deviation=	5.413610697
Variance=	29.30718078
Skewness=	0.485088111
Kurtosis=	3.913638397

<b>Percentile Values</b>	
5Perc=	55.32083511
10Perc=	57.03125
15Perc=	58.31041718
20Perc=	59.22916794
25Perc=	59.98125076
30Perc=	60.73125076
35Perc=	61.37916565
40Perc=	62.01874924
45Perc=	62.59166718
50Perc=	63.24166489
55Perc=	63.95416641
60Perc=	64.59791565
65Perc=	65.31874847
70Perc=	66.0708313
75Perc=	66.8833313
80Perc=	67.78125
85Perc=	68.97916412
90Perc=	70.42082977
95Perc=	72.91249847

<b>Target Values:</b>	
Value#1=	42
Prob#1=	0.00%
Value#2=	45
Prob#2=	0.02%
Value#3=	48
Prob#3=	0.05%
Value#4=	51
Prob#4=	0.50%
Value#5=	54
Prob#5=	2.60%
Value#6=	57
Prob#6=	9.84%
Value#7=	60
Prob#7=	25.14%
Value#8=	63
Prob#8=	48.05%
Value#9=	66
Prob#9=	69.60%
Value#10=	69
Prob#10=	85.09%

<b>Simulation Statistics</b>	
Date: 7/31/99 at 14:49	
Iterations: 10000	
Simulations: 1	
Worksheet: EXAM4.MPP	
Output Range: duration_p	
Cell: Duration 1	
Minimum=	42.25
Maximum=	92.56874847
Mean=	63.62653417
Std Deviation=	5.413610697
Variance=	29.30718078
Skewness=	0.485088111
Kurtosis=	3.913638397

<b>Percentile Values</b>	
5Perc=	55.32083511
10Perc=	57.03125
15Perc=	58.31041718
20Perc=	59.22916794
25Perc=	59.98125076
30Perc=	60.73125076
35Perc=	61.37916565
40Perc=	62.01874924
45Perc=	62.59166718
50Perc=	63.24166489
55Perc=	63.95416641
60Perc=	64.59791565
65Perc=	65.31874847
70Perc=	66.0708313
75Perc=	66.8833313
80Perc=	67.78125
85Perc=	68.97916412
90Perc=	70.42082977
95Perc=	72.91249847

<b>Target Values:</b>	
Value#1=	72
Prob#1=	93.42%
Value#2=	75
Prob#2=	97.13%
Value#3=	78
Prob#3=	98.73%
Value#4=	81
Prob#4=	99.57%
Value#5=	84
Prob#5=	99.80%
Value#6=	87
Prob#6=	99.90%
Value#7=	90
Prob#7=	99.97%
Value#8=	93
Prob#8=	100.00%
Value#9=	96
Prob#9=	100.00%
Value#10=	99
Prob#10=	100.00%



**Table D.5 Results of MCS for Example 4.5**

<b>Simulation Statistics</b>	
Date: 8/3/99 at 13:34	
Iterations: 10000	
Simulations: 1	
Worksheet: EXAM5.MPP	
Output Range: duration_p	
Cell: Duration 1	
Minimum=	58.91041565
Maximum=	124.4604187
Mean=	81.76537563
Std Deviation=	9.609289841
Variance=	92.33845125
Skewness=	0.439385036
Kurtosis=	2.920759189
<b>Percentile Values</b>	
5Perc=	67.63957977
10Perc=	69.75208282
15Perc=	71.5395813
20Perc=	73.05000305
25Perc=	74.4770813
30Perc=	75.83958435
35Perc=	77.07291412
40Perc=	78.36042023
45Perc=	79.68958282
50Perc=	81.0395813
55Perc=	82.375
60Perc=	83.7104187
65Perc=	85
70Perc=	86.59166718
75Perc=	88.31874847
80Perc=	90.08125305
85Perc=	92.01249695
90Perc=	94.66041565
95Perc=	98.43958282
<b>Target Values:</b>	
Value#1=	60
Prob#1=	0.05%
Value#2=	63
Prob#2=	0.41%
Value#3=	66
Prob#3=	2.71%
Value#4=	69
Prob#4=	8.24%
Value#5=	72
Prob#5=	16.33%
Value#6=	75
Prob#6=	26.94%
Value#7=	78
Prob#7=	38.75%
Value#8=	81
Prob#8=	49.88%
Value#9=	84
Prob#9=	61.08%
Value#10=	87
Prob#10=	71.25%

<b>Simulation Statistics</b>	
Date: 8/3/99 at 13:38	
Iterations: 10000	
Simulations: 1	
Worksheet: EXAM5.MPP	
Output Range: duration_p	
Cell: Duration 1	
Minimum=	58.91041565
Maximum=	124.4604187
Mean=	81.76537563
Std Deviation=	9.609289841
Variance=	92.33845125
Skewness=	0.439385036
Kurtosis=	2.920759189
<b>Percentile Values</b>	
5Perc=	67.63957977
10Perc=	69.75208282
15Perc=	71.5395813
20Perc=	73.05000305
25Perc=	74.4770813
30Perc=	75.83958435
35Perc=	77.07291412
40Perc=	78.36042023
45Perc=	79.68958282
50Perc=	81.0395813
55Perc=	82.375
60Perc=	83.7104187
65Perc=	85
70Perc=	86.59166718
75Perc=	88.31874847
80Perc=	90.08125305
85Perc=	92.01249695
90Perc=	94.66041565
95Perc=	98.43958282
<b>Target Values:</b>	
Value#1=	90
Prob#1=	79.83%
Value#2=	93
Prob#2=	86.92%
Value#3=	96
Prob#3=	92.07%
Value#4=	99
Prob#4=	95.48%
Value#5=	102
Prob#5=	97.33%
Value#6=	105
Prob#6=	98.61%
Value#7=	108
Prob#7=	99.32%
Value#8=	111
Prob#8=	99.63%
Value#9=	114
Prob#9=	99.83%
Value#10=	117
Prob#10=	99.91%

# APPENDIX E

## RESULTS OF MCS FOR EXAMPLES 5.1-5.10

**Table E.1 Result of MCS for Example 5.1 (Iterations: 1000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 13:23	
Iterations: 1000	
Simulations: 1	
Worksheet: [EXAM511.XLS]Sheet1	
Output Range: c1	
Cell: \$C\$1	
Minimum=	-9663.2686
Maximum=	4444
Mean=	-2945.7387
Std Deviation=	2361.3011
Variance=	5575742.91
Skewness=	0.02243602
Kurtosis=	2.80546027
<b>Percentile Values</b>	
5Perc=	-6954.8208
10Perc=	-5949.603
15Perc=	-5434.144
20Perc=	-4960.5894
25Perc=	-4517.8452
30Perc=	-4151.8223
35Perc=	-3893.0293
40Perc=	-3598.2373
45Perc=	-3318.2656
50Perc=	-3059.0066
55Perc=	-2665.7585
60Perc=	-2366.0605
65Perc=	-2018.9436
70Perc=	-1600.5569
75Perc=	-1299.4365
80Perc=	-943.86066
85Perc=	-544.10931
90Perc=	176.533005
95Perc=	1103.59827
<b>Target Values:</b>	
Value#1=	0
Prob#1=	89.17%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.2 Result of MCS for Example 5.1 (Iterations: 5000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 13:43	
Iterations: 5000	
Simulations: 1	
Worksheet: [EXAM512.XLS]Sheet1	
Output Range: c1	
Cell: \$C\$1	
Minimum=	-10458.243
Maximum=	4630.61035
Mean=	-2950.4287
Std Deviation=	2229.77259
Variance=	4971885.82
Skewness=	-0.0279121
Kurtosis=	2.9146982
<b>Percentile Values</b>	
5Perc=	-6627.02
10Perc=	-5858.4907
15Perc=	-5253.0962
20Perc=	-4808.271
25Perc=	-4421.1377
30Perc=	-4096.1587
35Perc=	-3812.741
40Perc=	-3524.3831
45Perc=	-3251.7473
50Perc=	-2973.1616
55Perc=	-2665.7585
60Perc=	-2388.1514
65Perc=	-2088.7673
70Perc=	-1783.895
75Perc=	-1419.657
80Perc=	-1058.7205
85Perc=	-654.39386
90Perc=	-26.648571
95Perc=	724.312378
<b>Target Values:</b>	
Value#1=	0
Prob#1=	90.23%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.3 Result of MCS for  
Example 5.1 (Iterations: 10000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 14:06	
Iterations: 10000	
Simulations: 1	
Worksheet: [EXAM513.XLS]Sheet1	
Output Range: c1	
Cell: \$C\$1	
Minimum=	-11581.647
Maximum=	5108.83838
Mean=	-2984.3708
Std Deviation=	2228.0176
Variance=	4964062.44
Skewness=	0.00396843
Kurtosis=	2.97025509
<b>Percentile Values</b>	
5Perc=	-6642.9585
10Perc=	-5869.8999
15Perc=	-5293.2651
20Perc=	-4862.1792
25Perc=	-4479.042
30Perc=	-4149.73
35Perc=	-3836.416
40Perc=	-3555.2551
45Perc=	-3272.7834
50Perc=	-2991.3821
55Perc=	-2694.0796
60Perc=	-2418.0503
65Perc=	-2134.6563
70Perc=	-1815.8226
75Perc=	-1483.0277
80Perc=	-1110.149
85Perc=	-689.37946
90Perc=	-107.59158
95Perc=	682.096008
<b>Target Values:</b>	
Value#1=	0
Prob#1=	90.73%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.4 Result of MCS for  
Example 5.2 (Iterations: 1000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 14:18	
Iterations: 1000	
Simulations: 1	
Worksheet: [EXAM521.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-1765.0129
Maximum=	603.375488
Mean=	-715.58121
Std Deviation=	468.091297
Variance=	219109.462
Skewness=	0.19173856
Kurtosis=	2.58295499
<b>Percentile Values</b>	
5Perc=	-1476.3433
10Perc=	-1333.89
15Perc=	-1224.0652
20Perc=	-1129.718
25Perc=	-1043.0052
30Perc=	-977.72021
35Perc=	-916.95184
40Perc=	-856.27869
45Perc=	-804.07874
50Perc=	-745.21069
55Perc=	-689.03204
60Perc=	-621.82452
65Perc=	-547.14642
70Perc=	-483.59274
75Perc=	-397.47775
80Perc=	-301.58673
85Perc=	-202.57747
90Perc=	-64.112083
95Perc=	75.8667297
<b>Target Values:</b>	
Value#1=	0
Prob#1=	92.44%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.5 Result of MCS for  
Example 5.2 (Iterations: 5000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 14:30	
Iterations: 5000	
Simulations: 1	
Worksheet: [EXAM522.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-1913.1836
Maximum=	700.233521
Mean=	-696.90153
Std Deviation=	464.037291
Variance=	215330.608
Skewness=	0.14744288
Kurtosis=	2.55967529
<b>Percentile Values</b>	
5Perc=	-1443.7096
10Perc=	-1305.248
15Perc=	-1194.4382
20Perc=	-1110.3376
25Perc=	-1034.5133
30Perc=	-963.26733
35Perc=	-894.63483
40Perc=	-831.5816
45Perc=	-774.53729
50Perc=	-715.60284
55Perc=	-656.6582
60Perc=	-590.32068
65Perc=	-518.54053
70Perc=	-444.33749
75Perc=	-368.94357
80Perc=	-294.15656
85Perc=	-191.58217
90Perc=	-64.406837
95Perc=	92.8300095
<b>Target Values:</b>	
Value#1=	0
Prob#1=	92.42%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.6 Result of MCS for  
Example 5.2 (Iterations: 10000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 14:44	
Iterations: 10000	
Simulations: 1	
Worksheet: [EXAM523.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-1974.674
Maximum=	783.390686
Mean=	-690.91723
Std Deviation=	460.236729
Variance=	211817.846
Skewness=	0.1392014
Kurtosis=	2.58862764
<b>Percentile Values</b>	
5Perc=	-1428.3361
10Perc=	-1287.4854
15Perc=	-1179.9274
20Perc=	-1094.4569
25Perc=	-1022.5876
30Perc=	-952.84241
35Perc=	-886.06207
40Perc=	-826.45734
45Perc=	-771.19385
50Perc=	-711.24554
55Perc=	-651.71515
60Perc=	-584.80768
65Perc=	-517.15625
70Perc=	-445.14413
75Perc=	-368.27719
80Perc=	-287
85Perc=	-186.73816
90Perc=	-67.991859
95Perc=	99.0868988
<b>Target Values:</b>	
Value#1=	0
Prob#1=	92.39%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.7 Result of MCS for  
Example 5.3 (Iterations: 1000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 15:48	
Iterations: 1000	
Simulations: 1	
Worksheet: [EXAM531.XLS]Sheet1	
Output Range: g1	
Cell: \$G\$1	
Minimum=	-36.762199
Maximum=	43.9394493
Mean=	6.89553458
Std Deviation=	10.7689985
Variance=	115.971329
Skewness=	0.01634484
Kurtosis=	3.12804619
<b>Percentile Values</b>	
5Perc=	-10.458655
10Perc=	-7.044415
15Perc=	-4.4867992
20Perc=	-2.3535042
25Perc=	-0.2849787
30Perc=	1.2991457
35Perc=	2.68727422
40Perc=	4.39856577
45Perc=	5.67128944
50Perc=	6.79565287
55Perc=	8.02162361
60Perc=	9.74863243
65Perc=	11.0057831
70Perc=	12.2635469
75Perc=	13.5652895
80Perc=	15.662323
85Perc=	18.4235859
90Perc=	20.8623314
95Perc=	24.7833309
<b>Target Values:</b>	
Value#1=	0
Prob#1=	25.69%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.8 Result of MCS for  
Example 5.3 (Iterations: 5000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 16:02	
Iterations: 5000	
Simulations: 1	
Worksheet: [EXAM532.XLS]Sheet1	
Output Range: g1	
Cell: \$G\$1	
Minimum=	-31.327232
Maximum=	41.8832474
Mean=	6.8539356
Std Deviation=	10.6934314
Variance=	114.349474
Skewness=	-0.029442
Kurtosis=	3.06841175
<b>Percentile Values</b>	
5Perc=	-10.821691
10Perc=	-6.8102102
15Perc=	-4.0151949
20Perc=	-1.9846771
25Perc=	-0.1901521
30Perc=	1.40919054
35Perc=	2.84251714
40Perc=	4.14949894
45Perc=	5.69266939
50Perc=	6.90265894
55Perc=	8.14382553
60Perc=	9.39259052
65Perc=	10.7588911
70Perc=	12.4248838
75Perc=	13.9434052
80Perc=	15.8373642
85Perc=	18.0796642
90Perc=	20.4335442
95Perc=	24.1932201
<b>Target Values:</b>	
Value#1=	0
Prob#1=	25.65%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.9 Result of MCS for  
Example 5.3 (Iterations: 10000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 16:37	
Iterations: 10000	
Simulations: 1	
Worksheet: [EXAM533.XLS]Sheet1	
Output Range: g1	
Cell: \$G\$1	
Minimum=	-33.284164
Maximum=	46.6574135
Mean=	6.86532694
Std Deviation=	10.6610837
Variance=	113.658706
Skewness=	-0.0132921
Kurtosis=	2.96907989
<b>Percentile Values</b>	
5Perc=	-10.858703
10Perc=	-6.7663484
15Perc=	-4.1733117
20Perc=	-2.0116925
25Perc=	-0.27588
30Perc=	1.28513455
35Perc=	2.73557949
40Perc=	4.13635159
45Perc=	5.55665779
50Perc=	6.78287411
55Perc=	8.18251228
60Perc=	9.53145504
65Perc=	10.9226713
70Perc=	12.3976498
75Perc=	13.8669104
80Perc=	15.8484955
85Perc=	17.9988232
90Perc=	20.7258568
95Perc=	24.5490437
<b>Target Values:</b>	
Value#1=	0
Prob#1=	25.95%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.10 Result of MCS for  
Example 5.4 (Iterations: 1000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 17:03	
Iterations: 1000	
Simulations: 1	
Worksheet: [EXAM541.XLS]Sheet1	
Output Range: f1	
Cell: \$F\$1	
Minimum=	-89.332573
Maximum=	271.938477
Mean=	96.017898
Std Deviation=	69.0112536
Variance=	4762.55312
Skewness=	-0.0056363
Kurtosis=	2.39654348
<b>Percentile Values</b>	
5Perc=	-15.579628
10Perc=	3.27033496
15Perc=	19.6893406
20Perc=	31.0949192
25Perc=	43.6366081
30Perc=	55.1318359
35Perc=	67.3055038
40Perc=	77.3235092
45Perc=	87.0890579
50Perc=	95.9907684
55Perc=	106.895538
60Perc=	115.357872
65Perc=	125.881012
70Perc=	135.079117
75Perc=	145.339279
80Perc=	156.526749
85Perc=	172.124573
90Perc=	189.901611
95Perc=	208.099625
<b>Target Values:</b>	
Value#1=	0
Prob#1=	9.10%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.11 Result of MCS for Example 5.4 (Iterations: 5000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 17:17	
Iterations: 5000	
Simulations: 1	
Worksheet: [EXAM542.XLS]Sheet1	
Output Range: f1	
Cell: \$F\$1	
Minimum=	-116.15926
Maximum=	275.643524
Mean=	93.9599403
Std Deviation=	69.6791869
Variance=	4855.18909
Skewness=	-0.078298
Kurtosis=	2.52703636
<b>Percentile Values</b>	
5Perc=	-23.315098
10Perc=	1.3281368
15Perc=	19.1741028
20Perc=	33.0485191
25Perc=	44.8157806
30Perc=	55.1193619
35Perc=	64.9777145
40Perc=	75.6762238
45Perc=	85.6387329
50Perc=	95.1725845
55Perc=	103.907097
60Perc=	114.104477
65Perc=	123.524048
70Perc=	133.38324
75Perc=	144.111679
80Perc=	156.369675
85Perc=	169.691925
90Perc=	186.106689
95Perc=	208.259949
<b>Target Values:</b>	
Value#1=	0
Prob#1=	9.72%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.12 Result of MCS for Example 5.4 (Iterations: 10000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 17:37	
Iterations: 10000	
Simulations: 1	
Worksheet: [EXAM543.XLS]Sheet1	
Output Range: f1	
Cell: \$F\$1	
Minimum=	-127.67753
Maximum=	282.725494
Mean=	95.0142941
Std Deviation=	70.1430636
Variance=	4920.04937
Skewness=	-0.0446689
Kurtosis=	2.49646264
<b>Percentile Values</b>	
5Perc=	-21.388271
10Perc=	1.32783759
15Perc=	18.6947327
20Perc=	32.5424118
25Perc=	45.142601
30Perc=	55.7441521
35Perc=	66.1949844
40Perc=	75.751297
45Perc=	85.3317871
50Perc=	95.5725327
55Perc=	104.484718
60Perc=	114.397163
65Perc=	124.680656
70Perc=	135.537979
75Perc=	146.773193
80Perc=	158.738312
85Perc=	171.751572
90Perc=	186.608719
95Perc=	209.063019
<b>Target Values:</b>	
Value#1=	0
Prob#1=	9.65%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.13 Result of MCS for Example 5.5 (Iterations: 1000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 17:45	
Iterations: 1000	
Simulations: 1	
Worksheet: [EXAM551.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-160.65192
Maximum=	124.311569
Mean=	-3.7361749
Std Deviation=	41.3966
Variance=	1713.67849
Skewness=	0.05138072
Kurtosis=	2.95067304
<b>Percentile Values</b>	
5Perc=	-70.225014
10Perc=	-57.168552
15Perc=	-48.304653
20Perc=	-38.011036
25Perc=	-31.725298
30Perc=	-26.526337
35Perc=	-20.255077
40Perc=	-14.833058
45Perc=	-10.215347
50Perc=	-3.8569243
55Perc=	0.35001042
60Perc=	5.86884212
65Perc=	12.8138599
70Perc=	17.3719234
75Perc=	23.1092033
80Perc=	30.693615
85Perc=	38.7512779
90Perc=	49.6102791
95Perc=	66.7696991
<b>Target Values:</b>	
Value#1=	0
Prob#1=	54.39%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.14 Result of MCS for Example 5.5 (Iterations: 5000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 17:53	
Iterations: 5000	
Simulations: 1	
Worksheet: [EXAM552.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-160.65192
Maximum=	148.971756
Mean=	-0.3452853
Std Deviation=	42.002376
Variance=	1764.19959
Skewness=	0.0761609
Kurtosis=	3.02896889
<b>Percentile Values</b>	
5Perc=	-68.356339
10Perc=	-54.369816
15Perc=	-43.661034
20Perc=	-35.294518
25Perc=	-28.06793
30Perc=	-22.309908
35Perc=	-16.653494
40Perc=	-11.367287
45Perc=	-6.2642765
50Perc=	-1.1310232
55Perc=	4.1264534
60Perc=	9.84256172
65Perc=	15.5110922
70Perc=	21.0051117
75Perc=	26.8546585
80Perc=	33.9089012
85Perc=	42.9630699
90Perc=	53.4009361
95Perc=	70.0624313
<b>Target Values:</b>	
Value#1=	0
Prob#1=	51.08%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	



**Table E.15 Result of MCS for Example 5.5 (Iterations: 10000)**

<b>Simulation Statistics</b>	
Date: 9/24/00 at 18:05	
Iterations: 10000	
Simulations: 1	
Worksheet: [EXAM553.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-160.65192
Maximum=	177.273682
Mean=	-0.5238962
Std Deviation=	41.6582346
Variance=	1735.40851
Skewness=	0.07595041
Kurtosis=	3.04112621
<b>Percentile Values</b>	
5Perc=	-68.127388
10Perc=	-53.999634
15Perc=	-43.830036
20Perc=	-35.482227
25Perc=	-28.557957
30Perc=	-22.551968
35Perc=	-16.880651
40Perc=	-11.4816
45Perc=	-6.1627154
50Perc=	-0.8212615
55Perc=	4.24390936
60Perc=	9.96992874
65Perc=	15.3333626
70Perc=	20.8457336
75Perc=	26.7856178
80Perc=	34.0170631
85Perc=	42.5923882
90Perc=	52.661541
95Perc=	68.4345779
<b>Target Values:</b>	
Value#1=	0
Prob#1=	50.81%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.16 Result of MCS for Example 5.6 (Iterations: 1000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 10:39	
Iterations: 1000	
Simulations: 1	
Worksheet: [EXAM561.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-47.996399
Maximum=	51.1980705
Mean=	-2.3541231
Std Deviation=	23.7667493
Variance=	564.858371
Skewness=	0.1090199
Kurtosis=	1.95346002
<b>Percentile Values</b>	
5Perc=	-37.880852
10Perc=	-34.202728
15Perc=	-30.397425
20Perc=	-25.957621
25Perc=	-22.333754
30Perc=	-19.043139
35Perc=	-15.023026
40Perc=	-11.519631
45Perc=	-7.5585423
50Perc=	-2.8802407
55Perc=	0.50922072
60Perc=	4.98452568
65Perc=	9.24302864
70Perc=	12.9785805
75Perc=	16.9875278
80Perc=	21.3547001
85Perc=	25.631813
90Perc=	30.0369415
95Perc=	36.193203
<b>Target Values:</b>	
Value#1=	0
Prob#1=	53.83%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.17 Result of MCS for Example 5.6 (Iterations: 5000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 11:01	
Iterations: 5000	
Simulations: 1	
Worksheet: [EXAM562.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-48.529606
Maximum=	52.2067642
Mean=	-0.4002599
Std Deviation=	24.0317788
Variance=	577.526391
Skewness=	0.04539308
Kurtosis=	1.97227798
<b>Percentile Values</b>	
5Perc=	-37.643738
10Perc=	-32.909523
15Perc=	-28.843691
20Perc=	-24.563255
25Perc=	-20.289772
30Perc=	-16.448608
35Perc=	-12.465761
40Perc=	-8.5529766
45Perc=	-4.783761
50Perc=	-0.6568476
55Perc=	3.14949346
60Perc=	7.46211147
65Perc=	11.4038429
70Perc=	15.5157652
75Perc=	19.4222965
80Perc=	23.1471195
85Perc=	27.5908833
90Perc=	31.7397041
95Perc=	38.1779861
<b>Target Values:</b>	
Value#1=	0
Prob#1=	50.95%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.18 Result of MCS for Example 5.6 (Iterations: 10000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 11:25	
Iterations: 10000	
Simulations: 1	
Worksheet: [EXAM563.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-48.529606
Maximum=	52.2906189
Mean=	-0.4922912
Std Deviation=	23.9656899
Variance=	574.354292
Skewness=	0.0407718
Kurtosis=	1.95787886
<b>Percentile Values</b>	
5Perc=	-37.643738
10Perc=	-32.819801
15Perc=	-28.803192
20Perc=	-24.756281
25Perc=	-20.634268
30Perc=	-16.693901
35Perc=	-12.692547
40Perc=	-8.7334299
45Perc=	-4.8551817
50Perc=	-0.5728708
55Perc=	3.32977605
60Perc=	7.62647724
65Perc=	11.4043121
70Perc=	15.4353952
75Perc=	19.2139816
80Perc=	23.1984386
85Perc=	27.3627319
90Perc=	31.5872669
95Perc=	37.6813889
<b>Target Values:</b>	
Value#1=	0
Prob#1=	50.81%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.19 Result of MCS for Example 5.7 (Iterations: 1000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 11:47	
Iterations: 1000	
Simulations: 1	
Worksheet: [EXAM571.XLS]Sheet16	
Output Range: f1	
Cell: \$F\$1	
Minimum=	-1998647
Maximum=	2985513.25
Mean=	514470.976
Std Deviation=	881480.748
Variance=	7.7701E+11
Skewness=	0.14943519
Kurtosis=	2.72049455
<b>Percentile Values</b>	
5Perc=	-913304.94
10Perc=	-631837.63
15Perc=	-447767.88
20Perc=	-290041.53
25Perc=	-100288.66
30Perc=	57381.6445
35Perc=	179916.516
40Perc=	279185.156
45Perc=	382915.031
50Perc=	475166.969
55Perc=	574169.563
60Perc=	709655.813
65Perc=	825439.875
70Perc=	950014.563
75Perc=	1074339.88
80Perc=	1242797.38
85Perc=	1472565.25
90Perc=	1741234.88
95Perc=	1990182.13
<b>Target Values:</b>	
Value#1=	0
Prob#1=	27.95%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.20 Result of MCS for Example 5.7 (Iterations: 5000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 12:05	
Iterations: 5000	
Simulations: 1	
Worksheet: [EXAM572.XLS]Sheet16	
Output Range: f1	
Cell: \$F\$1	
Minimum=	-2193586.8
Maximum=	4068546.75
Mean=	523407.019
Std Deviation=	891845.119
Variance=	7.9539E+11
Skewness=	0.26643133
Kurtosis=	3.04365304
<b>Percentile Values</b>	
5Perc=	-892600.5
10Perc=	-605240.13
15Perc=	-378396.88
20Perc=	-231269.05
25Perc=	-98312.578
30Perc=	26705.3047
35Perc=	143883.156
40Perc=	259587.344
45Perc=	375457.594
50Perc=	475400.656
55Perc=	584705.188
60Perc=	699491.313
65Perc=	821643.125
70Perc=	969100.938
75Perc=	1112838.13
80Perc=	1275270.13
85Perc=	1466090.5
90Perc=	1688293.5
95Perc=	2048684.75
<b>Target Values:</b>	
Value#1=	0
Prob#1=	29.11%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.21 Result of MCS for Example 5.7 (Iterations: 10000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 12:22	
Iterations: 10000	
Simulations: 1	
Worksheet: [EXAM573.XLS]Sheet16	
Output Range: f1	
Cell: \$F\$1	
Minimum=	-2380475
Maximum=	5149482
Mean=	517219.373
Std Deviation=	893205.308
Variance=	7.9782E+11
Skewness=	0.29551465
Kurtosis=	3.17899037
<b>Percentile Values</b>	
5Perc=	-885443.13
10Perc=	-600145.19
15Perc=	-395872.63
20Perc=	-238349.28
25Perc=	-102408.47
30Perc=	27819.041
35Perc=	141928.438
40Perc=	250856.906
45Perc=	358412.375
50Perc=	479925.344
55Perc=	583621.25
60Perc=	703642.375
65Perc=	821644.188
70Perc=	945794
75Perc=	1087090.75
80Perc=	1252198.38
85Perc=	1444897.63
90Perc=	1689089.75
95Perc=	2048360.25
<b>Target Values:</b>	
Value#1=	0
Prob#1=	28.89%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.22 Result of MCS for Example 5.8 (Iterations: 1000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 12:30	
Iterations: 1000	
Simulations: 1	
Worksheet: [EXAM581.XLS]Sheet16	
Output Range: f1	
Cell: \$F\$1	
Minimum=	-1981126
Maximum=	3031806.25
Mean=	514621.184
Std Deviation=	880510.711
Variance=	7.753E+11
Skewness=	0.23988034
Kurtosis=	2.77984541
<b>Percentile Values</b>	
5Perc=	-869915.56
10Perc=	-620364.31
15Perc=	-453524.41
20Perc=	-284014.34
25Perc=	-113780.21
30Perc=	52216.3164
35Perc=	180117.484
40Perc=	258820.094
45Perc=	364386
50Perc=	458445
55Perc=	560487
60Perc=	696399.063
65Perc=	808717.063
70Perc=	936092.688
75Perc=	1056983.88
80Perc=	1233062
85Perc=	1462355.88
90Perc=	1732165.25
95Perc=	2010415.63
<b>Target Values:</b>	
Value#1=	0
Prob#1=	28.29%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.23 Result of MCS for Example 5.8 (Iterations: 5000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 12:40	
Iterations: 5000	
Simulations: 1	
Worksheet: [EXAM582.XLS]Sheet16	
Output Range: f1	
Cell: \$F\$1	
Minimum=	-2128897.3
Maximum=	4257503.5
Mean=	523375.516
Std Deviation=	894724.361
Variance=	8.0053E+11
Skewness=	0.36094143
Kurtosis=	3.16190517
<b>Percentile Values</b>	
5Perc=	-864039.44
10Perc=	-589847.06
15Perc=	-370679.47
20Perc=	-234153.42
25Perc=	-104914.98
30Perc=	21002.9668
35Perc=	131612.125
40Perc=	246497.906
45Perc=	360741.938
50Perc=	458906.125
55Perc=	569098.75
60Perc=	682099.313
65Perc=	814899.625
70Perc=	958128.063
75Perc=	1097579.75
80Perc=	1267116.5
85Perc=	1461873.63
90Perc=	1696130.63
95Perc=	2092442.63
<b>Target Values:</b>	
Value#1=	0
Prob#1=	29.28%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.24 Result of MCS for Example 5.8 (Iterations: 10000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 13:33	
Iterations: 10000	
Simulations: 1	
Worksheet: [EXAM583.XLS]Sheet16	
Output Range: f1	
Cell: \$F\$1	
Minimum=	-2097735.8
Maximum=	5546798.5
Mean=	516957.624
Std Deviation=	895368.327
Variance=	8.0168E+11
Skewness=	0.39090599
Kurtosis=	3.3116238
<b>Percentile Values</b>	
5Perc=	-854943.69
10Perc=	-592739.94
15Perc=	-393540.13
20Perc=	-243081.69
25Perc=	-107496.58
30Perc=	18528.8984
35Perc=	132352.156
40Perc=	241213.563
45Perc=	345110.031
50Perc=	467906.906
55Perc=	569000.75
60Perc=	689756.25
65Perc=	808014.375
70Perc=	933878.5
75Perc=	1075468.38
80Perc=	1245072.75
85Perc=	1448588.75
90Perc=	1692789.88
95Perc=	2071704.13
<b>Target Values:</b>	
Value#1=	0
Prob#1=	29.26%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.25 Result of MCS for Example 5.9 (Iterations: 1000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 14:10	
Iterations: 1000	
Simulations: 1	
Worksheet: [EXAM591.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-195.91122
Maximum=	508.555878
Mean=	145.074344
Std Deviation=	112.616969
Variance=	12682.5816
Skewness=	-0.0006628
Kurtosis=	2.86244348
<b>Percentile Values</b>	
5Perc=	-45.621143
10Perc=	-6.7190733
15Perc=	18.4842529
20Perc=	49.6194305
25Perc=	72.2490082
30Perc=	89.2270279
35Perc=	105.184105
40Perc=	118.042801
45Perc=	130.619141
50Perc=	145.587784
55Perc=	161.142456
60Perc=	175.17804
65Perc=	190.37854
70Perc=	207.052505
75Perc=	220.240036
80Perc=	239.02887
85Perc=	257.609619
90Perc=	284.817474
95Perc=	330.901031
<b>Target Values:</b>	
Value#1=	0
Prob#1=	11.11%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.26 Result of MCS for Example 5.9 (Iterations: 5000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 14:19	
Iterations: 5000	
Simulations: 1	
Worksheet: [EXAM592.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-283.62384
Maximum=	592.517029
Mean=	148.434327
Std Deviation=	115.710122
Variance=	13388.8323
Skewness=	0.01177897
Kurtosis=	3.07661376
<b>Percentile Values</b>	
5Perc=	-41.55402
10Perc=	1.08793116
15Perc=	27.3380356
20Perc=	51.5654411
25Perc=	71.7176437
30Perc=	90.1675644
35Perc=	106.454956
40Perc=	120.967499
45Perc=	133.924591
50Perc=	147.552689
55Perc=	162.263626
60Perc=	175.91713
65Perc=	191.838547
70Perc=	207.556335
75Perc=	224.89263
80Perc=	245.061111
85Perc=	265.70929
90Perc=	295.758972
95Perc=	338.595337
<b>Target Values:</b>	
Value#1=	0
Prob#1=	9.88%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.27 Result of MCS for Example 5.9 (Iterations: 10000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 14:39	
Iterations: 10000	
Simulations: 1	
Worksheet: [EXAM593.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-362.9595
Maximum=	592.517029
Mean=	148.576017
Std Deviation=	114.949351
Variance=	13213.3533
Skewness=	-0.0341989
Kurtosis=	3.0508677
<b>Percentile Values</b>	
5Perc=	-41.008083
10Perc=	3.23528552
15Perc=	28.9392242
20Perc=	51.265564
25Perc=	71.1767197
30Perc=	89.2869415
35Perc=	105.011925
40Perc=	119.948166
45Perc=	133.584793
50Perc=	148.196335
55Perc=	162.825043
60Perc=	177.548248
65Perc=	193.304047
70Perc=	208.6064
75Perc=	226.459702
80Perc=	246.2854
85Perc=	267.327332
90Perc=	295.74173
95Perc=	335.970001
<b>Target Values:</b>	
Value#1=	0
Prob#1=	9.49%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.28 Result of MCS for Example 5.10 (Iterations: 1000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 14:47	
Iterations: 1000	
Simulations: 1	
Worksheet: [EXAM5101.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-25775.414
Maximum=	16596.1348
Mean=	-4844.6972
Std Deviation=	7382.84175
Variance=	54506352.4
Skewness=	0.25827092
Kurtosis=	2.92069759
<b>Percentile Values</b>	
5Perc=	-16236.593
10Perc=	-14313.745
15Perc=	-12473.52
20Perc=	-11225.131
25Perc=	-9960.2607
30Perc=	-8763.7695
35Perc=	-7778.561
40Perc=	-6980.0366
45Perc=	-6182.9785
50Perc=	-5055.2129
55Perc=	-4368.8569
60Perc=	-3285.2429
65Perc=	-2429.52
70Perc=	-1356.5184
75Perc=	-249.83247
80Perc=	1094.30762
85Perc=	2706.62866
90Perc=	5049.39258
95Perc=	8365.88965
<b>Target Values:</b>	
Value#1=	0
Prob#1=	75.78%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.29 Result of MCS for  
Example 5.10 (Iterations: 5000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 14:54	
Iterations: 5000	
Simulations: 1	
Worksheet: [EXAM510 2.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-28598.707
Maximum=	25319.082
Mean=	-4504.145
Std Deviation=	7217.02984
Variance=	52085519.7
Skewness=	0.32992361
Kurtosis=	3.24790517
<b>Percentile Values</b>	
5Perc=	-15647.145
10Perc=	-13409.619
15Perc=	-11934.222
20Perc=	-10636.951
25Perc=	-9523.6064
30Perc=	-8459.3477
35Perc=	-7509.7461
40Perc=	-6623.0894
45Perc=	-5711.1104
50Perc=	-4867.4346
55Perc=	-4075.0247
60Perc=	-3083.4956
65Perc=	-2118.9106
70Perc=	-1104.8951
75Perc=	33.4974098
80Perc=	1290.69324
85Perc=	2863.36426
90Perc=	4963.35352
95Perc=	7964.53711
<b>Target Values:</b>	
Value#1=	0
Prob#1=	74.85%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	

**Table E.30 Result of MCS for  
Example 5.10 (Iterations: 10000)**

<b>Simulation Statistics</b>	
Date: 9/25/00 at 15:06	
Iterations: 10000	
Simulations: 1	
Worksheet: [EXAM510 3.XLS]Sheet1	
Output Range: b1	
Cell: \$B\$1	
Minimum=	-28598.707
Maximum=	25319.082
Mean=	-4427.3228
Std Deviation=	7173.22026
Variance=	51455088.8
Skewness=	0.32141265
Kurtosis=	3.13209567
<b>Percentile Values</b>	
5Perc=	-15562.58
10Perc=	-13370.849
15Perc=	-11828.785
20Perc=	-10561.132
25Perc=	-9423.4854
30Perc=	-8389.4512
35Perc=	-7476.3335
40Perc=	-6564.1055
45Perc=	-5657.9888
50Perc=	-4802.6709
55Perc=	-3920.3535
60Perc=	-2965.554
65Perc=	-1981.9266
70Perc=	-1012.8314
75Perc=	70.6205521
80Perc=	1361.66223
85Perc=	2928.51611
90Perc=	5038.22363
95Perc=	8059.94971
<b>Target Values:</b>	
Value#1=	0
Prob#1=	74.62%
Value#2=	
Prob#2=	
Value#3=	
Prob#3=	
Value#4=	
Prob#4=	
Value#5=	
Prob#5=	
Value#6=	
Prob#6=	
Value#7=	
Prob#7=	
Value#8=	
Prob#8=	
Value#9=	
Prob#9=	
Value#10=	
Prob#10=	