Semi-Global Persistence and Stability for a Class of Forced Discrete-Time Population Models

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Abstract

We consider persistence and stability properties for a class of forced discrete-time difference equations with three defining properties: the solution is constrained to evolve in the non-negative orthant, the forcing acts multiplicatively, and the dynamics are described by so-called Lur'e systems, containing both linear and non-linear terms. Many discrete-time biological models encountered in the literature may be expressed in the form of a Lur'e system and, in this context, the multiplicative forcing may correspond to harvesting, culling or time-varying (such as seasonal) vital rates or environmental conditions. Drawing upon techniques from systems and control theory, and assuming that the forcing is bounded, we provide conditions under which persistence occurs and, further, that a unique non-zero equilibrium is stable with respect to the forcing in a sense which is reminiscent of input-to-state stability, a concept well-known in nonlinear control theory. The theoretical results are illustrated with several examples. In particular, we discuss how our results relate to previous literature on stabilization of chaotic systems by so-called proportional feedback control.

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1. Introduction

We consider persistence and stability properties of forced discrete-time nonlinear models which take the form

$$x^\nabla = Ax + bh(c^T x, u), \quad x(0) = x^0, \tag{1.1}$$

and which are constrained to evolve in the non-negative orthant of n-dimensional Euclidean space. Here x^∇ denotes the image of x under the left-shift operator, that is, $x^\nabla(t) = x(t+1)$ for all $t \in \mathbb{Z}_+$, where $\mathbb{Z}_+$ denotes the set of non-negative integers. Further, $x^0 \in \mathbb{R}_+^n$, $A \in \mathbb{R}_+^{n \times n}$, $b, c \in \mathbb{R}_+^n$ with c^T denoting the transpose of c, and $h : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a (nonlinear) function. The scalar-valued non-negative sequence u plays the role of a forcing term (sometimes termed and understood as a control or disturbance).

Models of the form (1.1) arise and have subsequently been studied in a variety of contexts, by a range of academic disciplines, and with a spectrum of techniques. The system (1.1) is an example of a positive system, and, in the special case that u is constant, it is a positive dynamical system, see [3, 28, 41, 42]. The state variables of positive systems are constrained to evolve in positive cones and, in the present context, the positive cone is simply $\mathbb{R}_+^n$, viewed as a subset of real n-dimensional Euclidean space, equipped with the usual partial ordering of componentwise inequality between vectors. Moreover, when

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$y \mapsto h(y, u)$ is a monotone increasing function, then (1.1) is an instance of a monotone dynamical system, see [33, 34, 60] and the references therein. Monotone dynamical systems are characterised by the property that the evolution map is order preserving with respect to the partial order defined by the positive cone. The study of positive and monotone systems is aided by comparison arguments and readily constructed classes of Lyapunov functions, such as so-called max- or sum-separable Lyapunov functions [52]. Of a plethora of potential references, we highlight [18] as an example which contains stability results for the zero equilibrium of general discrete-time monotone dynamical systems.

When $n = 1$, $A = 0$ and $b = c = 1$, then (1.1) reduces to the one-dimensional system

$$x^\nabla = h(x, u), \quad x(0) = x^0,$$

which has been analysed in a variety of physical and biological contexts [15, 67] — usually for constant u, the logistic map being a particularly well-studied example. We remark that even in this simplest case, system (1.1) can show very complicated dynamics [49]. In population models, the function h is commonly called the production function and frequently (but not always) is of the form $g(x, u)x$, where the function g captures the per-capita growth rate of the population. We will consider two situations: (i) the forcing only affects the per-capita growth rate, and (ii) the forcing in (1.1) affects the whole production function.

Another area in which (1.1) arises is systems & control theory and control engineering, where these models are often called Lur'e systems (after A.I. Lur'e, a Soviet scientist who made early contributions to the stability theory of continuous-time Lur'e systems). Today Lur'e systems are a common and important class of nonlinear control systems. Under the assumption that the nonlinearity h satisfies $h(0, u) = 0$, it follows that 0 is an equilibrium of (1.1). A so-called absolute stability criterion for (1.1) is a sufficient condition for the stability of this equilibrium, usually formulated in terms of the linear components (A, b, c) and related sector or boundedness conditions for h: stability is guaranteed for every nonlinearity h satisfying these sector or boundedness properties, thereby ensuring robustness of stability with respect to uncertainty in h. There is a large and contemporary body of work on absolute stability theory; see, for example, [27, 36, 37, 40, 44, 56, 57, 70, 72].

Lur'e systems capture density-independent as well as density-dependent vital or transition rates and have been used to address various aspects of population dynamics, see, for example, [6, 21, 22, 23, 25, 51, 62, 69], evidencing their suitability in this context. To the best of our knowledge, the paper [69] was the first to combine tools from positive dynamical systems and absolute stability theory and apply them to (unforced) models of the form (1.1) with two equilibria (the origin and a positive equilibrium). In applied contexts, such as population dynamics, these equilibria correspond to extinction and a non-zero, co-existent equilibrium, respectively. The results of [69] have been extended to classes of infinite-dimensional models in [51, 62] and to forced continuous-time Lur'e systems in [6]. Part of the appeal of [6, 69] is that a so-called "trichotomy of stability" is presented, describing in terms of the model parameters when, under certain assumptions, solutions either converge to the zero equilibrium, or a unique non-zero equilibrium, or diverge. Similar trichotomies have been established for various classes of monotone discrete-time dynamical systems: for finite-dimensional systems in [43] and [42, chapter 6] and for infinite-dimensional systems in [33, 62].

The model (1.1) also contains a forcing term u, the role of which we have not yet mentioned. Borrowing from the systems and control framework, there are two possible interpretations for the forcing u. On the one hand, it could represent "control actions" which are chosen, designed or determined by the modeller or end-user, corresponding to harvesting, culling, replanting or treating effort in a population model. In these cases, u is known or u may be generated by a suitable feedback law. On the other hand, u may represent a (possibly unwanted and unknown) disturbance over which the modeller or end-user has no control, but must account for. Predation or poaching or unaccounted and unmodelled environmental or demographic temporal variation all fall within this latter category for population models. From the perspective of management, particularly of a resource to be conserved, it is essential to understand the effect of both control actions and disturbances on persistence and stability properties. Persistence is a fundamental aspect of population modelling which has been incorporated into mainstream mathematical biology with detailed treatments from both deterministic [61] and stochastic [59] perspectives.

In the context of models of the form (1.1), we pursue three lines of enquiry: persistence, stability and control; the latter two being intimately related. Indeed, given the model (1.1), one may wish to know: how the choice of control (or forcing) effects equilibria and their attractivity and stability properties.

We comment that the word "stabilization" is used in control theory to describe the action of designing controls which make an unstable equilibrium stable (such as the upright position of a pendulum — the so-called inverted pendulum): the control action does not change the given equilibrium, but it changes its stability properties; see, for example, [10, 40, 64]. In the context of chaos control and its applications to population dynamics, stabilization often means reshaping possibly periodic or chaotic dynamics by choice of control to give rise to some desired equilibrium that then enjoys desired stability properties; including, for instance, [8, 20, 55].

Typically, the origin will be an equilibrium of the system (1.1) (for any control u) and in a population dynamics context it is of interest to investigate if solutions (or certain components) corresponding to non-zero initial conditions are "persistent", that is, bounded away or ultimately bounded away from zero. We derive conditions under which a semi-global uniform persistence property holds: here semi-globality relates to initial conditions and controls, whilst uniformity is with respect to time. Persistence results may alternatively be interpreted as the zero equilibrium being repelling (in a certain sense). Furthermore, we provide conditions which guarantee, for a given constant control u^*, the existence of a unique positive equilibrium x^* which is input-to-state stable. Input-to-state stability is a well-known control theoretic concept which is an appropriate notion of stability for forced systems: roughly speaking, input-to-state stability means that the map $(x^0, u) \mapsto x$ enjoys certain uniform continuity properties: in particular, if $\|x^0 - x^*\| + \sup_{t \geq 0} \|u(t) - u^*\|$ is small, then $\sup_{t \geq 0} \|x(t) - x^*\|$ is small and if $u(t) \to u^*$ as $t \to \infty$, then $x(t) \to x^*$ as $t \to \infty$ for every non-zero initial condition x^0. More details on input-to-state stability can be found further below and in the survey papers [19, 66].

We also provide sufficient conditions which guarantee that convergent forcing (with arbitrary limit, not necessarily equal to u^*) yields a convergent state (this property is known as converging-input converging-state property, cf. [7, 65]). Finally, we apply our stability theory in the context of proportional feedback control which has received considerable attention in chaos control and its application to theoretical ecology: the results obtained complement, extend and strengthen those in [9, 11, 12, 45, 48].

The paper is organised as follows. Section 2 gathers mathematical preliminaries. Section 3 describes the forced discrete-time non-linear models which we consider. Sections 4 and 5 contain the main contribution of the paper, a suite of persistence and stability results for the models under consideration. Finally, Section 6 contains several examples from population dynamics and chaos control.

2. Preliminaries

For $I \subseteq \mathbb{R}$, $J \subseteq \mathbb{R}^n$, let $\mathcal{F}(I, J)$ denote the set of all functions $I \to J$. For $w \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}^n)$, we set

$$\|w\|_{\infty, t} := \max\{\|w(s)\| : 0 \leq s \leq t\}, \quad t \in \mathbb{Z}_+,$$

and, if w is bounded, then we define

$$\|w\|_\infty := \sup\{\|w(t)\| : t \in \mathbb{Z}_+\}.$$

Recall that a matrix $A \in \mathbb{R}^{n \times n}$ is called Schur if every eigenvalue λ of A satisfies $|\lambda| < 1$. We will make use of the following classes of comparison functions:

$$\mathcal{K} := \{\varphi \in \mathcal{F}(\mathbb{R}_+, \mathbb{R}_+) : \varphi(0) = 0, \ \varphi \text{ is continuous and strictly increasing}\},$$
$$\mathcal{K}_\infty := \{\varphi \in \mathcal{K} : \varphi(s) \to \infty \ \text{ as } s \to \infty\}.$$

Finally, we denote by $\mathcal{KL}$ the set of all functions $\varphi : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ with the following properties: for each fixed $t \in \mathbb{Z}_+$, the function $\varphi(\cdot, t)$ is in $\mathcal{K}$, and, for each fixed $s \in \mathbb{R}_+$, the function $\varphi(s, \cdot)$ is non-increasing and $\varphi(s, t) \to 0$ as $t \to \infty$. The reader is referred to [39] for more details on comparison functions.

For the proof of the stability theorems in Section 5, we need an input-to-state stability result from control theory. To explain this result, consider the following system with additive forcing

$$x^\nabla = Ax + b[h(c^T x) + w], \quad x(0) = x^0 \in \mathbb{R}^n, \tag{2.1}$$

where $A \in \mathbb{R}^{n \times n}$, $b, c \in \mathbb{R}^n$, and $h : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity, such that $h(0) = 0$ and $w \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R})$ is a forcing function (or disturbance, input or control, as appropriate). Note that we are

not yet imposing any non-negativity assumptions on (2.1) — they are not required for general input-to-state stability theory, but will play a key role from Section 3 onwards.

Obviously, (2.1) can be thought of as a feedback system obtained by application of the feedback law $v = h(y) + w$ to the linear controlled and observed system

$$x^\nabla = Ax + bv, \quad y = c^T x. \tag{2.2}$$

Usually, v and y are called, respectively, the input (again, forcing or control) and output (also interpreted as a measurement or observation) of (2.2). Associated with (2.2) is the so-called transfer function

$$\mathbf{G}(z) := c^T(zI - A)^{-1}b, \tag{2.3}$$

a rational function in the complex variable z. Formally applying the Z-transform (denoted by $\mathcal{Z}$) to (2.2), we obtain

$$(\mathcal{Z}y)(z) = zc^T(zI - A)^{-1}x^0 + \mathbf{G}(z)(\mathcal{Z}v)(z).$$

The above identity shows that, in the frequency domain, the effect of the input on the linear dynamics is described by the product of the transfer function and the Z-transform of the input.

If A is Schur, then we set

$$\|\mathbf{G}\|_{H^\infty} := \sup_{|z|=1} |\mathbf{G}(z)| = \sup_{|z|\geq 1} |\mathbf{G}(z)|,$$

which is well-defined and finite. The second equality above follows from the maximum modulus principle applied to the function $\mathbf{H} : s \mapsto \mathbf{G}(1/s)$ on the closed unit disc, where $\mathbf{H}(0) := \lim_{|z|\to\infty} \mathbf{G}(z) = 0$.

Denoting the output of (2.2) corresponding to the initial condition $x(0) = 0$ by y_v, we have that

$$\sup\{\|y_v\|_{\ell^2(\mathbb{Z}_+)} : \|v\|_{\ell^2(\mathbb{Z}_+)} = 1\} = \|\mathbf{G}\|_{H^\infty}.$$

This identity is well-known in control theory and operator theory and it provides an appealing interpretation of $\|\mathbf{G}\|_{H^\infty}$ in time-domain terms.

The following result is a special case of [57, Theorem 13].

Theorem 2.1. *Consider the system (2.1) and denote its solution by $x(\cdot; x^0, w)$. Assume that A is Schur and at least one of the following condition holds: (i) the linear system (2.2) is controllable and observable, or, (ii) there exists $z \in \mathbb{C}$ such that $|z| = 1$ and $|\mathbf{G}(z)| < \|\mathbf{G}\|_\infty$. If there exists $\alpha \in \mathcal{K}_\infty$ such that*

$$\|\mathbf{G}\|_{H^\infty}|h(y)| \leq |y| - \alpha(|y|) \quad \forall\, y \in \mathbb{R},$$

then there exist $\varphi \in \mathcal{KL}$ and $\psi \in \mathcal{K}$, depending only on A, b, c and α, such that

$$\|x(t; x^0, w)\| \leq \varphi(\|x^0\|, t) + \psi(\|w\|_{\infty,t}) \,\forall\, t \in \mathbb{Z}_+, \forall\, x^0 \in \mathbb{R}^n, \forall\, w \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}). \tag{2.4}$$

We recall that the linear system (2.2) is said to be *controllable* if every initial state can be transferred to any other state by a suitable control function. More formally, denoting the solution of (2.2) by $x(\cdot; x^0, v)$, the system (2.2) is controllable if, for every initial state $x^0 \in \mathbb{R}^n$ and every "target" state $x^1 \in \mathbb{R}^n$, there exists $\tau \in \mathbb{Z}_+$ and an input function $v \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R})$ such that $x(\tau; x^0, v) = x^1$. Furthermore, (2.2) is *observable* if it has the property that an observation corresponding to zero control can be zero for all times only if the initial state is equal to zero, that is, the following implication holds:

$$\left(c^T x(t; x^0, 0) = 0 \quad \forall\, t \in \mathbb{Z}_+ \right) \Rightarrow x^0 = 0.$$

By a well-known result from control theory, we have that (2.2) is controllable if, and only if, the $n \times n$ matrix $(b, Ab, \ldots, A^{n-1}b)$ is invertible. Similarly, observability of (2.2) is equivalent to the invertibility of the $n \times n$ matrix $(c, A^T c, \ldots, (A^T)^{n-1}c)$. For more background on controllability and observability the reader is referred to, for example, [64, chapters 2 and 6].

If system (2.1) satisfies (2.4) (for some $\varphi \in \mathcal{KL}$ and $\psi \in \mathcal{K}$), then the zero equilibrium of the unforced ($w(t) \equiv 0$) system $x^\nabla = Ax + bh(c^T x)$ is said to be input-to-state stable (ISS). Frequently, it is also said that (2.1) is ISS. The ISS concept is a standard stability concept in nonlinear control theory. It was defined by Sontag in the 1989 paper [63] in the context of general forced (or controlled) nonlinear systems

(of which systems of the form (2.1) are a special case), and subsequently, a substantial Lyapunov theoretic ISS framework has been developed; see, for example, [19, 38, 66]. Note that (2.4) implies that the zero equilibrium of the unforced system $x^\nabla = Ax + bh(c^T x)$ is globally asymptotically stable (GAS). Simple counterexamples [57] show that there exist systems of the form (2.1) such that the zero equilibrium of the unforced system is GAS, but the system is not ISS and, moreover, that there exist initial conditions and bounded inputs which generate unbounded state trajectories.

3. Classes of forced positive discrete-time nonlinear models

We consider the following two forced difference equations:

$$x^\nabla = Ax + bg(uc^T x)c^T x, \quad x(0) = x^0 \in \mathbb{R}^n_+, \tag{3.1}$$

and

$$x^\nabla = Ax + bf(uc^T x), \quad x(0) = x^0 \in \mathbb{R}^n_+. \tag{3.2}$$

Here $g : (0, \infty) \to (0, \infty)$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nonlinearities and we shall always assume that the limit of $g(y)y$ as $y \downarrow 0$ exists and is finite. The function $u : \mathbb{Z}_+ \to \mathbb{R}_+$ is a forcing term. We will also consider the following variant of system (3.2):

$$x^\nabla = Ax + buf(c^T x), \quad x(0) = x^0 \in \mathbb{R}^n_+. \tag{3.3}$$

It will turn out that the analysis of boundedness, persistence and stability properties of (3.3) are very similar to that of (3.2). The above systems (3.1)–(3.3) may be thought of as feedback systems obtained by application of the feedback laws

$$v = g(uy)y, \quad v = f(uy), \quad \text{or} \quad v = uf(y),$$

respectively, to the linear controlled and observed system (2.2).

For a real matrix M, we write

$$M \geq 0 \text{ if every entry of } M \text{ is non-negative} \quad \text{and} \quad M \gg 0 \text{ if every entry of } M \text{ is positive.}$$

Furthermore, $M > 0$ means that $M \geq 0$ and $M \neq 0$. For real matrices M and N with the same format we write $M \geq N$, $M \gg N$ and $M > N$ if $M - N \geq 0$, $M - N \gg 0$ and $M - N > 0$, respectively. We recall that for $M \geq 0$, M is said to be primitive if there exists $k \in \mathbb{Z}_+$ such that $M^k \gg 0$. If M is a primitive matrix, then it is well-know from Perron-Frobenius theory (see [2, 50]), that the spectral radius r of M is a simple eigenvalue of M with an eigenvector $v \gg 0$. The normalized vector $(1/\|v\|)v$ is called the Perron vector of M. In particular, for any non-negative vector w such that $Mw = rw$, there exists a positive scalar λ such that $w = (\lambda/\|v\|)v$ and so $w \gg 0$. Obviously, if M is primitive, then M^T is also primitive and the Perron vector of M^T is called the left Perron eigenvector of M.

We impose the following assumptions on A, b and c.

(A1) $A \in \mathbb{R}^{n \times n}_+$ and A is Schur.

(A2) $b, c \in \mathbb{R}^n_+$, $b \neq 0$ and $c \neq 0$.

(A3) $A + bc^T$ is primitive, with τ the smallest number in $\mathbb{Z}_+$ such that $(A + bc^T)^\tau \gg 0$.

(A4) At least one of the following condition holds: (i) the linear system (2.2) is controllable and observable, or, (ii) there exists $z \in \mathbb{C}$ such that $|z| = 1$ and $|\mathbf{G}(z)| < \mathbf{G}(1)$, where $\mathbf{G}$ is given by (2.3).

We provide some commentary on these assumptions.

Remark 3.1. (i) In the scalar case $n = 1$ (that is, systems (3.1)–(3.3) evolve on a one-dimensional state space), assumptions **(A1)**–**(A4)** are satisfied provided that $A \in (0, 1)$ and $b, c > 0$.

(ii) In the higher dimensional case (that is, for general $n \in \mathbb{Z}_+$), if **(A1)** and **(A2)** hold, then there exists $\kappa^* > 0$ such that $A + \kappa^* bc^T$ is primitive if, and only if, $A + \kappa bc^T$ is primitive for all $\kappa > 0$. Moreover, given $\theta \in \mathbb{Z}_+$, there exists $\kappa^* > 0$ such that $(A + \kappa^* bc^T)^\theta \gg 0$ if, and only if, $(A + \kappa bc^T)^\theta \gg 0$ for all $\kappa > 0$. In particular, if **(A3)** is satisfied, then, for all $\kappa > 0$, the matrix $A + \kappa bc^T$ is primitive and $(A + \kappa bc^T)^\tau \gg 0$.

(iii) If **(A1)** and **(A2)** hold, then

$$\mathbf{G}(1) = c^T(I - A)^{-1}b = \sum_{j \in \mathbb{Z}_+} c^T A^j b \geq 0.$$

It can be shown that assumptions **(A1)**–**(A3)** together imply that $\mathbf{G}(1) > 0$. In this case, the number p defined by

$$p := \frac{1}{\mathbf{G}(1)} = \frac{1}{c^T(I - A)^{-1}b}, \tag{3.4}$$

is positive and finite. It is well known (and easy to show) that if **(A1)** and **(A2)** hold, then $\|\mathbf{G}\|_{H^\infty} = \mathbf{G}(1)$ and condition (ii) of assumption **(A4)** is equivalent to saying that $|\mathbf{G}(z)|$ is not constant on the unit circle. $\diamond$

Assumptions on the nonlinearities g and f will be specified in the boundedness, persistence and stability results below. It will be usually assumed that the forcing function u is in $\mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$, where $0 < u^- \leq u^+ < \infty$. From a robust control theoretic perspective, u^- and u^+ are often given more prominence than the resulting $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$ as they correspond to the "most extreme" permitted or expected disturbances. In other words, u^- and u^+ correspond to the worst case scenarios — although that depends on one's perspective and the interpretation of u.

Note that solutions of (3.1)–(3.3) are also solutions of a difference inclusion of the form

$$x^\nabla - Ax \in bF(c^T x), \tag{3.5}$$

where F is a suitably defined set-valued function. For example, if x is a solution of (3.2), then x also solves (3.5) with F given by

$$F(y) := \{f(vy) : v \in [u^-, u^+]\} \quad \forall\, y \geq 0.$$

Therefore, it is useful to consider stability properties of the difference inclusion (3.5).

Proposition 3.2. *Assume that* **(A1)**–**(A4)** *hold and F is a set-valued function defined on $\mathbb{R}_+$, the values of which are non-empty subsets of $\mathbb{R}_+$. If there exist $\theta \geq 0$ and $\alpha \in \mathcal{K}_\infty$ such that the set $\cup_{0 \leq y \leq \theta} F(y)$ is bounded and*

$$w \leq py - \alpha(y) \quad \forall\, y \geq \theta, \ \forall\, w \in F(y),$$

then there exist $\varphi \in \mathcal{KL}$ and $\psi \in \mathcal{K}$ such that, for every solution x of (3.5),

$$\|x(t)\| \leq \varphi(\|x(0)\|, t) + \psi(\beta_F) \quad \forall\, t \in \mathbb{Z}_+,$$

where

$$\beta_F := \sup_{\substack{y \in \mathbb{R}_+ \\ w \in F(y)}} \big[w - py + \alpha(y)\big] = \sup_{\substack{0 \leq y \leq \theta \\ w \in F(y)}} \big[w - py + \alpha(y)\big].$$

Proof. If the values of F are singletons (in which case F can be identified with a real-valued function), then the claim is an immediate consequence of [57, Corollary 17]. An inspection of the proof of [57, Corollary 17] shows that it extends in a straightforward way to set-valued nonlinearities and the proposition is a special case of such an extension. Note that the two suprema defining β_F are equal as the function $(w, y) \mapsto w - py + \alpha(y)$ is non-positive for all $w \in F(y)$ when $y \geq \theta$, but non-negative for all $w \in F(0)$. $\square$

4. Boundedness and persistence

In this section we prove boundedness and persistence results for the systems (3.1)–(3.3), which requires suitable assumptions on g and f, respectively. To this end, the nonlinearity g appearing in (3.1) is assumed to satisfy the following conditions.

(A5) $g : (0, \infty) \to (0, \infty)$ is continuous, the limit of $g(y)y$ as $y \downarrow 0$ exists and is finite,

$$\liminf_{y \downarrow 0} g(y) > p \quad \text{and} \quad \limsup_{y \to \infty} g(y) < p, \tag{4.1}$$

where p is given by (3.4).

In the following let u^- and u^+ be real numbers such that

$$0 < u^- \le u^+ < \infty, \tag{4.2}$$

which, throughout the paper, shall play the role of lower and upper bounds, respectively, for the forcing term u in (3.1)–(3.3).

The nonlinearity f appearing in (3.2) and (3.3) is required to satisfy the following assumption.

(A6) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $f(y) > 0$ for $y > 0$, and,

$$\liminf_{y \downarrow 0} \frac{f(y)}{y} > \frac{p}{u^-} \quad \text{and} \quad \frac{p}{u^+} y - f(y) \to \infty \text{ as } y \to \infty, \tag{4.3}$$

where p is given by (3.4).

We remark that for the stability results in Section 5 the above assumptions **(A5)** and **(A6)** will have to be somewhat strengthened. We note an asymmetry in that u^- and u^+ in (4.2) do not appear in **(A5)**, but do in **(A6)**. Despite this difference, u^- and u^+ do play a crucial role in determining boundedness, persistence and stability properties for each of the Lur'e systems (3.1)–(3.3).

The results in this section show that **(A5)** and **(A6)**, in combination with **(A1)**–**(A4)**, are sufficient to establish semi-global boundedness and persistence properties of (3.1)–(3.3). Beforehand, we provide some typical examples of functions which satisfy **(A5)** and **(A6)**.

Example 4.1. For the forced difference equations (3.2) and (3.3), consider the three nonlinear functions $f_i : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$f_1(y) = \frac{a_1 y}{a_2 + y}, \quad f_2(y) = ye^{-ay}, \quad f_3(y) = y^b, \tag{4.4}$$

where $a_1, a_2, a > 0$ and $b \in (0, 1)$. In population dynamics, the nonlinearity f_1 is often called of Beverton-Holt type [4] or of Holling type II [35]. The function f_2 is referred to as a Ricker nonlinearity [53] and f_3 is a power law. The functions $g_i : (0, \infty) \to \mathbb{R}_+$ in (3.1) corresponding to the above f_i are given by

$$g_1(y) = \frac{a_1}{a_2 + y}, \quad g_2(y) = e^{-ay}, \quad g_3(y) = y^{b-1}. \tag{4.5}$$

Table 4.1 contains (necessary and sufficient) conditions on the parameters appearing in g_i and f_i which ensure that **(A5)** or **(A6)** hold for given $p > 0$ and u^-, u^+ satisfying (4.2). Note that the limit of $f_i(y) = g_i(y)y$ exists (and is equal to zero) as $y \downarrow 0$ for each $i \in \{1, 2, 3\}$. The results displayed in the table are readily established by elementary analysis and so the details are omitted. We comment that the conditions at infinity are satisfied for all $u^+, p > 0$ as each function f_i is either bounded, or grows sublinearly (for sufficiently large arguments). Consequently, the functions g_i satisfy $g_i(y) \to 0$ as $y \to \infty$.

	Condition	g_1 or f_1	g_2 or f_2	g_3 or f_3
(A5)	liminf	$p < a_1/a_2$	$p < 1$	✓
	limsup	✓	✓	✓
(A6)	liminf	$p < u^- a_1/a_2$	$p < u^-$	✓
	divergence	✓	✓	✓

Table 4.1: Parameter values under which properties **(A5)** and **(A6)** hold for functions given by (4.4) and (4.5), for given $p > 0$ and u^-, u^+ satisfying (4.2). A tick indicates that the property holds for all possible parameter values.

We remark that in one-dimensional ($n = 1$) population models the Beverton-Holt nonlinearity is often expressed in the form

$$f(y) = \frac{\rho y}{1 + \frac{(\rho - 1)}{K} y}, \quad y \ge 0,$$

where $\rho > 1$ and $K > 0$ denote the inherent growth rate and carrying capacity, respectively (see, for instance, [17]). Obviously, $f = f_1$ from (4.4) with $a_1 := K\rho/(\rho - 1)$ and $a_2 := K/(\rho - 1)$.

Theorem 4.2. *Consider the initial-value problem* (3.1) *and let* $x(\cdot; x^0, u)$ *denote the solution. Assume that* **(A1)**–**(A5)** *hold. Furthermore, let* $\Gamma \subseteq \mathbb{R}_+^n$ *be compact and such that* $0 \notin \Gamma$, *and let* u^- *and* u^+ *be real numbers satisfying* (4.2).

(1) *There exists* $\gamma > 0$ *such that*

$$\|x(t; x^0, u)\| \leq \gamma \quad \forall\, t \in \mathbb{Z}_+, \ \forall\, x^0 \in \Gamma, \ \forall\, u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+]). \tag{4.6}$$

(2) *There exists* $\delta > 0$ *such that*

$$\|x(t; x^0, u)\| \geq \delta \quad \forall\, t \in \mathbb{Z}_+, \ \forall\, x^0 \in \Gamma, \ \forall\, u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+]). \tag{4.7}$$

(3) *There exists* $\eta > 0$ *such that*

$$c^T x(t + \tau; x^0, u) \geq \eta \quad \forall\, t \in \mathbb{Z}_+, \ \forall\, x^0 \in \Gamma, \ \forall\, u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+]), \tag{4.8}$$

where $\tau \in \mathbb{Z}_+$ *is as in* **(A3)**.

The constants γ, δ and η which appear in (4.6)–(4.8) depend on Γ and $u^\pm$. Statements (2) and (3) of Theorem 4.2 can be viewed as semi-global persistence results: $\|x(\cdot; x^0, u)\|$ and $c^T x(\cdot; x^0, u)$ "persist" in the sense that, given a compact set $\Gamma \subseteq \mathbb{R}_+^n \setminus \{0\}$ and u^- and u^+ satisfying $0 < u^- \leq u^+ < \infty$, $\|x(t; x^0, u)\|$ and $c^T x(t + \tau; x^0, u)$ are bounded away from 0, uniformly in time and uniformly for all $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$ and all $x^0 \in \Gamma$. The relevance of persistence in the context of population dynamics is obvious: its absence is equivalent to extinction. The persistence properties guaranteed by Theorem 4.2 overlap with persistency concepts (for systems without forcing) introduced in [26, 61, 62, 71] (see also the stochastic persistence notion discussed in [59]). In particular, it follows immediately from statements (2) and (3) of Theorem 4.2 that system (3.1) is strongly $\|\cdot\|$-persistent and strongly c^T-persistent, respectively, in the sense of [61, Definition 3.1]. We note that Theorem 4.2 guarantees that

$$\liminf_{t \to \infty} c^T x(t; x^0, u) > 0$$

holds for every non-zero $x^0 \in \mathbb{R}_+^n$ (and not just for $x^0 \in \mathbb{R}_+^n$ such that $c^T x^0 > 0$, cf. [61, Definition 3.1]) and all $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$. Furthermore, we point out that statement (2) of Theorem 4.2 implies that system (3.1) is persistent with respect to $\{0\}$ in the sense of [26].

We proceed to comment on the concept of uniform strong persistency as given in [61, Definition 3.1]. For example, uniform strong c^T-persistency would require that, for the "uncontrolled" system 4.2 (that is, $u^- = u^+ = 1$), there exists $\eta > 0$ such that, for every $x^0 \in \mathbb{R}_+^n$ with $c^T x^0 > 0$,

$$\liminf_{t \to \infty} c^T x(t; x^0, 1) \geq \eta. \tag{4.9}$$

Theorem 4.2 establishes that, for every compact $\Gamma \subset \mathbb{R}_+^n$ not containing 0, there exists a suitable positive constant η such that (4.9) holds for all $x^0 \in \Gamma$. For all practical purposes this semi-global result is sufficient: for any given application context, there exists a compact set $\Gamma \subset \mathbb{R}_+^n$ such that every practically relevant initial condition will belong to Γ. Moreover, we remark that Theorem 4.2 provides uniformity with respect to time, an aspect which is not included in the persistency concepts in [61, 62, 71], but which is nevertheless relevant as it relates to properties of the transient dynamics, an important issue in ecology, see [30]. In the current paper, uniformity with respect to time is crucial in the proof of the stability results in Section 5. To conclude the discussion on persistency, we point out that statements (2) and (3) of Theorem 4.2 do not guarantee that all the components of $x(t; x^0, u)$ are (eventually) bounded away from 0 and therefore does not guarantee *stage persistence* (cf.[61, Section 7.4]). We remark however that the stability results in Section 5, under suitable assumptions on the forcing function u, do imply stage persistence (see the commentary after Theorem 5.5).

Proof of Theorem 4.2. (1) Define a set-valued nonlinearity F by

$$F(0) := \{g_0/v : v \in [u^-, u^+]\} \text{ and } F(y) := \{g(vy)y : v \in [u^-, u^+]\} \ \forall\, y > 0,$$

where $g_0 := \lim_{y\downarrow 0}(g(y)y)$. By **(A5)**, $g(y) > 0$ for all $y > 0$ and

$$\limsup_{y\to\infty} g(y) < p.$$

Hence there exists $\theta > 0$ and $q \in (0, p)$ such that

$$0 < w \le qy \quad \forall\, y \ge \theta,\ \forall\, w \in F(y).$$

It is clear that, for every $x^0 \in \Gamma$ and every $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$, the solution $x(\cdot\,; x^0, u)$ is also a solution of the difference inclusion

$$x^\nabla - Ax \in bF(c^T x),$$

and the claim follows from Proposition 3.2 with $\alpha \in \mathcal{K}_\infty$ given by $\alpha(s) = (p-q)s$.

(2) & (3) By the first condition in (4.1), there exists $y^\sharp > 0$ such that

$$g(y) \ge p \quad \forall\, y \in [0, u^+ y^\sharp]. \tag{4.10}$$

Let $\gamma > 0$ be as in statement (1), and set $y^\dagger := \max\{y^\sharp, \|c\|\gamma\}$ and

$$\lambda := \inf\{g(y) : u^- y^\sharp \le y \le u^+ y^\dagger\} > 0. \tag{4.11}$$

Let $x^0 \in \Gamma$, $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$ and write

$$x(t) := x(t; x^0, u) \quad \text{and} \quad y(t) := c^T x(t; x^0, u) \qquad \forall\, t \in \mathbb{Z}_+. \tag{4.12}$$

For given $t \in \mathbb{Z}_+$, we have either $y(t) \in [0, y^\sharp]$ or $y(t) > y^\sharp$.

CASE 1: $y(t) \in [0, y^\sharp]$. By (4.10), $g(u(t)y(t)) \ge p$, and so

$$x(t+1) \ge (A + pbc^T)x(t). \tag{4.13}$$

It follows from stability radius theory that the spectral radius of $A + pbc^T$ is equal to 1 (see, for example, [32, Theorem 3.4] or [69, Lemma 3.2]) and, since $A + pbc^T$ is primitive, Perron-Frobenius theory guarantees that there exists $\zeta = (\zeta_1, \ldots, \zeta_n)^T \gg 0$ such that $\zeta^T(A + pbc^T) = \zeta^T$, see, for example, [2, 50]. Consequently, invoking (4.13),

$$\zeta^T x(t+1) \ge \zeta^T x(t). \tag{4.14}$$

CASE 2: $y(t) > y^\sharp$. By statement (1) and definition of $y^\dagger$, $y(t) \le y^\dagger$ and so, by (4.11),

$$g(u(t)y(t))y(t) \ge \lambda y(t) \ge \lambda y^\sharp.$$

As a consequence, $x(t+1) \ge Ax(t) + \lambda y^\sharp b$, and so,

$$\zeta^T x(t+1) \ge \lambda y^\sharp \zeta^T b > 0. \tag{4.15}$$

Combining (4.14) and (4.15) (the outcomes of the considerations in Cases 1 and 2), we conclude that

$$\zeta^T x(t+1) \ge \min\{\zeta^T x(t), \lambda y^\sharp \zeta^T b\} \quad \forall\, t \in \mathbb{Z}_+,$$

whence

$$\zeta^T x(t) \ge \min\{\zeta^T x^0, \lambda y^\sharp \zeta^T b\} \ge \varepsilon \quad \forall\, t \in \mathbb{Z}_+, \tag{4.16}$$

where $\varepsilon > 0$ is the minimum of $\min_{\xi \in \Gamma}(\zeta^T \xi)$ and $\lambda y^\sharp \zeta^T b$. It follows that there exists $\delta > 0$ such that (4.7) holds.

Furthermore, setting $\mu := \min\{\lambda, p\} > 0$ and appealing to (4.10) and (4.11), we obtain

$$x(t+1) \ge (A + \mu bc^T)x(t) \quad \forall\, t \in \mathbb{Z}_+,$$

and hence, with τ from **(A4)**,

$$c^T x(t+\tau) \ge c^T(A + \mu bc^T)^\tau x(t) \quad \forall\, t \in \mathbb{Z}_+.$$

By **(A4)**, every component of the row $c^T(A + \mu b c^T)^\tau$ is positive and thus, letting ω denote the minimum of these n components, we have that $\omega > 0$, and

$$c^T x(t + \tau) \geq \omega \sum_{i=1}^{n} x_i(t) \quad \forall\, t \in \mathbb{Z}_+.$$

Finally, invoking (4.16),

$$\sum_{i=1}^{n} x_i(t) \geq \frac{\varepsilon}{\nu} \quad \forall\, t \in \mathbb{Z}_+,$$

where $\nu := \max\{\zeta_1, \ldots, \zeta_n\}$, we arrive at

$$c^T x(t + \tau) \geq \frac{\varepsilon \omega}{\nu} \quad \forall\, t \in \mathbb{Z}_+,$$

completing the proof. $\qquad\square$

We now turn our attention to systems (3.2) and (3.3).

Theorem 4.3. *Let u^- and u^+ be real numbers satisfying (4.2), and assume that* **(A1)**–**(A4)** *and* **(A6)** *hold. Furthermore, let $\Gamma \subseteq \mathbb{R}_+^n$ be compact and such that $0 \notin \Gamma$.*

(1) *Consider the initial-value problem (3.2) and let $x(\cdot\,; x^0, u)$ denote the solution. Then there exist $\gamma > 0$, $\delta > 0$ and $\eta > 0$ such that (4.6)–(4.8) hold.*

(2) *Statement (1) remains valid in the context of system (3.3).*

Proof. (1) Define a set-valued nonlinearity F by

$$F(y) := \{f(vy) : v \in [u^-, u^+]\} \quad \forall\, y \geq 0.$$

By **(A6)**, $f(y) \geq 0$ for all $y \geq 0$ and

$$py - f(vy) \geq \frac{p}{u^+} vy - f(vy) \to \infty \quad \text{uniformly in } v \in [u^-, u^+] \text{ as } y \to \infty.$$

Consequently,

$$\Big[py - \max_{u^- \leq v \leq u^+} f(vy) \Big] \to \infty \quad \text{as } y \to \infty,$$

which in turn implies the existence of a number $\theta > 0$ and $\alpha \in \mathcal{K}_\infty$ such that

$$0 \leq w \leq py - \alpha(y) \quad \forall\, y \geq \theta,\ \forall\, w \in F(y). \tag{4.17}$$

It is clear that, for every $x^0 \in \Gamma$ and every $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$, the solution $x(\cdot\,; x^0, u)$ is also a solution of the difference inclusion

$$x^\nabla - Ax \in bF(c^T x),$$

and it follows from Proposition 3.2 that there exists $\gamma > 0$ such that (4.6) is satisfied.

We proceed to show that (4.7) and (4.8) hold for positive δ and η. To this end, we note that, by the first condition in (4.3), there exists $y^\sharp > 0$ such that

$$\frac{f(y)}{y} \geq \frac{p}{u^-} \quad \forall\, y \in (0, u^+ y^\sharp],$$

and hence,

$$\frac{f(vy)}{y} \geq v \frac{p}{u^-} \geq p \quad \forall\, y \in (0, y^\sharp],\ \forall\, v \in [u^-, u^+]. \tag{4.18}$$

Setting

$$y^\dagger := \max\{y^\sharp, \|c\| \gamma\}, \tag{4.19}$$

it follows from **(A6)** that

$$\lambda := \inf\{f(vy)/y : y^\sharp \leq y \leq y^\dagger,\ u^- \leq v \leq u^+\} > 0. \tag{4.20}$$

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Let $x^0 \in \Gamma$, $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$ and define $x(t)$ and $y(t)$ as in (4.12). For given $t \in \mathbb{Z}_+$, we have either $y(t) \in [0, y^\sharp]$ or $y(t) > y^\sharp$.

CASE 1: $y(t) \in [0, y^\sharp]$. By (4.18), $f(u(t)y(t)) \geq py(t)$, and so

$$x(t+1) \geq (A + pbc^T)x(t).$$

We can now argue as in the proof of Theorem 4.2 to obtain

$$\zeta^T x(t+1) \geq \zeta^T x(t), \tag{4.21}$$

where $\zeta^T \gg 0$ is the left Perron eigenvector of $A + pbc^T$ corresponding to the spectral radius of $A + pbc^T$, which is an eigenvalue and is equal to 1.

CASE 2: $y(t) > y^\sharp$. By statement (1) and definition of $y^\dagger$, $y(t) \leq y^\dagger$ and so, by (4.20),

$$f(u(t)y(t)) \geq \lambda y(t) \geq \lambda y^\sharp.$$

As a consequence,

$$\zeta^T x(t+1) \geq \lambda y^\sharp \zeta^T b > 0. \tag{4.22}$$

Combining (4.21) and (4.22) (the outcomes of the considerations in Cases 1 and 2), we conclude that

$$\zeta^T x(t+1) \geq \min\{\zeta^T x(t), \lambda y^\sharp \zeta^T b\} \quad \forall t \in \mathbb{Z}_+,$$

and the proof can now be completed by arguments identical to those used in the proof of Theorem 4.2.

(2) Defining a set-valued map F by

$$F(y) := \{vf(y) : v \in [u^-, u^+]\} \quad \forall y \geq 0,$$

it is straightforward to show that there exist $\theta > 0$ and $\alpha \in \mathcal{K}$ such that (4.17) is satisfied. The existence of a number $\gamma > 0$ such that (4.6) holds now follows in the same way as in the proof of statement (1).

To show that (4.7) and (4.8) are valid for suitable $\eta > 0$, we note that, by the first condition in (4.3), there exists $y^\sharp > 0$ such that

$$\frac{f(y)}{y} \geq \frac{p}{u^-} \quad \forall y \in (0, y^\sharp],$$

and thus

$$\frac{vf(y)}{y} \geq v\frac{p}{u^-} \geq p \quad \forall y \in (0, y^\sharp], \quad \forall v \in [u^-, u^+].$$

Furthermore, by (**A6**),

$$\mu := \inf\{f(y)/y : y^\sharp \leq y \leq y^\dagger\} > 0,$$

where $y^\dagger$ is given by (4.19). We see that $u(t)f(y(t)) \geq py(t)$ if $y(t) \in [0, y^\sharp]$ and $u(t)f(y(t)) \geq u^-\mu y^\sharp$ if $y(t) > y^\sharp$. We can now use arguments very similar to those used in the proof of statement (1) to establish the claim. $\qquad\square$

5. Stability

Having established boundedness and persistence results in Section 4, we turn attention to conditions which guarantee the existence of a "stable" non-zero equilibrium: the stability notion used here takes into account the forcing term u and is reminiscent of the input-to-state stability concept from nonlinear control theory, discussed in Section 2. As has already been mentioned, to derive useful stability results, we need to somewhat strengthen the assumptions (**A5**) and (**A6**) on g and f, respectively.

In the following, let p denote the constant given by (3.4) and let $u^* \in [u^-, u^+]$, where u^-, u^+ satisfy (4.2). The term u^* will play the role of a target or nominal control value, and recall u^- and u^+ denote the lower and upper limits of the forcing, respectively. If $1 \in [u^-, u^+]$, then the case wherein $u^* = 1$ corresponds to the unforced system.

We formulate the following condition for the nonlinearity g which appears in (3.1).

(**A5′**) Condition (**A5**) holds, and

$$|g(u^*y)y - g(u^*y^*)y^*| = |g(u^*y)y - py^*| < p|y - y^*| \quad \forall y > 0, \ y \neq y^*, \tag{5.1}$$

where y^* is the unique positive number such that $g(u^*y^*) = p$.

The existence of $y^* > 0$ such that $g(u^*y^*) = p$ follows from the continuity of g and (4.1), whilst uniqueness of y^* is a consequence of (5.1).

The inequality (5.1) is often called a sector condition, in this particular case for the function $y \mapsto g(u^*y)y$, an illustration of which is given in Figure 5.1.

Figure 5.1: Illustration of the sector condition (5.1). The graph of $y \mapsto g(u^*y)y$ must lie within the shaded region, that is, the graph is "sandwiched" between the lines specified by py and $-py + 2py^*$. The sector is "strict" in that the function $y \mapsto g(u^*y)y$ cannot intersect either of these lines apart from at $y = 0$ or $y = y^*$.
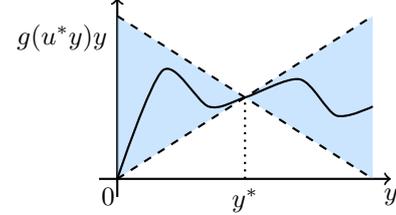
The following lemma shows that the existence of a unique non-zero equilibrium of the Lur'e system $x^\nabla = Ax + bg(u^*c^Tx)c^Tx$ is a consequence of (**A1**)–(**A4**) and (**A5′**).

Lemma 5.1. *Assume that* (**A1**)–(**A3**) *and* (**A5′**) *hold and set*

$$x^* := (I - A)^{-1}bpy^* \in \mathbb{R}_+^n. \tag{5.2}$$

Then $c^Tx^* = y^*$, $x^* \gg 0$ *and* x^* *is the unique non-zero equilibrium of* (3.1) *with* $u(t) \equiv u^*$, *where* $y^* > 0$ *is as in* (**A5′**).

Proof. It follows immediately from the definitions of x^* and p that $c^Tx^* = y^*$. Consequently, since $y^* > 0$, we conclude that $x^* \neq 0$, and hence, $x^* > 0$. Furthermore,

$$(A + pbc^T)x^* = x^* + (A - I + pbc^T)x^* = x^* - bpy^* + pb\mathbf{G}(1)py^* = x^*,$$

showing that x^* is an eigenvector of $A + pbc^T$ associated with the eigenvalue 1. As was pointed out in the proof of Theorem 4.2, the spectral radius of $A + pbc^T$ is equal to 1 and thus, invoking the primitivity of $A + pbc^T$, we conclude that x^* is a positive scalar multiple of the Perron vector of $A + pbc^T$. In particular, $x^* \gg 0$.

To show that x^* is an equilibrium, note that

$$x^* = (I - A)^{-1}bpy^* = (I - A)^{-1}bg(u^*y^*)y^* = (I - A)^{-1}bg(u^*c^Tx^*)c^Tx^*,$$

and so, $x^* = Ax^* + bg(u^*c^Tx^*)c^Tx^*$. As for uniqueness, let $x^\dagger$ be another non-zero vector in $\mathbb{R}_+^n$ satisfying

$$x^\dagger = Ax^\dagger + bg(u^*c^Tx^\dagger)c^Tx^\dagger. \tag{5.3}$$

Then $c^Tx^\dagger = \mathbf{G}(1)g(u^*c^Tx^\dagger)c^Tx^\dagger$, and thus, $g(u^*c^Tx^\dagger)c^Tx^\dagger = pc^Tx^\dagger$. Now $c^Tx^\dagger \neq 0$ (otherwise, in light of (5.3), 1 would be an eigenvalue of A which is not possible by (**A1**)), and so $g(u^*c^Tx^\dagger) = p$. Invoking uniqueness of y^*, we see that $c^Tx^\dagger = y^* = c^Tx^*$, whence

$$x^\dagger = (I - A)^{-1}bg(u^*c^Tx^\dagger)c^Tx^\dagger = (I - A)^{-1}bpy^* = x^*,$$

completing the proof. □

The above proof shows that the equilibrium x^* is a positive scalar multiple of the Perron vector of $A + pbc^T$ which is completely determined by the linear part of the Lur'e system. The effect of the nonlinearity on x^* is described by the scalar factor y^* in (5.2).

As before, let u^- and u^+ be real numbers such that (4.2) is satisfied, where the interval $[u^-, u^+]$ contains the ranges of the forcing functions under consideration and let $u^* \in [u^-, u^+]$. We impose the following condition on the nonlinearity f appearing in (3.2).

(A6′) Condition **(A6)** holds and

$$|f(u^*y) - f(u^*y^*)| = |f(u^*y) - py^*| < p|y - y^*| \quad \forall\, y > 0,\ y \neq y^*, \tag{5.4}$$

where y^* is the unique positive number such that $f(u^*y^*) = py^*$.

It follows from the continuity of f and (4.3) that there exists $y^* > 0$ such that $f(u^*y^*) = py^*$, and the uniqueness of y^* is a consequence (5.4). Inequality (5.4) is another sector condition, now for $y \mapsto f(u^*y)$, see Figure 5.1 and its caption. As a consequence of **(A6′)**, there exists a unique non-zero equilibrium of the Lur'e system $x^\nabla = Ax + bf(u^*c^Tx)$, which we record in the next lemma, the proof of which mirrors that of Lemma 5.1, and is thus omitted.

Lemma 5.2. *Assume that* **(A1)**–**(A3)** *and* **(A6′)** *hold and define x^* by (5.2). Then $c^Tx^* = y^*$, $x^* \gg 0$ and x^* is the unique non-zero equilibrium of (3.2) with $u(t) \equiv u^*$, where y^* is as in* **(A6′)**.

Similarly, we record the following conditions on the nonlinearity f appearing in (3.3).

(A6″) Condition **(A6)** holds and

$$|u^*f(y) - u^*f(y^*)| = |u^*f(y) - py^*| < p|y - y^*| \quad \forall\, y > 0,\ y \neq y^*, \tag{5.5}$$

where y^* is the unique positive number such that $u^*f(y^*) = py^*$.

It follows from the continuity of f and (4.3) that there exists $y^* > 0$ such that $u^*f(y^*) = py^*$, and the uniqueness of y^* is a consequence (5.5).

Paralleling Lemmas 5.1 and 5.2, we obtain the following lemma, again with proof omitted as it mirrors that of Lemma 5.1.

Lemma 5.3. *Assume that* **(A1)**–**(A3)** *and* **(A6″)** *hold and define x^* by (5.2). Then $c^Tx^* = y^*$, $x^* \gg 0$ and x^* is the unique non-zero equilibrium of (3.3) with $u(t) \equiv u^*$, where y^* is as in* **(A6″)**.

We comment that there is a certain symmetry between **(A6′)** and **(A6″)**, but the parameters involved change. To see this, fix $p > 0$, $0 < u^- \leq u^+ < \infty$, $u^* \in [u^-, u^+]$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$. Then f satisfies **(A6′)** for some $y^* > 0$ if, and only if, f satisfies **(A6″)** with y^* replaced by u^*y^*.

The next lemma provides sufficient conditions for **(A5)**, **(A6)**, **(A5′)**, **(A6′)** or **(A6″)** to hold. Given differentiable $f : \mathbb{R}_+ \to \mathbb{R}_+$, the derivative of f is denoted by f' and $f'(0)$ is interpreted as the right derivative of f at 0.

Lemma 5.4. *Let $g : (0, \infty) \to (0, \infty)$, $f : \mathbb{R}_+ \to \mathbb{R}_+$, and $p > 0$ be given. Let $u^-, u^+ \in \mathbb{R}$ satisfy (4.2) and let $u^* \in [u^-, u^+]$.*

(1) *Assume that g is continuously differentiable and $\lim_{y\downarrow 0} g(y)y$ exists. If $g'(y) \leq 0$ for all $y \in (0, \infty)$, $\lim_{y\downarrow 0} g(y) > p$ and $\lim_{y\to\infty} g(y) < p$, then* **(A5)** *holds.*

(2) *If, in addition to the assumptions in statement (1), $g'(y) < 0$ and $(g(y)y)' \geq 0$ for all $y \in (0, \infty)$, then* **(A5′)** *holds.*

(3) *Assume that f is continuously differentiable and $\lim_{y\to\infty} f'(y)$ exists. If $f(0) = 0$, $f'(0) > p/u^-$ and $\lim_{y\to\infty} f'(y) < p/u^+$, then* **(A6)** *holds.*

(4) *Assume that f is twice continuously differentiable. If $f(0) = 0$, $f'(0) > p/u^-$, $f'(y) \geq 0$, $f''(y) \leq 0$ for all $y \in \mathbb{R}_+$ and $\lim_{y\to\infty} f'(y) < p/u^+$, then* **(A6′)** *and* **(A6″)** *hold.*

The proof of Lemma 5.4 is relegated to the Appendix. Note that, in statement (1), the assumption that g is continuously differentiable with $g' \leq 0$ implies the existence of both the limits, where the limit as $y \downarrow 0$ may be infinite. Existence of the limit of $f'(y)$ as $y \to \infty$ in statement (4) is ensured as it is assumed that $f' \geq 0$ and $f'' \leq 0$.

Table 5.1 provides conditions under which **(A5′)**, **(A6′)** and **(A6″)** hold for the three nonlinear functions considered in Example 4.1. The results displayed in the table can be established by a combination of elementary analysis and applications of Lemma 5.4. The details are omitted.

We are now in the position to state and prove stability theorems relating to the non-zero equilibria of systems (3.1)–(3.3).

	g_1	g_2	g_3
(A5$'$)	$p < a_1/a_2$	$\mathrm{e}^{-2} \leq p < 1$	✓

	f_1	f_2	f_3
(A6$'$) or **(A6$''$)**	$p < a_1 u^-/a_2$	$u^+\mathrm{e}^{-2} \leq p < u^-$	✓

Table 5.1: Parameter values under which properties **(A5$'$)**, **(A6$'$)** and **(A6$''$)** hold for the functions given by (4.4) and (4.5), for given $p > 0$, u^-, u^+ satisfying (4.2) and $u^* \in [u^-, u^+]$. A tick indicates that the property holds for all possible parameter values.

Theorem 5.5. *Consider the initial-value problem* (3.1), *let* $x(\cdot\,; x^0, u)$ *denote the solution, let* u^- *and* u^+ *be real numbers satisfying* (4.2), *and let* $u^* \in [u^-, u^+]$. *Assume that* **(A1)**–**(A4)** *and* **(A5$'$)** *hold and let* $x^* \gg 0$ *be given by* (5.2).

(1) *Let* $\Gamma \subseteq \mathbb{R}^n_+$ *be compact and such that* $0 \notin \Gamma$. *There exist* $\varphi \in \mathcal{KL}$ *and* $\psi \in \mathcal{K}$ *such that, for all* $x^0 \in \Gamma$ *and all* $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$,

$$\|x(t; x^0, u) - x^*\| \leq \varphi(\|x^0 - x^*\|, t) + \psi(\beta(u)) \quad \forall t \in \mathbb{Z}_+, \tag{5.6}$$

with $\beta(u) := \sup\{y|g(u^*y) - g(u(t)y)| : t \in \mathbb{Z}_+,\ 0 \leq y \leq \gamma\|c\|\}$, *where* γ *is the constant from statement* (1) *of Theorem 4.2.*

(2) *Let* $u^\infty > 0$. *For every* $x^0 \in \mathbb{R}^n_+$, $x^0 \neq 0$, *and every* $u \in \mathcal{F}(\mathbb{Z}_+, (0, \infty))$ *such that* $u(t) \to u^\infty$ *as* $t \to \infty$, *we have that* $x(t; x^0, u) \to (u^*/u^\infty)x^*$ *as* $t \to \infty$.

Since $x^* \gg 0$, it is clear that statement (2) implies stage persistence (cf. [61, Section 7.4]), that is, for sufficiently large $t \in \mathbb{Z}_+$ there is a positive lower bound for each of the n components of $x(t)$.

Note that if $(u_k)_{k \in \mathbb{N}}$ is a sequence in $\mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$ such that u_k converges to (the constant function) u^* in the sup-norm as $k \to \infty$, then $\lim_{k \to \infty} \beta(u_k) = 0$. Consequently, $\psi(\beta(u_k)) \to 0$ as $k \to \infty$, and thus, for large t and k, the state $x(t; x^0, u_k)$ is close to x^*. Finally, it is clear that $(u^*/u^\infty)x^*$ is an equilibrium of the system

$$x(t+1) = Ax(t) + bg(u^\infty c^T x(t))c^T x(t), \tag{5.7}$$

and an inspection of the proof below shows that this equilibrium is "globally asymptotically stable" in the sense that it is stable and attracts every solution of (5.7) with initial condition in $\mathbb{R}^n_+ \setminus \{0\}$.

Proof of Theorem 5.5. (1) Let $x^0 \in \Gamma$ and $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$, set $h(y) := g(u^*y)y$ and write $x(t) := x(t; x^0, u)$. Then

$$x^\nabla = Ax + b(h(c^T x) + v), \quad \text{where } v := (g(uc^T x) - g(u^* c^T x))c^T x.$$

Since $c^T x^* = y^*$ and $x^* = Ax^* + bh(y^*)$, it follows that the function $\tilde{x}(t) := x(t) - x^*$ satisfies

$$\tilde{x}(t+1) = A\tilde{x}(t) + b[h(c^T \tilde{x}(t) + y^*) - h(y^*)] + bv(t), \quad \forall t \in \mathbb{Z}_+. \tag{5.8}$$

By Theorem 4.2, there exists $\eta \in (0, y^*)$ such that

$$c^T \tilde{x}(t + \tau) \geq -y^* + \eta \quad \forall t \in \mathbb{Z}_+, \tag{5.9}$$

where τ is as in **(A3)**. Defining

$$\tilde{h} : \mathbb{R} \to \mathbb{R}, \ y \mapsto \begin{cases} h(y + y^*) - h(y^*), & \text{for } y \geq -y^* + \eta \\ h(\eta) - h(y^*), & \text{for } y < -y^* + \eta, \end{cases}$$

it follows from (5.8) and (5.9) that

$$\tilde{x}(t+1) = A\tilde{x}(t) + b[\tilde{h}(c^T \tilde{x}(t)) + v(t)], \quad \forall t \in \tau + \mathbb{Z}_+. \tag{5.10}$$

By assumption **(A5$'$)**,

$$|\tilde{h}(y)| < p|y| \ \forall y \in \mathbb{R}, \ y \neq 0, \quad \text{and} \quad p|y| - \tilde{h}(y) \to \infty \quad \text{as } |y| \to \infty,$$

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and so, there exists $\alpha \in \mathcal{K}_\infty$ such that

$$|\tilde{h}(y)| \leq p|y| - \alpha(|y|) \quad \forall\, y \in \mathbb{R}. \tag{5.11}$$

Invoking **(A1)** and **(A2)**, it is easy to show that $\mathbf{G}(1) = \|\mathbf{G}\|_{H^\infty}$, implying that $p = 1/\|\mathbf{G}\|_{H^\infty}$. Combining this with (5.11) and appealing to Theorem 2.1, we conclude that system (5.10) is input-to-state stable, that is, there exist $\varphi_0 \in \mathcal{KL}$ and $\psi_0 \in \mathcal{K}$ (depending only on A, b, c, g, Γ, u^- and u^+) such that

$$\|\tilde{x}(t+\tau)\| \leq \varphi_0(\|\tilde{x}(\tau)\|, t) + \psi_0(\|v\|_\infty) \quad \forall\, t \in \mathbb{Z}_+. \tag{5.12}$$

Furthermore, since the function $y \mapsto h(y + y^*) - h(y^*)$ is linearly bounded on $[-y^*, \infty)$, we obtain from (5.8) that

$$\|\tilde{x}(t)\| \leq \kappa_1 \|\tilde{x}(0)\| + \kappa_2 \|v\|_\infty \quad t = 1, 2, \ldots, \tau, \tag{5.13}$$

where the positive constants κ_1 and κ_2 depend only on A, b, c and g. Noting that, for all $s_1, s_2 \in \mathbb{R}_+$ and for all $t \in \mathbb{Z}_+$,

$$\varphi_0(s_1 + s_2, t) \leq \varphi_0(2s_1, t) + \varphi_0(2s_2, t) \leq \varphi_0(2s_1, t) + \varphi_0(2s_2, 0),$$

it follows from (5.12) and (5.13) that

$$\|\tilde{x}(t+\tau)\| \leq \varphi_0(2\kappa_1 \|\tilde{x}(0)\|, t) + \varphi_0(2\kappa_2 \|v\|_\infty, 0) + \psi_0(\|v\|_\infty) \quad \forall\, t \in \mathbb{Z}_+. \tag{5.14}$$

Define $\varphi : \mathbb{R}_+ \times \mathbb{Z}_+ \to \mathbb{R}_+$ by

$$\varphi(s,t) := \begin{cases} \kappa_1 s + \varphi_0(2\kappa_1 s, 0), & \text{for } t = 1, \ldots, \tau - 1 \\ \varphi_0(2\kappa_1 s, t - \tau), & \text{for } t = \tau, \tau+1, \ldots \end{cases}$$

and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by $\psi(s) := \psi_0(s) + \varphi_0(2\kappa_2 s, 0) + \kappa_2 s$. It is clear that $\varphi \in \mathcal{KL}$ and $\psi \in \mathcal{K}$, and, appealing to (5.13) and (5.14), we arrive at

$$\|\tilde{x}(t)\| \leq \varphi(\|\tilde{x}(0)\|, t) + \psi(\|v\|_\infty) \quad \forall\, t \in \mathbb{Z}_+.$$

Combining this with

$$\|v\|_\infty \leq \sup\{|y(g(u^*y) - g(u(t)y))| : t \in \mathbb{Z}_+,\, 0 \leq y \leq \gamma\|c\|\} = \beta(u),$$

shows that (5.6) holds, completing the proof of statement (1).

(2) We proceed in two steps.

STEP 1. In this step, we prove the claim for $u^\infty = u^*$. Let $x^0 \in \mathbb{R}^n_+$ with $x^0 \neq 0$ and let $u \in \mathcal{F}(\mathbb{Z}_+, (0, \infty))$ be such that $\lim_{t\to\infty} u(t) = u^*$. Then there exists $0 < v^- \leq v^+ < \infty$ such that $u(t) \in [v^-, v^+]$ for all $t \in \mathbb{Z}_+$. By statements (1) and (2) of Theorem 4.2 (with v^- and v^+ playing the role of u^- and u^+, respectively), there exists a compact set $K \subseteq \mathbb{R}^n_+$ such that $0 \notin K$ and $x(t, x^0, u) \in K$ for all $t \in \mathbb{Z}_+$. By statement (1) of Theorem 5.5, there exist $\varphi \in \mathcal{KL}$ and $\psi \in \mathcal{K}$ such that, for all $s, t \in \mathbb{Z}_+$,

$$\|x(t; x(s; x^0, u), u_s) - x^*\| \leq \varphi(\|x(s; x^0, u) - x^*\|, t) + \psi(\beta(u_s)),$$

where $u_s(t) = u(t+s)$ for all $t \in \mathbb{Z}_+$. Now $x(t+s; x^0, u) = x(t; x(s; x^0, u), u_s)$ and so, for all $s, t \in \mathbb{Z}_+$,

$$\|x(t+s; x^0, u) - x^*\| \leq \varphi(\|x(s; x^0, u) - x^*\|, t) + \psi(\beta(u_s)).$$

Given $\varepsilon > 0$, there exists $\sigma \in \mathbb{Z}_+$ such that $\psi(\beta(u_\sigma)) \leq \varepsilon/2$ (where we have used that $\beta(u_s) \to 0$ as $s \to \infty$). Since φ is in $\mathcal{KL}$, we can choose $\theta \in \mathbb{Z}_+$ such that $\varphi(\|x(\sigma; x^0, u) - x^*\|, t) \leq \varepsilon/2$ for all $t \geq \theta$. Consequently,

$$\|x(t; x^0, u) - x^*\| \leq \varepsilon \quad \forall\, t \in (\sigma + \theta) + \mathbb{Z}_+,$$

which proves the claim for the case wherein $u^\infty = u^*$.

STEP 2. Let $u^\infty > 0$. We will reduce this case to the special case which has been dealt with in Step 1. To this end, let $x^0 \in \mathbb{R}^n_+$, $x^0 \neq 0$ and let $u \in \mathcal{F}(\mathbb{Z}_+, (0, \infty))$ be such that $u(t) \to u^\infty > 0$ as $t \to \infty$. Setting $\tilde{x}(t) := (u^\infty/u^*)x(t; x^0, u)$ and $\tilde{u}(t) := (u^*/u^\infty)u(t)$, we have that $\tilde{u}(t) \to u^*$ as $t \to \infty$ and

$$\tilde{x}(t+1) = A\tilde{x}(t) + bg(\tilde{u}(t)c^T\tilde{x}(t))c^T\tilde{x}(t) \quad \forall\, t \in \mathbb{Z}_+.$$

By Step 1, we have that $\tilde{x}(t) \to x^*$ as $t \to \infty$ and so $x(t; x^0, u) \to (u^*/u^\infty)x^*$ as $t \to \infty$, completing the proof. $\qquad\square$

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We turn now our attention to the stability of the forced Lur'e systems (3.2) and (3.3).

Theorem 5.6. *Let u^- and u^+ be real numbers satisfying (4.2), let $u^* \in [u^-, u^+]$ and assume that* **(A1)**–**(A4)** *hold.*

(1) *Consider the initial-value problem (3.2) with solution denoted by $x(\cdot\,; x^0, u)$, assume that* **(A6′)** *holds and let $x^* \gg 0$ be given by (5.2).*

    (a) *Let $\Gamma \subseteq \mathbb{R}_+^n$ be compact and such that $0 \notin \Gamma$. There exist $\varphi \in \mathcal{KL}$ and $\psi \in \mathcal{K}$ such that, for all $x^0 \in \Gamma$ and all $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$, (5.6) holds with $\beta(u)$ given by*

$$\beta(u) := \sup\{|f(u^*y) - f(u(t)y)| : t \in \mathbb{Z}_+, \ 0 \leq y \leq \gamma\|c\|\}, \tag{5.15}$$

    where γ is as in Theorem 4.3.

    (b) *For every $x^0 \in \mathbb{R}_+^n$, $x^0 \neq 0$, and all $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$ such that $u(t) \to u^*$ as $t \to \infty$, we have that $x(t; x^0, u) \to x^*$ as $t \to \infty$.*

(2) *Consider the initial-value problem (3.3) with solution denoted by $x(\cdot\,; x^0, u)$, assume that* **(A6″)** *holds and let $x^* \gg 0$ be given by (5.2). Then the conclusions of statement (1) remain valid in the context of system (3.3), provided that $\beta(u)$ is replaced by $\tilde{\beta}(u) := \theta \sup\{|u(t) - u^*| : t \in \mathbb{Z}_+\}$, where $\theta := \max\{|f(y)| : 0 \leq y \leq \gamma\|c\|\}$.*

The arguments of the proof of Theorem 5.5 can be invoked, mutatis mutandis, to prove Theorem 5.6. We leave the details to the reader.

6. Applications and examples

We present two classes of examples in the subsequent two subsections. The first subsection addresses structured population models which, by their very nature, involve multiple states (and so $n > 1$). The second considers one-dimensional difference equations in a context of chaos control.

6.1. Stage-structured populations

Example 6.1. Consider the following model for a population at time-step $t \in \mathbb{Z}_+$ partitioned into $n \in \mathbb{N}$ discrete stage-classes

$$\left.\begin{aligned}
x_1(t+1) &= s_1 x_1(t) + g\Big(\sum_{j=1}^n u(t)c_j x_j(t)\Big)\sum_{j=1}^n c_j x_j(t) \\
x_i(t+1) &= s_i x_i(t) + h_{i-1}x_{i-1}(t) \quad i \in \{2, 3, \ldots, n\} \\
x(0) &= \begin{pmatrix} x_1^0 & x_2^0 & \ldots & x_n^0 \end{pmatrix}^T
\end{aligned}\right\}. \tag{6.1}$$

Here $x_i(t)$ denotes the abundance of the i-th stage-class at time-step t. The s_i and h_i are probabilities (or proportions) denoting survival (or stasis) within stage-classes and movement into subsequent stage-classes, respectively. As such $s_i \in [0, 1]$, $h_j \in (0, 1]$ and $s_j + h_j \leq 1$ for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n-1\}$. The case $s_i + h_i = 1$ corresponds to the absence of mortality in the i-th stage-class of the population. In an age-based model with stage-classes corresponding to age in time-steps, $s_i = 0$ for all $i \in \{1, 2, \ldots, n-1\}$, and s_n may be non-zero if the final stage-class denotes "individuals above a certain age". The constants c_i are non-negative and capture the fecundity of the i-th stage-class. The function g represents per-capita recruitment rates. The product term

$$g\Big(\sum_{j=1}^n u(t)c_j x_j(t)\Big)\sum_{j=1}^n c_j x_j(t),$$

in (6.1) models the recruitment into the population at time $t + 1$. We assume, as is often the case, that reproduction adds individuals into the first stage-class, which typically denotes eggs, juveniles or seeds, in an insect, animal or plant model, respectively. If g is assumed constant then there is no

density-dependence and (6.1) may be expressed as a matrix population projection model [13] which, for example, have been used to model albatross [1], anchovy [47] and trout [31] populations. Finally, the forcing function u seeks to model the effect of temporal environmental or demographic fluctuations which, in the current model, specifically affect recruitment only.

When $n = 1$, then (6.1) reduces to the class of models considered in [17], which has also been examined in [57, Section 5.2]. For general n, (6.1) may be written in the form of (3.1) with

$$
A := \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ h_1 & s_2 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & h_{n-1} & s_n \end{pmatrix}, \quad b := \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad c := \begin{pmatrix} c_1 \\ \vdots \\ \vdots \\ c_n \end{pmatrix}.
$$

Clearly, $\sigma(A) = \{s_1, s_2, \ldots, s_n\}$, so that $r(A) \leq 1$. We will make the realistic assumption that $s_j < 1$ for all $j \in \{1, 2, \ldots, n\}$, meaning that $r(A) < 1$ and thus **(A1)** holds. We assume that $c_n \neq 0$, so that **(A2)** is definitely satisfied. We note that $c_n = 0$ means that the final stage-class does not contribute to the life-cycle of the rest of the population, and so leads to a reducible model; a case we avoid. Assumption **(A3)** holds if at least one of the s_j are positive or if the integers in the set $\{j \ : \ c_j > 0\}$ are coprime (see [5, Theorem 2.2.3 and Remark 2.2.4]).

We note that the pair (A, b) is controllable since $h_j > 0$ for all $j \in \{1, \ldots, n-1\}$. Observability of the pair (c^T, A) is not guaranteed for all parameter values, and must be checked for each specific example to show that **(A4)** (i) holds. Alternatively, a straightforward calculation shows that

$$
\mathbf{G}(z) = \begin{pmatrix} c_1 & c_2 & \cdots & c_n \end{pmatrix} \begin{pmatrix} z - s_1 & 0 & \cdots & 0 \\ -h_1 & z - s_2 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & -h_{n-1} & z - s_n \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}
$$

$$
= \sum_{j=1}^{n} c_j \prod_{i=1}^{j} \frac{h_{i-1}}{z - s_i}, \quad \text{where } h_0 := 1,
$$

and if $|\mathbf{G}(z)|$ is not constant on the unit circle (which is easily verified graphically), then **(A4)** (ii) holds.

If $u(t) \equiv 1$, then, by Theorem 5.5, the asymptotic behaviour of (6.1) is determined by the function g and the scalar $p := 1/\mathbf{G}(1)$, see also [69, Theorem 2.1]. Biologically, the number p is the reciprocal of the inherent net reproductive number; see [16, p. 7], or [21, Section 1] for a helpful discussion. Briefly, if $g(y)y < py$ for all $y > 0$, then zero is the only equilibrium of (6.1) and it is globally asymptotically stable. If, in fact, $g(y)y < \rho y$, for some $\rho \in (0, p)$, then the zero equilibrium is globally exponentially stable. Every positive $y^* > 0$ such that $g(y^*) = p$ gives rise to a non-zero equilibrium $x^* = (I - A)^{-1} b p y^*$. More specifically, if g satisfies **(A5$'$)**, then there is a unique positive equilibrium by Lemma 5.1 and, by statement (1) of Theorem 5.5, for each compact $\Gamma \subseteq \mathbb{R}_+^n$ with $0 \notin \Gamma$, there exists $\varphi \in \mathcal{KL}$ such that

$$
\|x(t; x^0, 1) - x^*\| \leq \varphi(\|x^0 - x^*\|, t) \quad \forall x^0 \in \Gamma, \ \forall t \in \mathbb{Z}_+.
$$

A consequence is that x^* is a stable equilibrium and attractive with domain of attraction equal to $\mathbb{R}_+^n \setminus \{0\}$.

When bounded temporal variation is included in (6.1), meaning that $u(t) \in [u^-, u^+]$, where u^-, u^+ satisfy (4.2), then under the same assumptions as for stability in the unforced case, namely **(A1)**–**(A4)** and **(A5$'$)**, statement (1) of Theorem 5.5 implies that the deviation of $x(t)$ from x^* is bounded in the uniform manner (5.6). Moreover, statement (2) of Theorem 5.5 ensures that if $u(t) \to u^\infty > 0$ as $t \to \infty$, then $x(t) \to (u^*/u^\infty) x^*$ as $t \to \infty$, demonstrating that the magnitude of the equilibrium x^*, but not its distribution across the population stages, may be adjusted by convergent forcing.

For the purposes of a numerical simulation, we use a matrix model for trout from [31], but assume that recruitment is density-dependent. The model has $n = 4$ stage-classes (based on combined age and size,

see [31] for details) and is described by (6.1) with

$$A := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.22 & 0.16 & 0 & 0 \\ 0 & 0.16 & 0.23 & 0 \\ 0 & 0 & 0.23 & 0.35 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c^T := \begin{pmatrix} 0 & 0 & 15 & 25 \end{pmatrix}. \tag{6.2}$$

As already discussed, assumptions **(A1)**–**(A3)** hold by virtue of the model structure. To verify **(A4)**, we note that the parameter values in (6.2) ensure that the $n \times n$ matrix $(c, A^T c, \ldots, (A^T)^{n-1}c)$ is invertible, and so the pair (c^T, A) is observable.

The matrix $A + bc^T$ has spectral radius 1.07, which corresponds to unbounded asymptotic growth of the density-independent model

$$x(t+1) = (A + bc^T)x(t), \quad x(0) = x^0, \quad t \in \mathbb{Z}_+. \tag{6.3}$$

Here $\mathbf{G}(1) = 1.2977$ so that $p = 1/\mathbf{G}(1) = 0.7706 < 1$. We use a Beverton-Holt nonlinearity (see Example 4.1) to model density-dependence g and set $a_1 = a_2 := \kappa/(1/p - 1) > 0$, where $\kappa > 0$ is a positive constant. The reason for the choice of the parameters a_1 and a_2 is threefold. First, since $a_1 = a_2$, we have that $a_1/a_2 = 1 > p$ so that by Table 5.1, assumption **(A5′)** holds and consequently Theorem 5.5 is applicable. Second, for small $y > 0$, $g(y) \approx a_1/a_2 = 1$, and so the model (6.1)–(6.2) approximates the linear system (6.3), considered in [31], at low population abundances. Third, the unique $y^* > 0$ such that $g(y^*) = p$ is equal to κ, which thus denotes the population's cumulative weighted fecundity at equilibrium, and may be thought of as a net reproductive carrying-capacity in this stratified population modelling context. With $\kappa = 200$, it follows that the limiting population distribution x^* is given by

$$x^* = (I - A)^{-1}bpy^* \approx \begin{pmatrix} 154 & 40 & 8 & 3 \end{pmatrix}^T \quad \text{with} \quad c^T x^* = y^* = \kappa = 200.$$

Figure 6.1 shows sample state-error trajectories $\|x(t) - x^*\|$ of (6.1)–(6.2) with $u(t) \equiv 1$, for a variety of (random) x^0. We see that the trajectories converge to x^* over time.
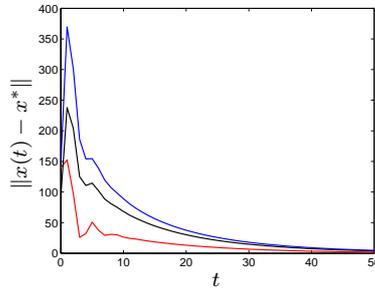
Figure 6.1: Numerical simulations of the uncontrolled ($u(t) \equiv 1$) trout model (6.1)–(6.2) for three different initial conditions.

Since, by construction

$$g(uy) = \frac{a_1}{a_2 + uy} = \frac{1}{1 + \frac{1-p}{p\kappa}uy} \quad \forall\, u > 0, \quad \forall\, y \in \mathbb{R}_+, \tag{6.4}$$

non-constant u may be used to model temporal variation of the constant κ, which may correspond to seasonal environmental fluctuations. Each choice of fixed u such that $0 < u \neq 1$ amounts to replacing κ by κ/u. Figure 6.2 (a) shows sample state-error trajectories $\|x(t) - x^*\|$ for $u = k_i v$, where $k_i \in \{0.25, 0.5, 1\}$ and $v \in \mathcal{F}(\mathbb{Z}_+, [0.7, 1.3])$ is (pseudo)random, but bounded. Bounded variation around x^* is now observed. Finally, Figure 6.2 (c) contains state-error trajectories $\|x(t) - (1/u^\infty)x^*\|$ for three (pseudo)random, but convergent $u \in \mathcal{F}(\mathbb{Z}_+, [0.1, 4])$, with limit $u^\infty := 2$, plotted in Figure 6.2 (b). $\diamondsuit$

Of course, not all stage-structured models correspond to one of the three Lur'e systems (3.1)–(3.3) that we have considered. However, the next example demonstrates that in certain cases simple transformations ensure that our results are applicable.

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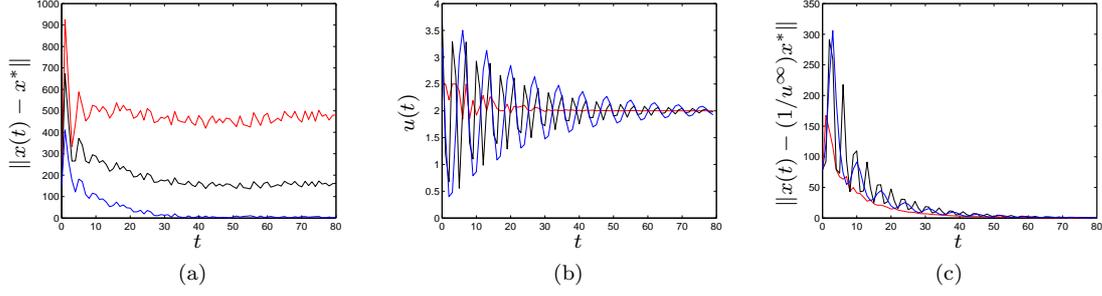
Figure 6.2: Numerical simulations of the trout model (6.1)–(6.2). Panel (a): state error $\|x(t; x^0, u) - x^*\|$ against t for one fixed initial condition and three forcing functions $u = k_i v$. Panel (b): three convergent controls $u(t)$ against t, each with limit $u^\infty = 2$. Panel (c): resulting state errors $\|x(t; x^0, u) - (1/u^\infty)x^*\|$ against t, with respective controls from panel (b).

Example 6.2. Consider the following stage-structure model with two age-classes (juveniles and adults):

$$
\left.\begin{aligned}
x_1(t+1) &= h(x_2(t)), & x_1(0) &= x_1^0 \\
x_2(t+1) &= (1-\gamma)s_1 x_1(t) + s_2 x_2(t), & x_2(0) &= x_2^0
\end{aligned}\right\} \quad t \in \mathbb{Z}_+ , \tag{6.5}
$$

where $x_1(t)$ and $x_2(t)$ denote the number of juveniles and adults at time step t, respectively, $\gamma \in [0,1)$ is a fixed harvesting rate and $s_1, s_2 \in (0,1]$ are fixed survival proportions. Further, the function h in (6.5) is the Ricker function $y \mapsto h(y) = \alpha y e^{-\beta y}$ (see Example 4.1) with $\alpha > 1$ and $\beta > 0$.

System (6.5) corresponds to the juvenile-only harvesting scenario studied in [73]. Indeed, [73] considers three different harvesting scenarios: a juvenile-only harvest, an adult-only harvest, and a harvest where both stages are targeted in equal proportion. Here, we only consider the first harvesting scenario, but we allow temporal variation in the harvesting rate, as opposed to constant harvesting rates adopted in [73], see also [25, 46]. Recall our perspective, discussed in the Introduction, that the temporal variation in the harvesting rate may be exactly known, and even explicitly chosen (a control action), or may be subject to unknown disturbances. In either case, our framework assumes that the temporal variation is bounded, with known lower and upper bounds u^- and u^+, respectively. Consequently, the model we will be considering is given by

$$
\left.\begin{aligned}
x_1(t+1) &= h(x_2(t)), & x_1(0) &= x_1^0 \\
x_2(t+1) &= v(t)s_1 x_1(t) + s_2 x_2(t), & x_2(0) &= x_2^0
\end{aligned}\right\} \quad t \in \mathbb{Z}_+ , \tag{6.6}
$$

where $v(t) \in [u^-, u^+]$, for some $0 < u^- \le u^+ < 1$. The system (6.6) is not of the form (3.1), (3.2) or (3.3). However, setting $w = \beta x_2$ and $f(y) = y e^{-y}$, w satisfies the following second-order non-autonomous scalar difference equation

$$
w(t+2) = s_2 w(t+1) + v(t+1)s_1 \alpha f(w(t)) . \tag{6.7}
$$

Defining $\xi_1 = w$, $\xi_2 = w^\nabla$ and $u = v^\nabla$, then equation (6.7) can be rewritten as

$$
\left.\begin{aligned}
\xi_1(t+1) &= \xi_2(t), & \xi_1(0) &= \xi_1^0 \\
\xi_2(t+1) &= s_2 \xi_2(t) + u(t)s_1 \alpha f(\xi_1(t)), & \xi_2(0) &= \xi_2^0
\end{aligned}\right\} \quad t \in \mathbb{Z}_+ , \tag{6.8}
$$

for some $\xi_1^0, \xi_2^0 \ge 0$. In matrix form (6.8) may be expressed as

$$
\xi^\nabla = \begin{pmatrix} 0 & 1 \\ 0 & s_2 \end{pmatrix} \xi + u \begin{pmatrix} 0 \\ s_1\alpha \end{pmatrix} f\big(\begin{pmatrix} 1 & 0 \end{pmatrix} \xi \big), \quad \xi(0) = \begin{pmatrix} \xi_1^0 \\ \xi_2^0 \end{pmatrix} , \tag{6.9}
$$

which is evidently of the form (3.3) with

$$
A = \begin{pmatrix} 0 & 1 \\ 0 & s_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ s_1\alpha \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f : \mathbb{R}_+ \to \mathbb{R}_+, \ f(y) = y e^{-y} .
$$

Assumption **(A1)** is satisfied if, and only if, $s_2 < 1$. Assumption **(A2)** is trivially satisfied and **(A3)** is satisfied as

$$
A + bc^T = \begin{pmatrix} 0 & 1 \\ s_1\alpha & s_2 \end{pmatrix} \quad \text{and} \quad (A + bc^T)^2 = \begin{pmatrix} s_1\alpha & s_2 \\ \alpha s_1 s_2 & s_2^2 + \alpha s_1 \end{pmatrix} \gg 0 .
$$

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The transfer function $\mathbf{G}$ is given by $\mathbf{G}(z) = (s_1\alpha)/(z(z-s_2))$ for $z \in \mathbb{C}$ and we see that

$$|\mathbf{G}(z)| = \left| \frac{s_1\alpha}{z(z-s_2)} \right| = \frac{s_1\alpha}{|z-s_2|},$$

is not constant on the unit circle as $s_2 > 0$, showing that **(A4)** holds.

Noting that $p = 1/\mathbf{G}(1) = (1-s_2)/(\alpha s_1)$, it follows from Table 5.1 that assumption **(A6″)** holds, if u^-, u^+ and $p = (1-s_2)/(\alpha s_1)$ are such that

$$u^+ e^{-2} \leq (1-s_2)/(\alpha s_1) < u^-. \tag{6.10}$$

If (6.10) holds, then, for each $u^* \in [u^-, u^+]$,

$$\xi^* := (I-A)^{-1}bpy^* = \frac{1}{1-s_2}\begin{pmatrix} \alpha s_1 \\ \alpha s_1 \end{pmatrix}py^* = \ln\left(\frac{u^*\alpha s_1}{1-s_2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

is a unique positive equilibrium of (6.9) with $u(t) \equiv u^*$ and statement (2) of Theorem 5.6 applies. Here

$$y^* = \ln\left(\frac{u^*}{p}\right) = \ln\left(\frac{u^*\alpha s_1}{1-s_2}\right) > 0,$$

is the unique positive solution of $py^* = u^*f(y^*)$. $\diamond$

6.2. Chaos control

Here we consider the one-dimensional difference equation

$$x^\nabla = f(x), \quad x(0) = x^0, \tag{6.11}$$

where $x \in \mathcal{F}(\mathbb{Z}_+, \mathbb{R}_+)$ and $f : \mathbb{R}_+ \to \mathbb{R}_+$, typically from the perspective that (6.11) has chaotic or complex dynamics. We consider the utility of so-called proportional feedback (PF) control, proposed in this context in [48], to create a desirable unique positive equilibrium which is asymptotically stable. The proportional feedback method replaces (6.11) by

$$\left.\begin{array}{ll} x^\nabla = f(v), & x(0) = x^0 \\ v = ux \end{array}\right\} \tag{6.12}$$

where $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$ for some u^-, u^+ satisfying (4.2). The choice $u(t) \equiv u^*$ in (6.12), where $0 < u^* \in [u^-, u^+]$, corresponds to a constant control effort or intensity u^*. We note that (6.12) is a special case of the Lur'e system (3.2) with $n = 1$, $A = 0$ and $b = c = 1$, so that assumptions **(A1)**–**(A4)** are automatically satisfied.

From a practical point of view it is desirable that the choice of $u(t) \equiv u^*$ in (6.12) gives rise to a difference equation with (at most) only two equilibria: zero (if $f(0) = 0$) and a non-zero equilibrium, the latter is desired to be stable and to attract every solution with positive initial condition. Such a situation is important as the existence of alternative attractors could result in control producing an opposite effect to that which is intended [24]. It is known that the above objective may be achieved by PF control for a range of u^* if f satisfies certain conditions. Usual assumptions rely on monotonicity conditions [9] or on imposing that the Schwarzian derivative of f has negative sign [12, 45]. Here, we present a novel result for achieving the desired control objective. First we provide sufficient conditions on f, relevant for the class of models under consideration, which guarantee the existence of a range of constant controls u^* for which **(A6′)** holds. To maintain the focus of the present section on properties of the difference equations (6.11) and (6.12), we relegate the proof of the next lemma to the Appendix.

Lemma 6.3. *Let $0 < u^- \leq u^+ < \infty$ and $u^* \in [u^-, u^+]$. Assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, and satisfies*

(1) *$f(0) = 0$, $f(y) > 0$ on $(0, \infty)$;*

(2) *f has a unique maximum, achieved at $y_M > 0$ and $f'(y) > 0$ on $(0, y_M)$ and $f'(y) < 0$ on (y_M, ∞) (f is unimodal).*

Assume further that one of the following conditions holds:

(3) *there exists a unique $y_I > y_M$ such that $f''(y) < 0$ on $(0, y_I)$ and $f''(y) > 0$ on (y_I, ∞) (y_I is a unique inflection point);*

(3′) *there exist positive z_I and y_I with $z_I < y_M < y_I$ and such that $f''(y) > 0$ on $(0, z_I)$, $f''(y) < 0$ on (z_I, y_I), and $f''(y) > 0$ on (y_I, ∞) (f has exactly two inflection points y_I and z_I).*

If
$$f'(0) > 1/u^- \quad and \quad u^+ f'(y_I) + 1 > 0, \tag{6.13}$$
then $(\mathbf{A6'})$ *holds with $p = 1$.*

Combining Theorem 5.6 and Lemma 6.3, we immediately obtain the main result of this section pertaining to the PF control system (6.12).

Corollary 6.4. *Let $0 < u^- \leq u^+ < \infty$ and $u^* \in [u^-, u^+]$. Assume that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable and satisfies conditions (1), (2) and (3) or (3′) of Lemma 6.3. If (6.13) is satisfied, then there exists a unique $x^* > 0$ such that $f(u^* x^*) = x^*$ and the following statements hold.*

(1) *For every compact set $\Gamma \subset (0, \infty)$, there exist $\varphi \in \mathcal{KL}$ and $\psi \in \mathcal{K}$ such that, for all $x^0 \in \Gamma$ and all $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$, the solution of (6.12), denoted by $x(\cdot\,; x^0, u)$, satisfies (5.6) with $\beta(u)$ given by (5.15).*

(2) *For every $x^0 > 0$ and all $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$ such that $u(t) \to u^*$ as $t \to \infty$, the solution of (6.12), denoted by $x(\cdot\,; x^0, u)$, satisfies $x(t; x^0, u) \to x^*$ as $t \to \infty$.*

Statements (1) and (2) of Corollary 6.4 imply in particular that, under the stated assumptions, there exists a unique positive equilibrium of (6.12) with $u(t) \equiv u^*$ which is stable and attracts every solution with a positive initial condition — these conclusions are similar to those in, for example, [9, 12, 45]. The conclusions of the above corollary also complement [11, Theorem 3.5], as it evidently captures the scenario wherein $u(t) \to u^*$. Furthermore, statement (1) of Corollary 6.4 provides the estimate (5.6) for the difference $|x(t; x^0, u) - x^*|$ which applies to every forcing function $u \in \mathcal{F}(\mathbb{Z}_+, [u^-, u^+])$, thereby addressing a situation which is not considered in [9, 11, 12, 45].

We illustrate the corollary with two different examples of population models from the literature.

Example 6.5. Consider the function $f_1 : \mathbb{R}_+ \to \mathbb{R}_+$ given by
$$f_1(x) = 100x(1 + x)^{-5}, \quad x \in \mathbb{R}_+, \tag{6.14}$$
a special case of the Hassell function [29], $x \mapsto \lambda x(1 + ax)^{-b}$. The assumptions of [9, Theorem 1] hold: f_1 has a positive global maximum at $y_M = 1/4$, $f_1''(y) < 0$ for all $y \in (0, y_M)$, and $y_M f_1(y) < y f_1(y_M)$ for all $y \in (y_M, \infty)$. Hence, [9, Theorem 1] guarantees that if
$$u^* \in (1/f_1'(0), y_M/f_1(y_M)) = (1/100, 125/4096) \approx (0.01, 0.0305),$$
then (6.12) with $u(t) \equiv u^*$ and $f = f_1$ given by (6.14) admits a unique positive equilibrium x^* satisfying $f(u^* x^*) = x^*$ which attracts every solution with positive initial condition.

Using Corollary 6.4 the above range of control intensities u^* for which the same conclusion holds can be extended. Indeed, it is not difficult to see that hypotheses (1)–(3) of Lemma 6.3 hold with $y_M = 1/4$ and $y_I = 1/2$. Consequently, the hypotheses of Corollary 6.4 hold for u^-, u^+ and u^* such that
$$u^* \in [u^-, u^+] \subset (1/f_1'(0), -1/f_1'(y_I)) = (1/100, 729/6400) \approx (0.01, 0.1139), \tag{6.15}$$
where we have used that $f'(y_I) = -6400/729$. Moreover, note that the conclusions of Corollary 6.4 are stronger than those of [9, Theorem 1], and do not require that u in (6.12) is constant. Indeed, statement (1) ensures that for each u^* as in (6.15) the resulting unique equilibrium x^* is input-to-state stable, in the sense that (5.6) holds. Statement (2) ensures that for every $x^0 > 0$, if $u(t) \to u^*$ as $t \to \infty$, then $x(t) = x(t; x^0, u) \to x^*$ as $t \to \infty$. $\diamondsuit$

Example 6.6. The function $f_2 : \mathbb{R}_+ \to \mathbb{R}_+$ given by

$$f_2(x) = x\mathrm{e}^{3.5(1-4x/3.5)-2/(1+8x)}, \quad x \in \mathbb{R}_+ \tag{6.16}$$

is a special case of the function $x \mapsto x\mathrm{e}^{r(1-x/K)-m/(1+sx)}$ considered in [58]. With such a choice of $f = f_2$, elementary calculus shows that the per-capita growth rate function $x \mapsto f_2(x)/x$ is strictly increasing on the interval $(0, 1/8)$ and strictly decreasing on $(1/8, \infty)$, that is, the population model (6.11) exhibits a so-called Allee effect (see, for example, [14, 54, 68] for detailed discussions of Allee effects). The non-monotonic behaviour of the per-capita growth rate is illustrated in Figure 6.3. We claim that the Allee effect on (6.12) is weak for all u^-, u^+ satisfying (4.2) with $u^- > \mathrm{e}^{-1.5} \approx 0.2231$ (which, in particular, includes the uncontrolled version (6.11)), meaning that the origin is still repulsive and there is not a critical population threshold below which the population goes extinct; see, for example, [68]. To establish these assertions, we reiterate that (6.12) is a special case of (3.2), with $n = 1$, $A = 0$, $b = c = 1$ and $p = 1$ so that **(A1)**–**(A4)** hold. Since

$$\liminf_{x \downarrow 0} \frac{f_2(x)}{x} = f_2'(0) = \mathrm{e}^{1.5}, \tag{6.17}$$

and f_2 is bounded, it follows that **(A6)** holds for all u^-, u^+ satisfying (4.2) with $u^- > 1/f_2'(0) = \mathrm{e}^{-1.5}$. Thus, statement (2) of Theorem 4.3 applies to (6.12) for all $u^- > \mathrm{e}^{-1.5}$ which includes system (6.11), the uncontrolled ($u(t) \equiv 1$) version of (6.12).
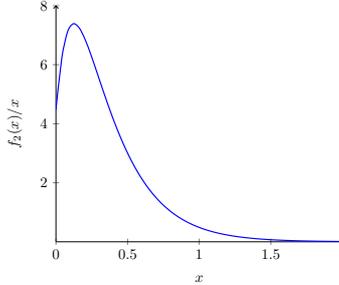
Figure 6.3: Graph of the per-capita growth function f_2 given by (6.16) which exhibits an Allee effect — the function increases as the population size increases in an interval of small population sizes.

The presence of a weak Allee effect means that the results in [9, 11, 12, 45] do not apply. We claim that Corollary 6.4 is applicable to (6.12) with f given by f_2 from (6.16), however. To that end, it may be shown that the sign of f_2' coincides with the sign of the cubic polynomial $-256x^3 + 28x + 1$ which has three real roots: two negative and one positive. Therefore, f_2 is unimodal and attains a maximum at $y_M \in (0.3473, 0.3474)$. On the other hand, the sign of f_2'' coincides with the sign of the polynomial $8192x^5 - 2304x^3 - 576x^2 + 18x + 3$. This polynomial has only two positive real roots $z_I \in (0.0764, 0.0765)$ and $y_I \in (0.6221, 0.6222)$. Therefore, we can guarantee that hypotheses (1), (2) and (3') of Lemma 6.3, and thus the hypotheses of Corollary 6.4, hold for u^-, u^+ and u^* such that

$$u^* \in [u^-, u^+] \subset (\mathrm{e}^{-1.5}, 1/2.37) \approx (0.2231, 0.4219). \tag{6.18}$$

To establish (6.18), we have used (6.13), (6.17) and the readily derived bound: $f_2'(y_I) > -2.37$.

We note that 1 does not belong to open interval in (6.18), so that Corollary 6.4 does not apply for $u^* = 1$. In fact, for our parameter choices, assumption **(A6')** does not hold when $u^* = 1$, in which case, $x^* = 3/8 + \sqrt{3}/4$ is an unstable equilibrium of the feedback system (6.12) (equivalently, of system (6.11)) with $f = f_2$, see Figure 6.4 (a). Figure 6.4 (b) illustrates the sector condition (5.4) in **(A6')** holding for $u^* = 0.4$, which belongs to the interval in (6.18), and thus Corollary 6.4 applies. For $u^* = 0.4$, the stable equilibrium x^* is given by the positive solution of the quadratic equation in x,

$$64(u^*)^2 x^2 + (48 - 16v)u^* x - 3 + 2v = 0,$$

where $v = -\ln(u^*)$, which is approximately equal to 1.3847. $\diamondsuit$
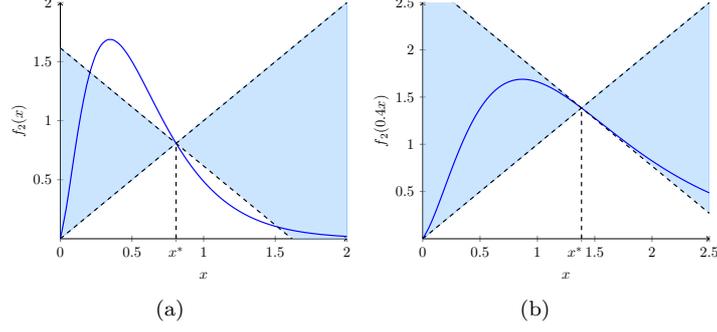
Figure 6.4: Sector conditions for the function f_2 given by (6.16). The sectors have slope $\pm p = \pm 1$ and the intersection of $f(u^* x)$ with the line $l(x) = x$ gives rise to a positive equilibrium x^* of (6.12). Panel (a): the sector condition (5.4) in $(\mathbf{A6'})$ fails for $u = u^* = 1$ (uncontrolled) and the positive equilibrium x^* is unstable. Panel (b): the sector condition (5.4) in $(\mathbf{A6'})$ does hold for $u^* = 0.4 \in (e^{-1.5}, 1/2.37)$ and the resulting equilibrium $x^* \approx 1.3847$ is stable.

AppendixA. Proof of Lemma 5.4

(1) Statement (1) is obvious and there is nothing to prove.

(2) Assumption $(\mathbf{A5})$ holds by statement (1). It follows from the hypotheses and the continuity of g that, for every $u^* \in [u^-, u^+]$, there exists $y^* > 0$ such that $g(u^* y^*) = p$. Furthermore, y^* is unique as $g' < 0$, and we have

$$g(u^* y) > p \quad \forall\, y \in (0, y^*) \quad \text{and} \quad g(u^* y) < p \quad \forall\, y \in (y^*, \infty). \tag{A.1}$$

Since $y \mapsto g(u^* y)y$ is non-decreasing and invoking (A.1), we obtain

$$|g(u^* y)y - g(u^* y^*)y^*| = g(u^* y^*)y^* - g(u^* y)y < p(y^* - y) = p|y - y^*| \quad \forall\, y \in (0, y^*),$$

and, furthermore,

$$|g(u^* y)y - g(u^* y^*)y^*| = g(u^* y)y - g(u^* y^*)y^* < p(y - y^*) = p|y - y^*| \quad \forall\, y \in (y^*, \infty).$$

We now conclude that $(\mathbf{A5'})$ holds.

(3) We have

$$\frac{p}{u^-} < f'(0) = \lim_{y \downarrow 0} \frac{f(y) - f(0)}{y} = \lim_{y \downarrow 0} \frac{f(y)}{y} = \liminf_{y \downarrow 0} \frac{f(y)}{y}.$$

The function

$$h : \mathbb{R}_+ \to \mathbb{R}, \quad y \mapsto \frac{p}{u^+} y - f(y)$$

is differentiable and

$$h'(y) = \frac{p}{u^+} - f'(y), \quad y \in \mathbb{R}_+.$$

By hypothesis $L := \lim_{y \to \infty} f'(y) < p/u^+$, and so $\lim_{y \to \infty} h'(y) > 0$, showing that $h(y) \to \infty$ as $y \to \infty$, establishing $(\mathbf{A6})$.

(4) Fix $u^* \in [u^-, u^+]$ and let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $h(y) := f(u^* y)$. Clearly, $h(0) = f(0) = 0$, $h' \geq 0$ and $h'' \leq 0$ on $(0, \infty)$. Note further that

$$h'(0) = u^* f'(0) > u^* p/u^- \geq p,$$

and

$$\lim_{y \to \infty} h'(y) = u^* \lim_{y \to \infty} f'(y) < u^* p/u^+ \leq p.$$

The sector bound (5.4) now follows from [6, Proposition 4.7] applied to h, thus establishing $(\mathbf{A6'})$. Finally, by the commentary after the statement of Lemma 5.3, it is clear that $(\mathbf{A6''})$ also holds. $\qquad\square$

AppendixB. Proof of Lemma 6.3

Fix $u^* \in [u^-, u^+]$ and note that the bounds in (6.13) imply that $[u^-, u^+] \subset (1/f'(0), -1/f'(y_I))$. We need to check that **(A6)** and the sector condition (5.4) in **(A6′)** hold. By hypothesis, f is continuous and $f(y) > 0$ for $y > 0$. Since f is assumed (right) differentiable at $y = 0$, the first condition in (4.3) follows immediately from the first inequality in (6.13). Since f is continuous and assumed to have a unique global maximum, attained at $y_M < \infty$, it follows that f is bounded and hence the second condition in (4.3) also holds. We conclude that **(A6)** is satisfied.

We proceed to prove (5.4). Define $h \colon \mathbb{R}_+ \to \mathbb{R}_+$ by $h(y) := f(u^*y) - y$. Note that $h(0) = 0$ and

$$h'(0) = u^* f'(0) - 1 \geq u^- f'(0) - 1 > 0,$$

where we made use of (6.13). Hence $h(y) > 0$ for all sufficiently small $y > 0$. Since f is bounded, it follows that $h(y) < 0$ for $y > 0$ sufficiently large. By the intermediate value theorem it follows that $h(y) = 0$ has positive solutions. Since $h'(0) > 0$, it is clear that there exists a minimal positive solution $y^* > 0$ of $h(y) = 0$. We claim that (i) h has no other positive zeros (meaning that y^* is the unique positive number satisfying $f(u^*y^*) = y^*$) and (ii) the sector condition (5.4) holds. To see this, we consider two cases.

CASE 1: Assumption (3) holds. In this case, h is strictly concave on $[0, y_I/u^*]$. Thus, if $y^* \in (0, y_I/u^*]$, then y^* is necessarily the only zero of h in $[0, y_I/u^*]$, and furthermore, since h decreases on $(y_I/u^*, \infty)$, it follows that y^* is the unique positive zero of h. Now assume that $y^* > y_I/u^*$. To prove that y^* is the unique zero of h, it is sufficient to establish that h is strictly decreasing on $(y_I/u^*, \infty)$. Since $f'(y_M) = 0$, $f'(y) < 0$ for $y > y_M$, $y_M < y_I$, and, by assumption (3), we have that

$$|f'(u^*y)| \leq |f'(y_I)| = -f'(y_I) \quad \forall y > y_M/u^*, \tag{B.1}$$

whence, in light of (6.13), (B.1) and the inequality $u^* \leq u^+$, it follows that

$$h'(y) = u^* f'(u^*y) - 1 \leq u^* |f'(u^*y)| - 1 \leq -u^+ f'(y_I) - 1 < 0 \quad \forall y > y_I/u^*.$$

Hence, h is strictly decreasing on $(y_I/u^*, \infty)$, implying that that y^* is the unique zero of h.

In either situation, since $y^* > 0$ is the unique positive solution of $f(u^*y) = y$, the first condition in (6.13) implies that

$$f(u^*y) - f(u^*y^*) > y - y^* = -|y - y^*| \quad \forall y \in (0, y^*), \tag{B.2}$$

and

$$f(u^*y) - f(u^*y^*) < y - y^* = |y - y^*| \quad \forall y \in (y^*, \infty). \tag{B.3}$$

In light of (B.2), (B.3) and since $f(0) = 0$, to establish (5.4), it now suffices to verify that y^* is the only positive solution of

$$f(u^*y) = 2y^* - y. \tag{B.4}$$

Seeking a contradiction, suppose that $0 < y^\dagger \neq y^*$ also solves (B.4). Then

$$f(u^*y^*) - f(u^*y^\dagger) = 2y^* - y^* - (2y^* - y^\dagger) = y^\dagger - y^*,$$

and, by the mean-value theorem, there exists a number ξ between y^* and $y^\dagger$ such that

$$-1 = \frac{f(u^*y^*) - f(u^*y^\dagger)}{y^* - y^\dagger} = u^* f'(u^*\xi). \tag{B.5}$$

Invoking assumption (2) shows that $\xi > y_M/u^*$. However, in light of (6.13), (B.1) and (B.5), we now arrive at the contradiction

$$-1 = u^* f'(u^*\xi) = -u^* |f'(u^*\xi)| \geq u^* f'(y_I) \geq u^+ f'(y_I) > -1,$$

completing the argument.

CASE 2: Assumption (3′) holds. Note that if $y^* > y_I/u^*$, then y^* is the unique zero of h, as h is strictly decreasing on $(y_I/u^*, \infty)$. If $y^* \leq y_I/u^*$, then, seeking a contradiction, assume that $y^\sharp > 0$ is another, different, zero of h, which then necessarily satisfies $y^* < y^\sharp$. Since $h(0) = 0$, applying Rolle's theorem

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twice, it follows that there exist z^* and $z^\sharp$ such that $0 < z^* < y^* < z^\sharp < y^\sharp$ and $h'(z^*) = h'(z^\sharp) = 0$. Consequently, applying Rolle's theorem again, there exists $w \in (z^*, z^\sharp)$ such that

$$h''(w) = (u^*)^2 f''(u^* w) = 0. \tag{B.6}$$

Now $z_I/u^* < z^*$ because $h'(0) > 0$, and thus, by assumption (3'), we have that h' does not vanish on $(0, z_I/u^*]$. Moreover, $h'(z^\sharp) = 0$ implies that $f'(u^* z^\sharp) > 0$, and so $z^\sharp < y_M/u^* < y_I/u^*$ by assumption (2). Hence, $u^* w \in (z_I, y_I)$ which leads to a contradiction with (B.6), as $f'' < 0$ on (z_I, y_I).

The proof that the sector condition (5.4) holds is omitted as it is identical to that in Case 1. $\qquad\square$

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