Timber Plate Dynamics with Space-Time Finite Element

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Summary

A new space-time finite element formulation for dynamic analysis of timber plates is presented. The method is based on an unconventional Hamilton variational principle and a special technique in time domain interpolation is employed. For the simple case of solid wood plates under sinusoidal driving forces the results of the proposed method are compared with the results of the methods Newmark-β and Wilson-θ as well as the exact solutions. The results show that the proposed space-time finite element method exhibits improved accuracy and precision in the dynamic analysis.

1. Introduction

While finite element methods have been widely used to solve time-dependent problems, most procedures have been based upon semi-discretizations; i.e. the spatial domain is discretized using finite elements, producing a system of ordinary differential equations in time which in turn is discretized using finite difference methods with ordinary differential equations. These procedures have been particularly successful in solving structural dynamics problems.

However, the accumulation of error from one increment to the next brings enormous difficulties in designing algorithms that accurately capture discontinuities or sharp gradients in the solution. In structural dynamic analysis, when high frequency loads are involved, such as machinery vibration and seismic actions, improving accuracy necessitates decreasing step increments, which greatly increases computational costs. Another disadvantage of semidiscrete methods is the difficulty of employing adaptive mesh techniques. Since, for semidiscrete methods, the spatial domain is discretized first and then the temporal discretization is used for the resultant set of ordinary differential equations, the corresponding space-time discretization is structured. The space-time discretization arising from the semidiscrete approach consists of rectangular subdomains of the space-time domain. In contrast, space-time finite element methods, in which the spatial and temporal domains are simultaneously discretized, accommodate unstructured meshes in the space-time domain. This mesh may be considered to arise from an adaptive mesh refinement strategy in which both spatial and temporal refinement can occur to accurately capture the stress waves, stress concentrations, etc. In regions where the solution is smooth, the mesh is relatively coarse, while a finer mesh is employed near the stress wave fronts. Thus, an accurate solution may be obtained without resorting to a uniformly refined (and computationally expensive) mesh.

In this paper, a new space-time finite element formulation suitable for dynamic analysis of timber plates is presented, where special techniques are employed for the time domain interpolation. The results of the proposed method are compared with the results of the methods Newmark-β and Wilson-θ for the case of single plates under the action of sinusoidal forces. The results show that the proposed space-time finite element method exhibits the improved accuracy and precision in the dynamic analysis when compared to the exact solution.

2. Governing Equations of Plate Dynamics

Equations governing flexural motions of plate structures (two-dimensional theory) can be deduced from the three-dimensional equations of elasticity, which include velocity-displacement relations, momentum-velocity relations, and equations of motion (see Reissner, 1945 and Mindlin, 1951.)

3. Unconventional Hamilton Variational Principles

In the course of his work on optics, Hamilton also began to consider the possibility of developing an analogous theory for the dynamic of systems of particles (Hamilton 1834, 1835). Hamilton's principle is an "integral principle"', which means that it considers the entire motion of a system between time t_1 and t_2. Hamilton's principle describes the time end-value problem, but, for practical engineering problems, such as earthquake engineering, it is difficult to get the structure's response at time t_2 and, naturally, the initial conditions are always given at time t_0.

Here, a novel approach for deriving an unconventional Hamilton variational principle is presented, which can fully characterise the initial-value problem of plate dynamics. By employing this principle, a space-time finite element method for plate dynamics analysis is presented by utilising the finite element discretization of the time domain as well as the usual finite element discretization of the spatial domain.

3.1 Generalized virtual work principle

The conventional virtual work principle is a well-known principle in continuum mechanics. In this section, a more general virtual work principle for problems of plate dynamics is proposed.

If $p, L_x, L_y, M_x, M_y, M_{xy}, Q_x, Q_y, w, \psi_x$ and ψ_y are the independent and arbitrary functions, the Ostrovski-Gauss formula can be rewritten as the following equation through the integration by parts,

$$\widehat{\Pi}_1 + \widehat{\Pi}_2 + \widehat{\Pi}_3 + \widehat{\Pi}_4 = 0 \qquad [1]$$

where

$$\widehat{\Pi}_1 = \int_0^{t_1} \iint_\Omega \left[p\dot{w} + L_x\dot{\theta}_x + L_y\dot{\theta}_y + M_x\theta_{x,x} + M_y\theta_{y,y} + M_{xy}(\theta_{x,y} + \theta_{y,x}) - Q_x(w_{,x} - \theta_x) - Q_y(w_{,y} - \theta_y) \right] dxdydt$$

$$\widehat{\Pi}_2 = \int_0^{t_1} \iint_\Omega \left[w(\dot{p} - Q_{x,x} - Q_{y,y}) + \theta_x(\dot{L}_x + M_{x,x} + M_{xy,y} - Q_x) + \theta_y(\dot{L}_y + M_{y,y} + M_{xy,x} - Q_y) \right] dxdydt$$

$$\widehat{\Pi}_3 = \int_0^{t_1} \int_{\partial\Omega} (Q_n w - M_n\theta_n - M_{ns}\theta_s) dsdt \text{ and } \widehat{\Pi}_4 = \iint_\Omega (w_0 p_0 + \theta_{x0}L_{x0} + \theta_{y0}L_{y0} - w_1 p_1 - \theta_{x1}L_{x1} - \theta_{y1}L_{y1}) dxdy$$

Eq. [1] is, actually, the generalised virtual work principle for the plate dynamics problem. It is also considered as one of the most important formulas given here, from which a series of unconventional Hamilton variational principles can be developed.

When $M_x, M_y, M_{xy}, Q_x, Q_y, p, L_x, L_y$ satisfy equations of motion, force boundary and initial conditions, w, θ_x, θ_y satisfy velocity-displacement, strain-displacement relations, displacement and initial conditions, Eq. [1] can be simplified as follows:

$$\int_0^{t_1}\iint_\Omega (fw + m_x\theta_x + m_y\theta_y)dxdydt + \int_0^{t_1}\iint_{\partial\Omega}(Q_n w - M_n\theta_n - M_{ns}\theta_s)dsdt - \iint_\Omega (w_1 p_1 + \theta_{x1}L_{x1} + \theta_{y1}L_{y1} - \tilde{w}_0\tilde{p}_0 - \tilde{\psi}_{x0}\tilde{L}_{x0} - \tilde{\theta}_{y0}\tilde{L}_{y0})dxdy = \int_0^{t_1}\iint_\Omega (M_x\kappa_x + M_y\kappa_y + 2M_{xy}\kappa_{xy} + Q_x\gamma_x + Q_y\gamma_y - pv - L_x\omega_x - L_y\omega_y)dxdydt \quad [2]$$

Eq. [2] is actually the conventional virtual work principle. It is very interesting that the variables are coupled in this equation, in the following ways: (f, m_x, m_y) and, (w, θ_x, θ_y), (p, L_x, L_y) and (v, ω_x, ω_y), and $(M_x, M_y, M_{xy},\ Q_x, Q_y)$ and $(\kappa_x, \kappa_y, \kappa_{xy}, \gamma_x, \gamma_y)$.

3.2 Five-field and two field unconventional Hamilton variational principle

From Eq. [1] and [2], the five-field unconventional Hamilton variational principle can be derived.

Theorem 1: $\partial\Pi_5 = 0$ or $\partial\Gamma_5 = 0$, if and only if p, L_x, L_y, v, ω_x, ω_y, w, θ_x, θ_y, $\kappa_x, \kappa_y, \kappa_{xy}, \gamma_x, \gamma_y$, $M_x, M_y, M_{xy}, Q_x, Q_y$ are the solutions to the plate initial-boundary problem. Where the functions Π_5 and Γ_5 are

$$\Pi_5 = \int_0^{t_1}\iint_\Omega \{K - p(v - \dot{w}) - L_x(\omega_x - \dot{\theta}_x) - L_x(\omega_x - \dot{\theta}_x) - U + M_x(\kappa_x + \theta_{x,x}) + M_y(\kappa_y + \theta_{y,y}) + M_{xy}\left[2\kappa_{xy} + (\theta_{x,y} + \theta_{y,x})\right] + Q_x[\gamma_x - (w_{,x} - \theta_x)] + Q_y\left[\gamma_y - (w_{,y} - \theta_y)\right] + fw + m_x\theta_x + m_y\theta_y\}dxdydt + \Pi_{IB} + \overset{\circ}{\Pi}$$

$$\Gamma_5 = \int_0^{t_1}\iint_\Omega \Big[K^* - B - V - A - (Q_{x,x} + Q_{y,y} + f - \dot{p})w - (-M_{x,x} - M_{y,y} + Q_x + m_x - \dot{L}_x)\theta_x - (-M_{xy,x} - M_{y,y} + Q_y + m_y - \dot{L}_y)\theta_y\Big]dxdydt + \Gamma_{IB} + \overset{\circ}{\Gamma}.$$

For proof of theorem refer to Zhang (2004). Theorem 1 is the completed five-field unconventional Hamilton variation principle. However, constraints in practical engineering dynamic problems will yield the five-field principle to a two-field one in which $\left(p, L_x, L_y\right)$ and $\left(w, \theta_x, \theta_y\right)$ are variational variants

$$\Pi_2\left(p_x, L_x, L_y; w, \theta_x, \theta_y\right) = \int_0^{t_1}\iint_\Omega \left[p\dot{w} - \frac{1}{2\rho h}p^2 + L_x\dot{\theta}_x - \frac{1}{2\rho J}L_x^2 + L_y\dot{\theta}_y - \frac{1}{2\rho J}L_y^2 - U^\circ + fw + m_x\theta_x + m_y\theta_y\right]dxdydt + \int_0^{t_1}\left\{-\int_{\partial\Omega_2+\partial\Omega_3}\bar{M}_n\theta_n ds + \int_{\partial\Omega_3}\left(-\bar{M}_{ns}\theta_s + \bar{Q}_n w\right)ds\right\}dt + \iint_\Omega\left(\tilde{p}_0 w_0 - \overset{\circ}{p}_1 w_1 + \tilde{L}_{x0}\theta_{x0} - \overset{\circ}{L}_{x1}\theta_{x1} + \tilde{L}_{y0}\theta_{y0} - \overset{\circ}{L}_{y1}\theta_{y1}\right)dxdy. \quad [3]$$

Eq. [3] is the variational principal with displacement and momentum fields. It is also the variational principle in the phase domain for plate dynamics.

4 Space and Time Domain Discretization

The first step of the implementation in the space-time finite element method is to discretize the structures in space domain. This step is a normal finite element procedure. In this paper, an eight node Reissner-Mindlin plate element (see Zienkiewicz 2000, vol. 2, page 173) is employed for space domain discretization.

4.1 Space domain discretization

The interpolations of displacement and momentums are:

$$w=\sum_{i=1}^{8}N_w w_i\ ,\ \theta_x=\sum_{i=1}^{8}N_{xi}\theta_{xi}\ ,\ \theta_y=\sum_{i=1}^{8}N_{yi}\theta_{yi}\ ,\ p=\sum_{i=1}^{8}N_{wi}p_i\ ,\ L_x=\sum_{i=1}^{8}N_{xi}L_{xi}\ ,\ L_y=\sum_{i=1}^{8}N_{yi}L_{yi} \qquad [4]$$

Substituting Eq. [4] into [3], gives:

$$\Pi_2=\int_0^{t_1}\Big[\{p\}^T\left[M^t\right]\{\dot q\}-\frac{1}{2}\{p\}^T\left[K^p\right]\{p\}-\frac{1}{2}\{q\}^T\left[K\right]\{q\}- \\ \{\overset{\circ}{\dot q}\}^T\left[C\right]\{q\}+\{F\}^T\{q\}\Big]dt+\{\overset{\circ}{p}_0\}^T\left[M^t\right]\{q_0\}-\{\overset{\circ}{p}_1\}^T\left[M^t\right]\{q_1\}, \qquad [5]$$

Next step is the discretization in time domain, which is based on Eq. [5]

4.2 Space domain discretization

For an arbitrary time step, $[t_i, t_{i+1}]$, let $t_0=t_{i+1}-t_i$. Thus, the local time coordinate within this time step can be defined as $t\in[0,t_0]$. This time step is divided into m small intervals. The length of each span is H, and $t_0=mH$. Let the normalized local time coordinates be τ, $\tau=t/H$, and $\tau\in[0,m]$. In the current time step, the interpolation in the time domain is assumed as follows:

$$\{q(t)\}=\left[\Phi(\tau)\right]\{a\}\ ,\ \{p(t)\}=\left[\Phi(\tau)\right]\{b\} \qquad [6]$$

Here, the fifth order of Lagrangian polynomial is employed. Substituting Eq. [6] into Eq. [5] gives

$$\Pi_2=\{b\}^T\left[M_t^t\right]\{a\}-\frac{1}{2}\{b\}^T\left[K_t^p\right]\{b\}-\frac{1}{2}\{a\}^T\left[K_t\right]\{a\}-\{\overset{\circ}{a}\}^T\left[C_t\right]\{a\}+ \\ \{F_t\}^T\{a\}+\{\overset{\circ}{b}\}^T\left[M_0^t\right]\{a\}-\{\overset{\circ}{b}\}^T\left[M_1^t\right]\{a\} \qquad [7]$$

Where

$$\left[M_t^t\right]=\left(\int_0^m[\varphi]^T[\dot\varphi]d\tau\right)\otimes\left[M\right],\ \left[K_t^p\right]=\left(H\int_0^m[\varphi]^T[\varphi]d\tau\right)\otimes\left[K^p\right],\ \left[K_t\right]=\left(H\int_0^m[\varphi]^T[\varphi]d\tau\right)\otimes\left[K\right]$$

$$\left[C_t\right]=\left(\int_0^m[\dot\varphi]^T[\varphi]d\tau\right)\otimes\left[C\right],\ \left[M_0^t\right]=\left([\varphi(0)]^T[\varphi(0)]\right)\otimes\left[M^t\right],\ \left[M_1^t\right]=\left([\varphi(m)]^T[\varphi(m)]\right)\left[M^t\right]$$

$\left[\varphi\right]=\left[\varphi_1 \;\; \varphi_2 \;\; \cdots \;\; \varphi_s\right]$, and the superscript $\dot{}$ represents the derivation in respect to τ ; $\otimes$ represents the Kronecker product.

As a variation to [7], viz., $\delta\Pi_2 = 0$, with rearrangement, it can be obtained as the following:

$$[R]_{2ns\times 2ns}\{\Delta\}_{2ns\times 1} = \{P\}_{2ns\times 1} \qquad [8]$$

Where $[R]_{2ns\times 2ns} = \begin{bmatrix} [R_{11}]_{ns\times ns} & [R_{12}]_{ns\times ns} \\ [R_{21}]_{ns\times ns} & [R_{22}]_{ns\times ns} \end{bmatrix} = \begin{bmatrix} \left([K_t]+[K_t]^T\right)/2+[C_t]^T & [N_m]^T-[N_t]^T \\ -[N_t] & \left([M_t]+[M_t]^T\right)/2 \end{bmatrix}$,

$\{\Delta\}_{2ns\times 1} = [\{a\},\{b\}]^T$, $\{P\}_{2ns\times 1} = [\{F_t\},\{0\}]^T$

Eq. [8] shows the recursive scheme of the proposed space-time finite element. The displacement and momentum values at the end of the last time step serve as the initial vectors of the current time step. By solving the linear equations, Eq.[8], for each step, we obtain the nodal displacement and momentum at each sample points within the current time step can be calculated. After that, by processing the stress process in the spatial domain, we establish the internal forces of the place structures can then be determined. Thus, plate dynamic analysis is completed.

5. Numerical Investigations

To demonstrate the proposed space-time finite element method for the plate dynamic problem, a series of numerical examples have been designed and analysed consisting of square plates with different boundary conditions and span/depth ratios. Harmonic loading is applied on the centre of the plate, which permits, for this demonstration, calculation of the exact solution for comparison.

Consider a square solid timber plate with either four clamped edges (shown in Fig 1) or four simply supported edges (shown in Fig 6). Here the plate is solid timber with the properties of strength class is C16 under service class 2. The length of each side of the plate is 1500 mm with four different thicknesses: h = 60 mm, 120 mm, 180 mm and 240mm. The harmonic load applied at the centre of the plate is described by the following function $f(t) = 9.8\sin\left(500t\right)$ kN (80Hz). It is assumed that, at time t_0, the initial displacement, is $\tilde{w}_0 = 0$ and the initial momentum is $\tilde{p}_0 = 0$. In the spatial domain, the plate is divided with a 2×2 mesh, while in the time domain, up to the fourth order Lagrange polynomial interpolations are employed for interpolation. Figures 2 to 5 and Figures 7 to 10 show comparisons of the results from the proposed method with those from the mode superposition method, Wilson-θ method and Newmark-β method.

Full modes are employed in the mode superposition method, and for each single degree, Duhamel's Integral is used. Thus, the results from the mode superposition method are considered to be the exact solution to this problem. Results from the proposed method agree very well with those of the exact solution, whereas the results from Wilson-θ and Newmark-β method depart from those of the mode superposition results due to the accumulation of error from increment to increment. For systems closest to resonance (Fig 2 and Fig 8) the difference in predicted amplitude and period are clear. For the other systems, the difference in behaviour at peak displacements are clear.

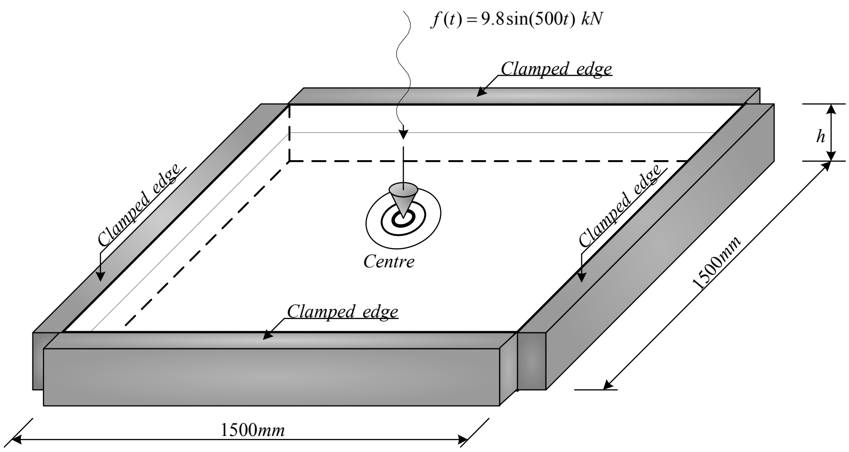

Fig 1 A square plate with four clamped edges

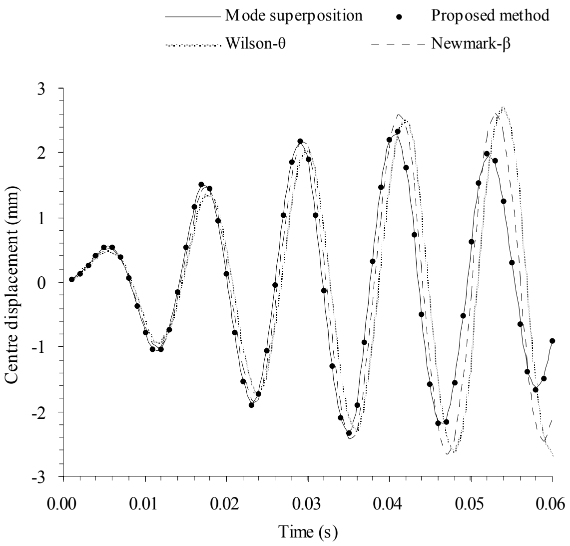

Fig 2 Centre displacement of a square plate with four clamped edges ($h = 60mm$)

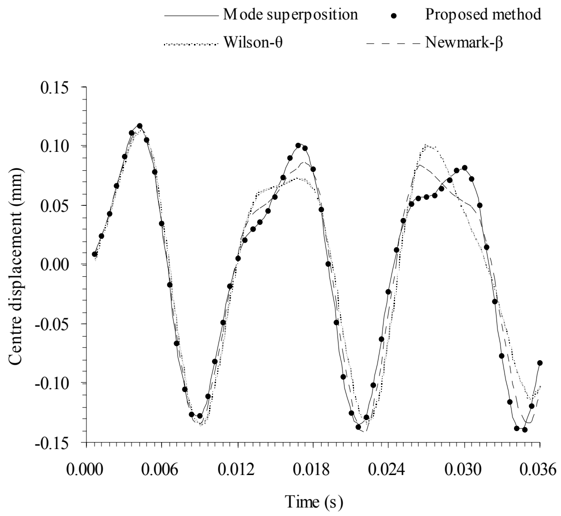

Fig 3 Centre displacement of a square plate with four clamped edges ($h = 120mm$)

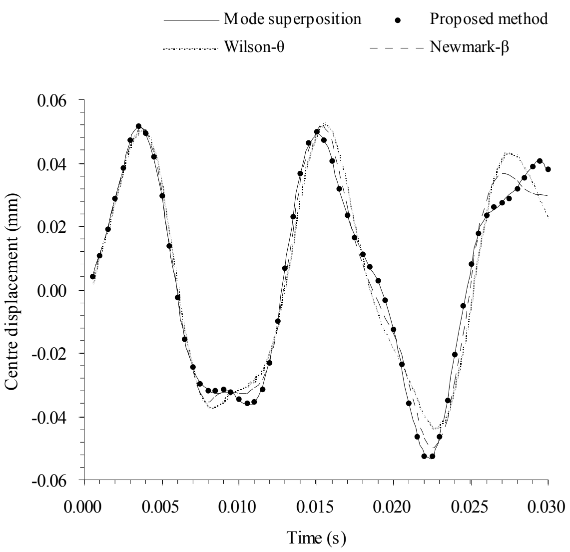

Fig 4 Centre displacement of a square plate with four clamped edges ($h = 180mm$)

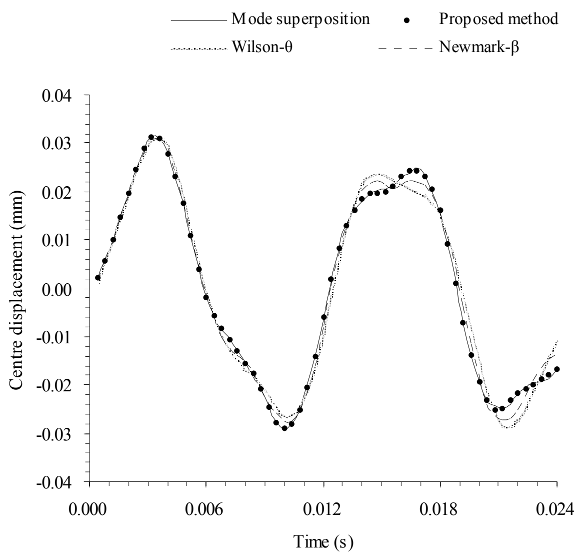

Fig 5 Centre displacement of a square plate with four clamped edges ($h = 240mm$)

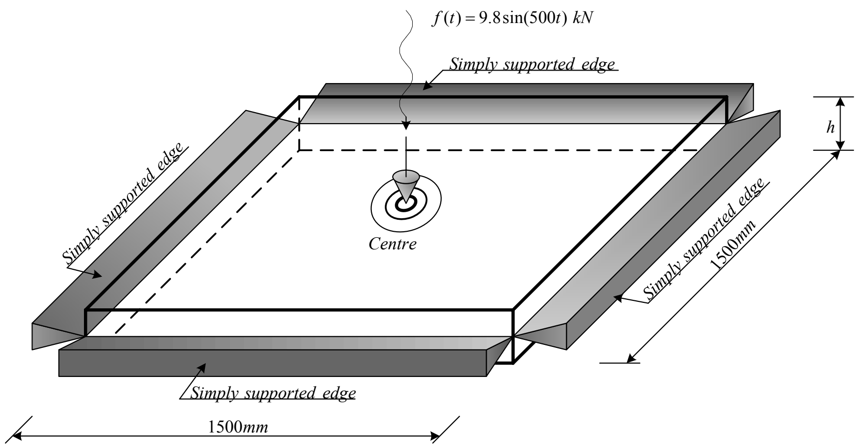

Fig 6 A square plate with four simply supported edges

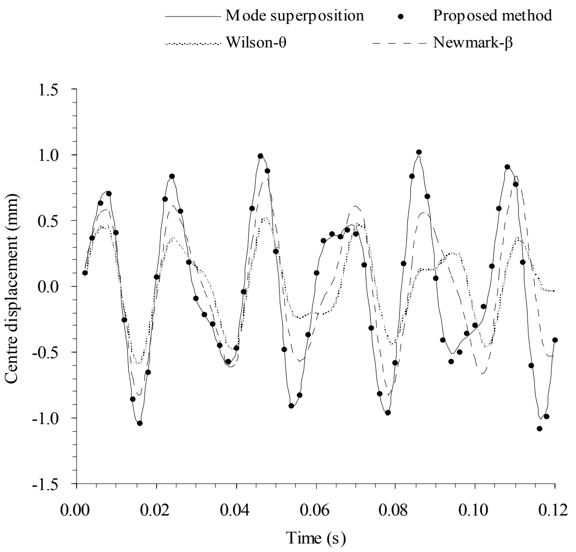

Fig 7 Centre displacement of a square plate with four simply supported edges ($h = 60mm$)

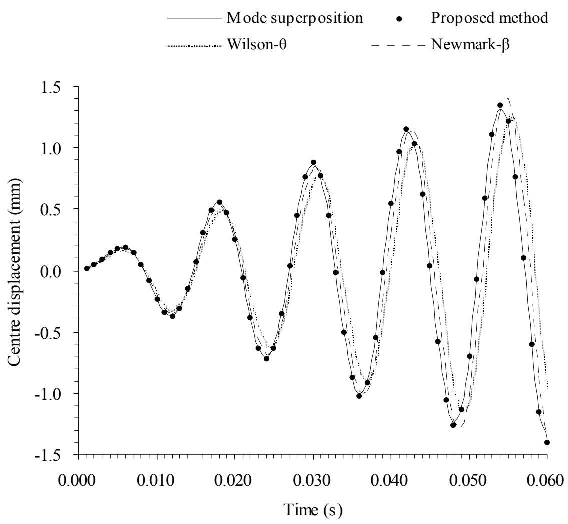

Fig 8 Centre displacement of a square plate with four simply supported edges ($h = 120mm$)

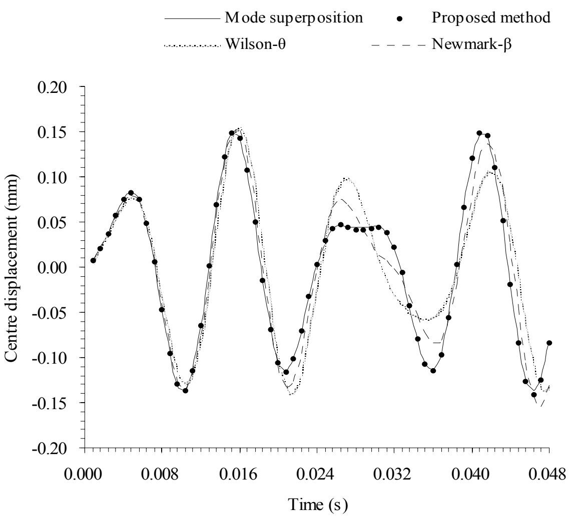

Fig 9 Centre displacement of a square plate with four simply supported edges ($h = 180mm$)

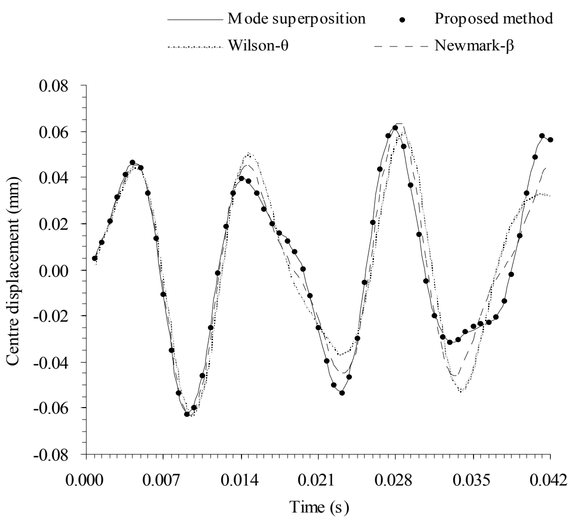

Fig 10 Centre displacement of a square plate with four simply supported edges ($h = 240mm$)

6. Concluding Remarks

Although the Wilson-θ and Newmark-β methods are widely used in the plate dynamic analysis, these semidiscretization approaches have difficulties in accurately capturing discontinuities or sharp gradients in the solution (for example due to sharp changes in driving force or structural form). In this paper, a new space-time finite element method for timber plate structure elasto-dynamics capable of overcoming this limitation has been presented. A novel approach for deriving unconventional Hamilton variational principles was presented beginning from a generalized virtual work principle. Mathematically, this virtual work principle is an identical equality. Consequently, the resulting variational principles are built on a firm mathematical foundation.

Based on the proposed unconventional Hamilton variation principles, the plate structures are discretized in the space domain using the eight node Reissner-Mindlin plate element. Then, the time domain is discretized by employing Lagrangian polynomial interpolation. Similar to the finite element method in the space domain, a set of simultaneous equations are constructed. By solving these simultaneous equations, the response of the plate structure can be obtained.

Numerical results in the test problems presented in this paper agree very well with the results from the mode superposition method (considered as the exact solution), whereas, the results from the Wilson-θ and Newmark-β methods depart from those of the mode superposition results due to error accumulation. Another interesting observation is, for thinner plates, where the amplitude is larger the proposed method captures the response accurately, but the results from Wilson θ and Newmark β methods are most in error. Therefore, it is believed that the proposed method generates unconditionally stable, higher-order and more accurate algorithms. The proposed method will allow calculation of more accurate solutions for earthquake analysis of timber plate structures for design of active control systems, and allow more computationally efficient solution for cases of machine vibration than present methods.

7. Acknowledgement

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8. References

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