MODELLING THE EFFECTS OF ROAD PRICING

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ABSTRACT

The classical road tolling problem is to toll network links such that under the principles of Wardropian User Equilibrium (UE) assignment, a System Optimising (SO) flow pattern is obtained. Such toll sets are however non-unique and further optimisation is possible: for example, *minimal revenue* tolls create the desired SO flow pattern at minimal additional cost to the user. In the case of deterministic assignment, the minimal revenue toll problem is capable of solution by various methods, such as linear programming (Bergendorff et al, 1997) and by reduction to a multi-commodity max-flow problem (Dial, 2000). However, these methods are restricted in their application to small networks or problems of special structure.

However, it is generally accepted that deterministic models are less realistic than stochastic, and thus it is of interest to investigate the principles of tolling under stochastic modelling conditions. This thesis develops methodologies to examine the minimal revenue toll problem in the case of Stochastic User Equilibrium. In examining the case of Stochastic User Equilibrium the 'desired flow pattern' to be created must first be determined. The classical economics solution of replacing cost flow functions with marginal cost flow functions does not generally result in the total network cost being minimised in the stochastic case. Thus tolls which are analogous to Marginal Social Cost Pricing (MSCP) in the deterministic case do not give the System Optimal solution but do result in drivers being charged for their externalities. If the true system optimal flow pattern is desired it may be possible to derive tolls which are unrelated to MSCP. This thesis examines tolling methodologies to achieve or to closely approach the deterministic SO solution under SUE and examines both path-based exact methods and link-based heuristic methods.

It may however be considered to be more desirable in the stochastic case to produce instead a 'Stochastic System Optimum' (SSO) where the *perceived* total network cost is minimised. The SSO solution may be achieved by applying MSCP-tolling under SUE (Maher et al, 2005). This thesis examines tolling solutions for inducing the Stochastic System Optimal (SSO) under SUE and compares such toll sets with those derived to induce SO flows.

Initial work was based on the assumption of a fixed demand stochastic equilibrium model. It is clear that imposing tolls on a network, will directly affect demand as well as being able to influence route choice and this paper investigates tolling under Stochastic User Equilibrium with elastic demand (SUEED). Elastic demand may be readily included in stochastic equilibrium models (Maher et al, 1999) and MSCP tolls may be easily derived by using marginal cost functions in an SUEED algorithm. In the case of deterministic assignment with elastic demand a Social Optimum with Elastic demand (SOED) has been formulated where economic benefit is to be maximised; under this formulation it has been shown that all toll-sets associated with the flow/demand pattern which minimises the SOED objective function generate the same revenue and that marginal social cost price tolls are such a toll set (Hearn and Yildirim, 2002; Larsson and Patriksson; 1998).

This thesis extends the SOED formulation to SSOED (Stochastic Social Optimum with Elastic Demand) and discusses the extension of the fixed revenue result to tolling to achieve SSOED under SUEED. It further examines the case where the economic benefit is not to be maximised and tolling may be used to achieve an SSO flow pattern for a particular OD matrix where minimal revenue tolls may be desired. The earlier heuristic used to create toll sets to induce the 'true SO' flow pattern under stochastic assignment methods presupposes that the desired flow pattern is fixed and may be determined. In the case of elastic demand, further iteration is required to account for the change in the 'desired flow pattern' as each link toll is increased and the fixed demand heuristic is extended to include this. This heuristic may also be utilised with a minor modification to produce toll sets to induce SSO solutions under SUEED and these are compared to toll sets derived to induce SO flow patterns under SUEED.

Numerical results are given for small toy networks and logit-based SUE is used primarily for reasons of mathematical tractability although the heuristics derived are equally valid for use with any stochastic assignment method.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Introduction

Road Tolling is a commonly used term, but can be used to describe quite different situations. For example there are many instances of 'toll roads' particularly in continental Europe, whereby a charge is made for travel along usually a section of high quality trunk road. Similarly a charge may be made to use a short length of road, primarily in the case of a tunnel or a bridge as is common in the UK. Such charges are usually fixed, and payment is made at a toll booth at one end of the charged section, either electronically, or by actual payment at the booth. Such road tolling is not however the subject of this thesis, but rather road tolling as applied to a town or city centre, i.e. not a single link, but a connected network of links. Such road tolling is often referred to as congestion charging, as its aim is often given to be the easing of city centre congestion.

Congestion charging by means of implementing road user tolls, has been much discussed, but has been implemented in relatively few cities. Toll rings exist and are operational in Oslo and Bergen (and formerly in Trondheim) in Norway, and area charging schemes exist in Singapore and London. Prior to the London scheme becoming operational these schemes whilst resulting in some reduction in congestion (which in the case of Singapore is supported by strict regulations upon car ownership), were mainly used as methods of raising revenue (AFFORD project) and as such the tolls are set at a low level and were fixed related to a point of entry at a toll ring (although the Trondheim scheme was refined and has since recently ended). Schemes in the UK such as in London (Richards et al, 1996), and the proposed Edinburgh scheme (City of Edinburgh Council 1999, MVA consultancy, 1999) also proposed either a toll ring or a set of toll rings and sectors, and although the relief from congestion is a major policy objective, the revenue created would still appear to be of prime importance (in Edinburgh the Local Transport Strategy would have been dependent on the amount of revenue which could have been raised).

Intuitively it would seem logical that if road tolls are to be implemented, that they should in some way be optimal; that is they should be as effective as possible with regard to specified criteria. It may be a political objective to maximise revenue, within limits of political acceptability, whilst not seeking particularly to discourage road users or to lessen congestion, which would lead to relatively cheap tolls. If instead congestion reduction were the primary objective, tolls would be set very high to discourage usage, an extreme case of which would be to completely restrict traffic and impose high fines for violation (dependant on relevant demand elasticities).

If optimality is desired however, suitable criteria must first be defined. Theoretically this is often considered by fixing network demand, and then considering how that traffic may be assigned throughout the network such that the overall network cost is minimised, this flow pattern occurs where marginal cost flow relations are used in place of the 'actual' cost flow relations. Marginal costs are the rate by which the total cost on a link increases with each additional unit of flow, and thus give the additional cost that each driver imposes on the network. This results in the System Optimal solution.

The theoretical economic basis for road tolling which requires that drivers should pay for their externalities is well known and has been so for many decades (e.g. Pigou, 1920; Walters, 1954; Vickrey, 1955). Thus from an economic perspective, the optimal tolling solution seeks to minimise total network travel cost, and is associated with

System Optimal flow patterns, the 'marginal cost' tolls (the difference between marginal link costs and 'actual' link costs) being set to replicate the system optimal flows. Theoretical marginal cost tolls in which a toll is placed on every link have been perceived a) to be problematic to implement and b) to impose tolls which are too high to be implemented politically, although they would imply the best solution regarding congestion reduction. That the practical problems in imposing a link based charging system have led to the pursuit of 'sub-optimal' cordon based schemes (Meng et al, 1999) is understandable, but the current advances in technology regarding electronic payment and vehicle tracking make such schemes a viable alternative for the future (although efforts have been made to 'optimise' cordon schemes with respect to certain criteria (May et al, 2000). Tolls which are high clearly create political problems regarding public acceptance and equity, although there is evidence to show that there is a greater public appreciation for the merits of road charging (particularly from city centre residents) if the revenue could be suitably hypothecated (Glaister et al 1998).

This project aims to develop methodologies and algorithms to determine toll sets which are required to produce desired flow patterns, in particular the examination of the minimal revenue toll problem. Various solutions to this problem have been examined and compared for the case of deterministic assignment and this project aims to extend existing work to examine the minimal revenue toll problem in the case of Stochastic User Equilibrium. It is aimed to produce solutions which do not require full path enumeration.

1.2 Thesis structure

Chapter 2 of this thesis examines the background to road user charging and discusses some of the existing congestion charging schemes worldwide, and some of the trials and modelling which have been conducted. Issues surrounding equity and acceptability are discussed in relation to some of the existing or proposed charging schemes.

Chapter 3 reviews the literature related to static traffic assignment modelling and road tolling. Firstly static assignment models are reviewed for both the deterministic and stochastic cases. The concept of the System Optimal flow pattern is defined and elastic demand is included in the assignment models. Secondly tolling to achieve different levels of network optimisation are discussed for both deterministic and stochastic assignment modelling.

Chapter 4 concentrates on tolling methodologies to achieve or to closely approach the deterministic system optimal flow pattern under stochastic user equilibrium assignment (SUE). Exact path-based methods are presented, inherent problems are discussed and a link-based heuristic algorithm is derived to create 'low-revenue' tolls. Toy-networks are used to illustrate numerical results in this chapter and in following chapters.

Chapter 5 defines the Stochastic System Optimal (SSO) solution and describes how this flow pattern is created by use of marginal social cost pricing under SUE. Exact and heuristic methods to obtain minimal-revenue toll sets which achieve this SSO under SUE are presented. Numerical results are compared to those obtained in Chapter 4.

Chapter 6 extends the fixed demand methodology presented in Chapter 4 to include elastic demand. The nature of the System Optimal under elastic demand is discussed and a tolling methodology to achieve deterministic SO flow patterns for varying demand under SUEED is presented.

Chapter 7 extends to definition of the SSO from Chapter 5 to Stochastic System Optimal with Elastic Demand (SSOED). Marginal social cost pricing is examined, and the possibility of reduced revenue tolls presented. Numerical results are compared to those obtained in Chapter 6.

Chapter 8 summarises the thesis and presents conclusions. Issues surrounding link-based charging schemes are discussed and opportunities for further work are given.

1.3 Research Methodology

In producing toll sets to induce desired flow patterns some direct analytical methods were used for small test-networks using logit-based SUE. For intermediate test networks logit-based SUE was still used but iterative methods were used within spreadsheets (EXCEL). For larger networks existing software for stochastic and deterministic assignment was utilised. A current working version of the SAM stochastic assignment method (NEWSAN ©Maher) was obtained, and familiarisation undertaken. Either logit-based or probit-based stochastic assignments could be determined using this software; logit-based were used primarily for comparative purposes but the heuristics derived apply equally to probit-based assignment (SAM or Monte-Carlo). Desired SO flow patterns were obtained from either existing WARDROP (©Maher) software or from NEWSAN with a value of the stochastic variability parameter such that the stochastic assignment tended towards the deterministic. The toll sets were derived by utilising excel as an interface to compare the existing flow pattern (from NEWSAN) with the desired SO flow pattern (from WARDROP). It would be desirable to combine the tolling heuristic into the FORTRAN 77 code for NEWSAN, but this was not completed as part of this project.

Initially familiarisation with SATURN software was undertaken, but whilst SATURN could have been utilised rather than SAM to produce stochastic assignments, it was not used in this project.

1.4 Publications

Parts of this thesis have appeared in publications; three peer refereed papers and two conference papers. The refereed papers are reproduced in Appendix B (as per university regulations).

Maher M.J., Stewart K. and Rosa A. (2005). Stochastic Social Optimum Traffic Assignment, *Transportation Research B,* **39(8)**, 753-767, defines and derives the SSO solution and is referenced in Chapter 5. Tolling solutions to achieve the SSO are also presented.

Stewart, K. and Maher, M.J. (2006). Minimal revenue network tolling: system optimisation under stochastic assignment, in Hearn D.W, Lawphongpanich S, Smith M, (Eds.), *Mathematical and Computational Models for Congestion Charging, Applied Optimization*, Vol 101, New York, Springer, presents the heuristic as in Chapter 4 and gives comparative results for tolling to achieve SO and SSO.

Stewart K. (2007). Tolling Traffic Links under Stochastic Assignment:
Modelling the relationship between the number and price level of tolled links and optimal traffic flows. *Transportation Research A (in press),* discusses some of the policy implications of this work.

The content of Chapters 6 and 7 respectively is presented in Stewart and Maher (2005), at the 4th IMA International Conference on Mathematics in Transport and Stewart (2006) at the 38th Annual UTSG conference. A full list of presentations and papers related to the content of this thesis is given in Appendix B.4.

CHAPTER 2

General Issues Surrounding Road User Charging

2.1 Introduction

Road tolling or Road User Charging (RUC) as a general theme is currently particularly topical in the UK and whilst there are not many actual schemes in existence, there have been numerous trials and much public debate. Road tolling is used to describe a variety of different scenarios where road use is charged for at the 'point of use' rather than through Vehicle Excise Duty (VED) or general taxation. It can refer to road tolls (such as the recent M6 toll road (M6 toll, 2006)), to toll bridges or tunnels, or to area charging schemes. Charging for area schemes is often termed 'congestion charging' as it is usually implemented when a city has a particular problem with congestion. (Unlike toll-roads, bridges or tunnels where the charge is often imposed to pay back the cost of a new-piece of infrastructure, irrespective of levels of congestion). Congestion charging as a method of reducing congestion in cities has been on the political agenda for many years (e.g. Smeed, 1964) but operational city charging schemes have been relatively few. Established schemes have been operational in Singapore and Norway for many years but more general worldwide implementation of such schemes has only been occurring in the current decade, such as the recent London scheme.

These operational road user charging schemes have used a cordon (or area) system, which has the benefit of being transparent and easy to implement, and acts to discourage drivers from entering the controlled area, but once the driver is within the cordon, there is no additional incentive to choose a route which would be beneficial to the system as a whole.

Whilst operational schemes are cordon based, trials have however been carried out in which tolling schemes have been tested with road pricing measures such as: distance travelled, time spent travelling and congestion caused (Cambridge study (May and Milne, 2000), (Ison, 1998)), which demonstrate that the technology to implement a path or link based tolling system for urban areas does exist, and so such schemes may be feasible for actual implementation in the future.

There is also current political interest in the UK regarding more developed tolling schemes: The Commission for Integrated Transport recently published a report 'Paying for Road Use' (CfIT, 2002), which suggests the introduction of nationwide road user charging. The report is of particular interest in that it suggests the use of marginal social cost pricing on all roads (i.e. motorways, A Roads, minor roads, city centres etc), balanced by a reduction (or abolition), of Vehicle Excise Duty (VED), combined with a reduction in fuel duty, so that the result desired would be fiscal neutrality. Such a scheme would rely on charging for travel along a link, rather than passing across a cordon, and would therefore require similar technology to implement as would be required for say a minimal revenue toll scheme. Thus they are potentially suggesting a theoretical economics based approach, which might require a fully electronic GPS based location and payment system. More recent reports expanded on the same concept (ITC, 2003; DfT 2003). More recently in the UK the Transport Innovation Fund (TIF) was launched in July 2005 (LTT 421), to encourage Local Authorities in England and Wales to trial road pricing schemes. In November 2005 seven LAs were selected to take their plans forward with TIF funding (LTT 432), and these projects will be monitored over the coming year.

2.2 Operational Road User Charging Schemes and Trials

Despite having been on the road transport agenda in a variety of forms for the past 40 years the only established schemes which are currently operational are those in Singapore, and the Norwegian cities of Bergen and Oslo (Trondheim recently discontinued) (May and Milne 2000) and the recent UK schemes in London and Durham. Whilst the Norwegian schemes have been designed mainly to raise revenue, the Singapore scheme was aimed more at reducing congestion. The Oslo toll ring has been operational since 1990 (AFFORD project 1999), and the toll gates would appear to be located primarily to maximise revenue. Electronic seasonal passes are available and the system operates continuously; thus there is no more discouragement to trips at congested periods, than to trips at any time, and once a seasonal pass has been purchased, there is no particular disincentive to enter the toll ring as many times as desired. Indeed if such permits are available there may be an incentive to make more trips than required to get one's 'moneys worth'. The schemes in Norway are in fact designed to raise revenue and are thus not particularly useful models for the type of system desired in the UK which are generally promoted primarily as schemes to decrease congestion, (although raising revenue for public transport is often a major additional motivator, as for the unsuccessful Edinburgh scheme).

Recently the first UK road-user charging scheme became operational, in Durham on the 1st October 2002. Initial reports (LTT 351) stated that there was a 90% reduction in vehicles in the restricted area in the first five days of the schemes operation. The charge is £2 to enter the central area of Durham between 10am and 4pm, Monday to Saturday, and is controlled by a single entry point ticket machine. The central area of Durham, is however very small, and access easily restricted, so that this scheme is not particularly comparable with the proposed schemes for larger cities, which would require rather more infrastructure. Having successfully received TIF funding (LTT 432) to test the feasibility of expanding the existing scheme, the Durham scheme may well be more significant in future. One option is to place an additional charge on the main A-road through the city, and a second option is to expand the current cordon area.

It is of interest to consider some of the existing and proposed schemes in more detail, section 2.2.1 looks at the Singapore scheme, 2.2.2 the London scheme and 2.2.3 the rejected Edinburgh scheme. Section 2.2.4 looks at the Cambridge trials, 2.2.5 the progress project trials and 2.2.6 summarises the TIF funded trials.

2.2.1 Singapore

The Singapore scheme is interesting not only as it is aimed to reduce congestion, but more currently because it successfully utilises an electronic road pricing system at a time when UK studies still seem sceptical about practical implementation of electronic systems (Road Charging Options for London, 2000). Singapore initially used an Area Licensing Scheme (ALS) introduced in 1975, but replaced this with Electronic Road Pricing (ERP) in 1998 (Seik 2000). Singapore's ERP system is the first system of its kind to be used permanently and extensively. There are three main visible components to the scheme; the gantries, the in-vehicle unit (IU), and the smart card. The fitting of IUs for all motor vehicles was undertaken in 1997-98, and was free of charge and voluntary. However by July 1998, 96% of 680,000 vehicle owners had fitted their vehicles with the IUs (Seik 2000).

This is noteworthy as much opposition to schemes which rely on electronic pricing, such as link based tolling schemes, rests on the claim that the fitting of vehicles with the necessary devices would be impractical and that such devices would have to be fitted during the vehicles manufacture creating an unacceptable time lag (Ison 1998). The success of the fitting of the devices for the Singapore scheme contradicts such claims and would suggest that electronic schemes are currently feasible, not just some mythical possibility for the future.

The IUs are permanently fixed to the vehicle windscreen and are back-lit to display the cash balance in the smart card which is being used. The cards themselves are available at numerous outlets for purchase and may be conveniently topped up at automatic teller machines. The pre-payment nature of the system avoids issues of public privacy which are often raised in respect of charging schemes, records of individual vehicles only being kept where a violation occurs.

Whilst this scheme is still cordon based, it offers the best technological potential for a link based scheme, as detection beacons could potentially be situated on links where a toll was to be levied. Thus the implementation of a minimal revenue link based toll scheme may be feasible at a not so distant point in the future. Further it is the first existing scheme to use a variable charging system related to location and time of day. Whilst this does not vary dynamically in relation to actual congestion, but is based on historic data, the system allows for different tolls at different gantries and implements the tolls on a gradual basis with small cost increments building up to the maximum level. This reduces the adverse boundary effects which can be observed in London at the beginning and end of the charging period.

2.2.2 London

Whilst the first road user charging scheme to become operational in the UK is the scheme in Durham, the highest profile scheme is probably that which was introduced in London on Monday 17th February 2003 (Road Charging Options for London, 2000). It is possible that this high profile is due in part to the high profile of Ken Livingston (Mayor) whose support for road user charging is strong and who made it part of his election manifesto. Various possible scenarios were considered, involving not only road user charging but also workplace parking levies. Whilst a variety of different cordon and zoning schemes were considered, the final scheme, as per the charging order (in place since July 2001), is based on one central zone bounded by the inner ring road, as shown in figure 2.1. The scheme is operational from 0700 to 1900 on weekdays, and is based on the purchase of permit. Purchase of this license results in the inclusion of the registration of a vehicle onto a database and digital enforcement cameras then read the number plates of vehicles within the restricted area and match them to the database; fixed penalties are issued for non payment (£80, reduced to £40 for early payment, or raised to £120 if late, which is in line with penalties for other motoring offences). Such technology has been proved successful on toll roads in North America and Australia, and with the suggested number of enforcement cameras (90 fixed, 10 mobile), offers a 70-80% chance of catching offenders. The charge is made for travelling within the boundary roads, not on them, and it is possible to pay both in advance or on the day (also from 19th June 2006, it is also possible to pay until midnight on the next charging day, but there will be a £10 fee for such payments). To make the scheme more politically acceptable there are various exemptions and reduced rates for emergency vehicles, disabled drivers and residents. The net revenue that could be collected was hypothecated to provide a package of complementary measures, such as bus priority schemes, interchange improvements and safety improvements.

Figure 2.1: London Congestion Charge area (source: TfL)

News reports after the first day of operation stated there was a 25% reduction in traffic within the controlled area where traffic was shown to be free-flowing. This has settled into a 30% reduction in congestion (TfL, 2004b) and an approximately 15% reduction in circulating traffic (TfL, 2004a) . The charge was increased from an initial £5 to £8 (July 2005) and a Western Extension (figure 2.2) and further charge increase to £10 has been proposed (LTT443). There is some discussion regarding where residents of the charging areas will be allowed to drive, i.e. would a resident from the new western extension zone be permitted to drive throughout the whole charged area, or only in the western zone.

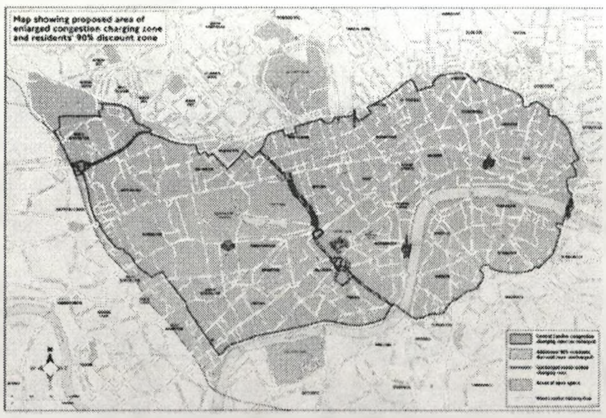


Figure 2.2: London Congestion Charge area: Western Extension (source: Tfl)

Other points of interest are the resulting modal shift. A 37% increase in bus users entering the charging zone was recorded between Autumn 2002 and Autumn 2003, this increased by a further 12% in the following year (Ma and Chatterjee, 2006). Whilst not a direct result of the charging scheme it should also be noted that bicycle levels have increased by about 35% post London bombs (June 2006) (M. Simpson, per. comm. 2006) as some commuters prefer to avoid the underground. Although this increase in bicycle use cannot be directly attributed to the implementation of the congestion charge the lower traffic volumes in the charging zone undoubtedly made cycling a more attractive proposition.

Despite the success of the Singapore ERP the London study initially concluded that it was unlikely that a fully electronic system which would comply with national standards could be made operational during the Mayor's first term of office (ending May 2004). It was recognised however that despite a high initial set up cost, such a scheme would have lower running and enforcement costs, and would have the potential for much greater flexibility of the charging scheme, such as higher charges at peak times. However the London scheme has now become embedded and has potential to develop (as per Singapore). Whilst there are no current plans in London to move away from area based charging, now the principle for charging has been established, more technologically advanced schemes may well become possible in the future. The charging levels have already altered since implementation and the existence of the planned western extension show that once schemes are operational it is possible for them to develop and evolve.

2.2.3 Edinburgh

Edinburgh had next to London the most advanced plans for introducing road user charging in the UK, but a political decision to assess local support by means of a public referendum (Feb 2005), led to the failure of these plans. The Edinburgh study (MVA

Consultancy, 1999), had focused mainly on the effect of different possible judgemental toll rings with different charges on reduction in congestion, and revenue raised which would be hypothecated to transport projects (City of Edinburgh Local Transport Strategy, 1999). 24 tests were carried out with possible flat rate charges of up to £4, including one type of scenario having two toll rings with separate charges. The scheme that was chosen to take forward was a double cordon scheme, with the outer cordon at the city bypass (operational 07.00-10.00) and the inner cordon around the historic city centre (operational 07.00-18.30), as illustrated in figure 2.3.

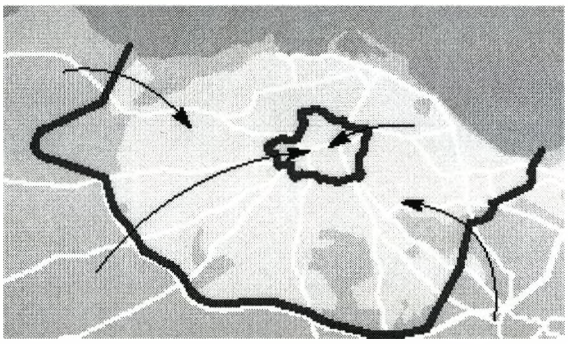


Figure 2.3: Edinburgh proposed double cordon (source TIE Ltd.)

A £2 cordon crossing charge was proposed whereby motorists would pay only for the first cordon they crossed in an inbound direction. As for the London scheme, once the charge was paid there would be no financial disincentive to minimise further driving within the cordons. If the charge were paid to enter the outer cordon from outwith Edinburgh, then no disincentive would be present to discourage motorists from driving within the historic city centre. Residents living within charging zones would also be permitted to drive uncharged as long as they did not cross an operational cordon in an inbound direction. As the outer cordon circled most of the city, this would mean that a large number of city residents would be able to drive uncharged as normal (as long as they avoided the central area). This scheme design was objected to by councils outwith the city of Edinburgh (i.e. West, East and Mid Lothians and Fife), as it was seen to materially penalise commuters from these areas, while income raised was planned to support public transport schemes focussed primarily within the City of Edinburgh. This obviously raised concerns about fairness and equity, particularly when it transpired that Edinburgh City Council intended to offer a charge exemption for crossing the outer cordon to city of Edinburgh council tax payers residing in Queensferry (just outwith the cordon). It is not clear whether the negative referendum result was wholly due to the unpopularity of congestion charging in general and this scheme in particular, or if it was purely a negative consequence of holding a referendum for this sort of political decision (Planning, 2005). It is possible though, that many of the perceived negative features of this cordon scheme would have been less severe had a different scheme been proposed. A link-based tolling scheme for instance would have avoided many of the equity issues raised, particularly if only a small number of links would have to bear a toll. Wider issues of equity and acceptability related to road user charging schemes will be discussed in sections 2.3 and 2.4.

2.2.4 Cambridge

Although it did not progress beyond the trial stage, Cambridge was the first UK city to actually carry out a field trial of an urban road pricing scheme. A scheme was considered from 1990-93, which would have involved fitting all the vehicles in the city and surrounding area (12-15 mile radius) with an electronic metering device, which would be connected to the vehicle's odometer. Outwith a cordon these devices would be dormant, but on entering the city they would be activated by fixed roadside beacons, and the vehicle would be charged (by means of a smart card debit similar to that used in Singapore) for each 'unit of congestion caused'. The metering device would be de-activated by the beacons on exiting the cordon. This scheme was directly aimed at reducing congestion as the metering devices would not commence charging until a calculation involving vehicle speed and distance travelled indicated that the network was congested.

Although the technology worked in practice in the trial there were serious reservations about extending it. The high initial costs of installing the infrastructure were off-putting particularly as there were no reliable estimates at that point in time about what the costs would actually be. It was also felt that installing the metering devices into the cars would be problematic, and such installation should ideally be carried out at manufacture. The later results from Singapore however suggest that fitting devices to cars may not be so problematic as might be perceived, particularly if the actual fitting of the device was carried out with no 'point of sale' charge to the user. A further reservation was that drivers would not be able to be certain what cost would be incurred prior to making the trip.

Although after the trial the above scheme was not implemented, Cambridge continues to experience high levels of congestion, and the potential for a road pricing scheme of some type is still being considered. A recent study (May, Milne 2000), used Cambridge as a case study to test four road pricing measures, based on cordons crossed, distance travelled, time spent travelling and time spent in congestion. Most of the indicators used showed that time based pricing performed the best and that overall cordon pricing was the least effective although the easiest to implement. This study also raised the issue of potential re-routing significantly reducing the overall benefit of charging schemes, in that traffic may be displaced to boundaries rather than actually reduced. More recently Cambridgeshire has secured TIF funding to progress their charging plans and are looking specifically at a fiscally neutral scheme (LTT432). Despite evidence that they may not be the best way of reducing congestion, cordon based schemes such as in London and rejected for Edinburgh are seen as attractive due to their relative ease of implementation and user transparency.

2.2.5 The Progress Project

The progress project (**P**ricing **RO**ad use for **G**reater **R**esponsibility, **E**fficiency and **S**ustainability in citie**S**), was a demonstration project (completed May 2004) led by Bristol City Council and sponsored by the European Commission. It includes the member countries as detailed on figure 2.4.

Figure 2.4: Progress Project Countries

The PROGRESS project states its objective to be:

> 'To demonstrate and evaluate the effectiveness and acceptance of integrated urban transport pricing schemes to achieve transport goals and to raise revenue.' (www.progress-project.org)

The project highlights the benefits of partner cities working together to test pricing schemes in real situations; co-ordination between sites is possible, common standards may be developed with regard to equipment and payment systems, results may be made available across Europe and could contribute to other cities implementing pricing schemes. A summary of the technology demonstrated in each member city is given in figure 2.5. As shown in the table, different cities have been researching different types of scheme; Rome and Trondheim were the only full scale implementations, Bristol, Genoa and Edinburgh were planning full scale implementations (although Edinburgh's was subsequently rejected), Copenhagen and Gothenburg ran demonstration trials and Helsinki completed a modelling study (Progress, 2004). Trondheim was unique in having an operational scheme (since 1991) at the start of the project, whilst Rome evolved an existing access restriction system into a charging scheme. As mentioned earlier, the scheme in Trondheim was originally introduced to raise revenue, but since its original implementation, it has evolved to become more equitable. Originally, it used a simple toll ring, but now it has a zone based system with differentiated prices dependent on the time of day. The revenues are hypothecated to transport investment, although not necessarily within Trondheim. The charging system has resulted in a decrease in traffic of around 10%, which is similar to the decrease for the London scheme.

Scheme concept	Road-pricing technology basis		
	DSRC – electronic tag	ANPR	GPS
Cordon (per trip)	Rome Helsinki	Bristol Genoa Rome	Copenhagen Bristol
Cordon (per day)		Edinburgh	
Zone (per trip)	Trondheim Helsinki		Copenhagen
Distance-based			Copenhagen Gothenburg Bristol

DSRC – Dedicated Short Range Communication (tags)
ANPR – Automatic Number Plate Recognition using digital cameras and OCR (Optical Character Recognition)
VPS – Vehicle Positioning System (GPS based)

Figure 2.5: Progess Project scheme concepts: (source:Progress, 2004)

The development of the scheme in Trondheim was of interest (despite the scheme now having recently been discontinued) as it demonstrates that initially simple schemes can evolve successfully post implementation, public objection to the scheme has also fallen considerably over time, suggesting that initially simple schemes such as for London have potential for development in the future as public acceptance increases.

Whilst the other cities were interested primarily in developing a scheme which they hoped to implement, Gothenburg and Copenhagen concentrated on testing a variety of schemes. (Helsinki concentrated on attitudinal surveying). The demonstrations conducted by Gothenburg and Copenhagen are of particular interest, as they focus on GPS technology, which could potentially be used to implement link based pricing schemes as opposed to sub-optimal cordon based schemes.

One of the listed goals of the progress project was (Progress, 2000):

> 'To develop and demonstrate integrated urban transport pricing schemes, based on the concept of marginal-cost pricing, in real urban situations.'

For a marginal cost pricing scheme (or similar) to be implemented, technology which could identify links would be required. Gothenburg used a fully functional vehicle positioning system in its trials, charges being levied on certain congestion prone major links. Such a scheme has similar technological requirements to the tolling methodology proposed in later chapters of this thesis.

2.2.6 Transport Initiative Fund (TIF) trials

After the unsuccessful referendum in Edinburgh there was concern that the political agenda for congestion charging in the UK might have suffered irreparably (e.g. Planning 2005), and that no further schemes would be supported. However the TIF funding awarded to seven English local authorities to promote congestion charging related trials (LTT 432) has maintained UK interest in such schemes. In fact the support for these trials is such that there is political thought that if they prove successful a nationwide scheme may prove unnecessary as a collection of locally based schemes may suffice (ibid). Bristol, Cambridge and Durham were awarded funds to progress their already advanced trials/schemes, and Greater Manchester, Shropshire, Tyne and Wear and West Midlands were awarded funds for development of new schemes including both distance-based and cordon-based pricing schemes and workplace parking levies. Greater Manchester and Tyne and Wear are both investigating distance based charging; the Greater Manchester scheme being coordinated with Norwich Union's 'pay as you drive' technology. Thus there continues to be considerable political interest in promoting charging schemes and potentially developing charging systems which are not necessarily cordon based. Manchester's work will also examine the effects on driver behaviour of a reward based scheme (for example by giving credits for public transport use), which might potentially have a significant impact on scheme acceptability.

2.3 Acceptability of Charging Schemes

Political acceptability and how to achieve it is a major consideration in the planning and successful implementation of road user charging schemes. Whilst to date most urban schemes have utilised toll rings, there is no reason to assume that link-based schemes may not be preferred in the future particularly if national road user charging schemes (be they revenue neutral or otherwise) are introduced. Issues of assessing acceptability may be problematic. In Edinburgh, despite a successful public enquiry, and whilst there was no legal need, the decision to assess public support by means of a dedicated public referendum was made. This referendum showed a high level of opposition to the scheme (about 3:1 with turnout 61.8%) and plans were consequently dropped (Gaunt, 2006). However opinion polls in other cities have shown that public opinion can be quite supportive of charging schemes when they are initially proposed, but that this support gradually diminishes up to the point where the scheme is actually introduced. Post implementation support and acceptability tends to rise again whilst people get 'used to' the changes. This has been the trend for most of the Norwegian toll rings (Tretvik, 2003), and specifically for Trondheim (Progress, 2004) where for one scheme 75% of opinion was negative prior to implementation, but two months into implementation negative opinion had dropped to below 50%, and after 2 years negative opinion was only about 30%. Thus if the Edinburgh scheme had been implemented it is reasonable to assume that negative opinion would have lessened with time.

The acceptability issue has been claimed by some to be analogous to the issues surrounding acceptability of parking charges, such charges are unpopular when first introduced, but tend to be accepted in time, particularly when such measures are so widely used that they become commonplace (Jones, 2003). The Stockholm trial

(Trivector, 2005) has been implemented as a trial from August 2005-July 2006, and a referendum will be held in September 2006 to decide on whether it will be made permanent. From the Norwegian experience it would seem that such a referendum would have more chance of success than that conducted in Edinburgh prior to scheme implementation, but the outcome clearly cannot be prejudged.

Issues surrounding acceptability include concerns regarding surveillance, payment procedure to be implemented, aesthetic issues, hypothecation and belief that the proposed scheme would actually solve traffic problems. At one site in Edinburgh up to two-thirds of the population did not believe that charging would have an effect on reducing traffic as they believed car use to be so embedded (Progress, 2004). Mayeres (2003) demonstrates that the type of hypothecation can have a significant impact on the acceptability of a proposed scheme, while Harsman (2003) comments on the significance of worries relating to privacy issues and surveillance, although the results from Copenhagen and Gothenburg (Progress, 2004) suggest that surveillance may not be as problematic as had previously been thought as people become accustomed to 'surveillance' from mobile phones or credit cards. Whilst issues surrounding acceptability continue to exist it would appear that they may be overcome with careful management of scheme details.

2.4 Equity considerations

There are two types of equity impacts which are commonly considered; Spatial (or horizontal) equity and social (or vertical) equity. It is perceived to be important that charging schemes are seen to be 'fair' in some way (Jones, 2003), and whilst economic marginal social cost price tolls might be viewed as fair from one perspective (in that they charge all road users for their externalities), the fact that such costs might be much harder to bear (i.e. a greater proportion of income), for lower income groups might make them 'socially' unfair, and that socially marginal journeys are not equal to economically marginal journeys. Thus low-income groups will be less willing to pay for road use and it is assumed that road pricing would primarily benefit the better-off (Schade, 2003). Furthermore the better off may often be able to be more flexible in their working arrangements and may be able to avoid a peak hour charge entirely (Frey, 2003).

The concept of link-based rather than cordon-based charging schemes might make charging more acceptable from a social equity perspective; the potential existence of untolled links and untolled routes imply that there will be a trade-off in routing based upon a person's own value of time, those of higher income groups may choose to take the quicker, tolled links in their route, whereas the lower income groups have the option of taking untolled links and possibly routes (depending on their journey), albeit with a greater time cost.

Issues of spatial equity arise from the differences in where people live and travel with respect to the charging points. This is a particularly problematic issue with respect to cordon or area schemes, as it is obviously possible for a person to live just outside a boundary and need to travel to just within it. This is particularly inequitable if the cordon charge is set up in such a way that a similar person who lived just within the boundary, with a similar trip distance did not have to pay (this would have been true of the proposed Edinburgh scheme discussed in section 2.2.3).

Again link-based schemes would considerably reduce the impact of spatial equity as it is primarily a cordon scheme problem. A system whereby everyone who used a particular link (at a particular time) being charged the same amount would certainly appear fairer.

2.5 Summary

Whilst various payment methods are being proposed and considered the cordon type scheme appears still to be the most favoured in the UK. The ease of potential implementation would appear to be the main reason for this, and such a transparent system would have the benefit of being easily understood by the users. The possibility of charging for particular links still seems to be an area for current and future research although the technology currently used in Singapore, and that used in the demonstration projects in Gothenburg and Copenhagen is such that link based schemes should be technologically possible. The development over time of the schemes in Singapore, Trondheim and London however give a positive indication that should simple schemes be primarily introduced in the UK, they could be developed in time as public acceptance of them increases (or perhaps as public dissatisfaction lessens).

Issues of vertical and spatial equity are problematic in relation to the acceptability of cordon-based schemes, but many of these issues would be reduced if a link-based charging scheme were to be implemented. In particular if a minimal-revenue type link-based scheme where alternative toll-free routes were often available was implemented, the social equity issues would be much reduced, as those people with a low value of time but small disposable income would be able to utilise toll-free routes at time rather than monetary cost.

CHAPTER 3

LITERATURE REVIEW

3.1 Introduction

Transportation planning models are traditionally stated as 4 stage models (Ortuzar and Willumsen, 1994). Such models are made up of firstly, trip generation (the number of trips that originate at each origin and terminate at each destination), secondly, trip distribution (the distribution of trips from each origin to specific destinations), thirdly, modal split (the distribution of trips between particular modes of transport, cars, buses, vans etc), and finally traffic assignment (the distribution of the generated trips to routes through the network). Outputs from the first three stages are required for the final stage of traffic assignment, in particular an OD (origin-destination) matrix. The work in this project is concerned with the fourth stage of modelling, the assignment stage.

In any form of modelling it is necessary to make certain assumptions when formulating the model. This often results in a relatively simple initial model which may be refined and adapted to represent reality more closely. For example it is clear that in reality traffic assignment is dynamic, and demand is constantly changing with time. Vehicles will enter a network and respond and make decisions which are based on the current conditions at that time. However static models are generally more mathematically tractable and are used as the basis for modelling within this project. Static models use a fixed set of demands, for example the average demands over a certain time period, often the morning peak.

It is also generally assumed that drivers are influenced by travel costs, in that they would wish to minimise their costs, and such costs are often taken to be equivalent to travel time. In the simplest case of traffic assignment it is assumed that all drivers wish to minimise their travel cost (or time), and that they have a perfect knowledge of the network. They all therefore identify the quickest (or shortest) route through the network for their particular journey, and choose that route. This would result (if cost were independent of flow) in what is known as All or Nothing assignment, i.e. that all traffic chooses a particular route, and so no traffic chooses other routes. This is obviously not the case as it is clear from observation of traffic in networks that route spreading occurs, and so models are accordingly refined to take this into account (defining cost flow relations), as in the case of deterministic assignment as detailed in section 3.2. Deterministic assignment models assume that drivers have perfect knowledge of network costs, and as this is not generally the case stochastic models, where drivers act to minimise their perceived costs, are preferred. Stochastic assignment models are reviewed in section 3.3. The concept of the system optimal where drivers are routed to optimise system flows rather than to minimise personal costs is the subject of section 3.4, and elastic demand in network assignment modelling is reviewed in section 3.5. Section 3.6 then discusses the concepts of 1st best (optimal) and 2nd best (suboptimal) tolling to achieve desired network flow patterns under both deterministic and stochastic assignment models, and section 3.7 summarises the chapter.

3.2 Deterministic assignment Models

Deterministic assignment is an assignment method which assumes drivers to have perfect knowledge of the network through which they travel, so that if each link has an associated cost function, then the driver is assumed to be able to determine the cheapest route for their particular journey and to act to minimise their cost (Wardrop's first principle (Wardrop, 1952)). These cost functions are said to be separable, in that they can be defined for a particular link (as opposed to route) under the assumption that junction effects are negligible in comparison with link effects and can thus be ignored. This sort of assignment is applicable to congested urban networks, where the cost functions incorporate capacity restraint. A commonly used cost function is that taken from the US Bureau of Public Roads (USBPR, 1964):

$$C_a(x_a) = c_a(0)[\ 1 + k(x_a/Y_a)^p] \qquad 3.1$$

where x_a is the flow, $c_a(0)$ is the free-flow cost, Y_a is the capacity and k and p are parameters; such as illustrated in figure 3.1.

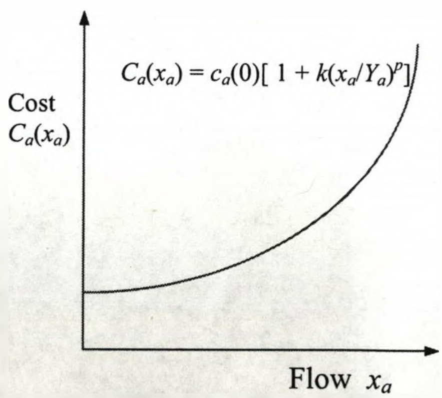


Figure 3.1: BPR cost-flow relation

Thus cost can be seen to increase as flow along a link increases, and this dependence of cost upon flow means that route spreading will occur, rather than an all-or-nothing solution, as an additional driver will seek the cheapest route taking into account the increased cost on links due to increased flows, and so the cheapest route will not always be the same as the cheapest route in free flow conditions.

The assumptions given above result in a traffic assignment known as User Equilibrium (UE) or Wardrop Equilibrium, (Sheffi 1985):

> 'For each OD pair, at user equilibrium, the travel time on all used paths is equal, and (also) less than or equal to the travel time that would be experienced by a single vehicle on any unused path.'

User Equilibrium can also be described as a selfish equilibrium, as every driver acts to minimise their personal cost.

The concept of UE can be easily illustrated by use of a simple 2 link network, such as that given below (for simplicity the value of p as in the BPR function is taken as 1). [The following notation will be used; (C_i , X_i : path-costs, path-flows), (c_i , x_i : link-costs, link-flows).]

There is a demand of 1000 vehicles between origin **1** and destination **2**.

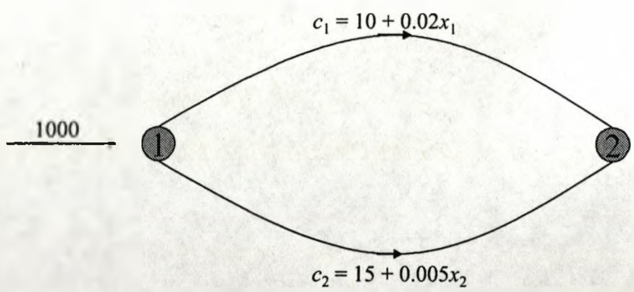


Figure 3.2: 2-link network

The principles of UE state that at equilibrium the travel time (or cost) should be equal for each OD pair. In this 2-link example each link is also a path, so $C_i = c_i$ and $X_i = x_i$ $\forall i$. Here therefore:

$$C_1 = C_2 = c_1 = c_2 \qquad 3.2$$

The demand of 1000 must be split between the two paths, so

$$x_1 + x_2 = 1000 \qquad 3.3$$

Equations (3.2) and (3.3) may then be readily solved to give the solution;

$$x_1 = 400 \qquad x_2 = 600 \qquad c_1 = c_2 = 18$$

That is 400 vehicles will use the upper path, and 600 the lower path, resulting in a common cost of 18 for either path. The total network travel cost (TNTC) for UE would therefore be 18000.

Clearly in the case of more complex networks the solution process to find the flow vector is non-trivial, established iterative algorithms being used (Sheffi 1985). The premise being that for any given set of cost flow functions, it is possible to carry out an all-or-nothing (AON) assignment, i.e. that where all of the traffic between any given OD pair being assigned to the cheapest route. In assigning the traffic in such a manner however, the traffic on each route will affect the cost on each route, so that the cheapest route (under congested conditions) will probably be different, thus under Wardrop's principle route spreading will occur. If initially the procedure is started with a vector of flows $\boldsymbol{x}^n$ and its associated costs $\boldsymbol{c}^n$, then if an AON loading is carried out based on these costs the flow pattern obtained $\boldsymbol{y}^n$ is referred to as the auxiliary of $\boldsymbol{x}^n$. $\boldsymbol{x}^{n+1}$ is then set to equal $\boldsymbol{y}^n$ and the procedure is repeated as in (3.4).

$$\boldsymbol{c}^0 \rightarrow \boldsymbol{x}^1 \rightarrow \boldsymbol{c}^1 \rightarrow \boldsymbol{y}^1 = \boldsymbol{x}^2 \rightarrow \boldsymbol{c}^2 \rightarrow \ldots.. \qquad 3.4$$

This is referred to as a 'hard speed change' and is unlikely to converge, but rather oscillates. To overcome this, a 'soft speed change' is used to move from the current flow pattern $\boldsymbol{x}^n$ to a weighted combination of $\boldsymbol{x}^n$ and $\boldsymbol{y}^n$ so that the new flow pattern $\boldsymbol{x}^{n+1}$ is given by (3.5).

$$\boldsymbol{x}^{n+1} = \boldsymbol{x}^n + \lambda_n (\boldsymbol{y}^n - \boldsymbol{x}^n) \qquad 3.5$$

where λ_n is the 'step length' i.e. the distance moved from x^n towards y^n. The method of successive averages (MSA), where $\lambda_n = 1/n$ is equivalent to an even split among the previously calculated routes, has been proved to converge to the UE solution (albeit slowly) (ibid). More efficient algorithms such as the Frank-Wolfe algorithm (ibid), where an optimum step length is utilised, rely on the Beckman formulation of an objective function for UE (3.6) (Beckman et al, 1956) and its formulation as an equivalent minimisation problem.

$$Z_{UE} = \int_0^{x_a} c_a(u)du \qquad 3.6$$

subject to

$$\sum_k f_k^{rs} = q_{rs} \quad \forall r,s$$

$$f_k^{rs} \geq 0 \quad \forall r,s \qquad 3.7$$

$$x_a = \sum_{rs}\sum_k f_k^{rs}\delta_{a,k}^{rs} \quad \forall k,r,s$$

Where f_k^{rs} is the flow on path k from origin r to destination s and $\delta_{a.k}^{rs}$ is an indicator variable which is set to 1 if that path passes through link a, and is otherwise zero. Both Frank-Wolfe and MSA may be used in a FORTRAN program WARDROP (© Maher), which has been utilised in later chapters to calculate deterministic assignment flow patterns.

3.3 Stochastic assignment Models

Route spreading, which is an observed phenomenon in traffic assignment, can be modelled by applying cost-flow relations to simulate congestion. In the deterministic case as discussed in section 3.2 it is assumed that drivers have perfect knowledge of these relations and act rationally to minimise their total travel costs. Stochastic assignment methods however assume that instead of drivers having a 'perfect' knowledge of the varying OD costs of a network; they have a variable perception of these costs.

Thus route choice can be determined by applying discrete choice models, based on concepts of utility maximisation and random utility (Sheffi 1985). Most models assume either a normal distribution for link costs (probit) or distribute traffic across a route set by use of the logistic function (logit); these methods are described in sections 3.3.1-3.3.3.

Traditionally deterministic assignment has been used to model congested urban networks, if the same methods are applied though to uncongested inter-urban networks they tend to result in an All-or-Nothing type solution which is unrealistic in practice. Stochastic methods may be used to successfully model inter-urban networks, but it is desirable to have a single method which will be capable of modelling both extremes (and the middle ground). Thus Stochastic User Equilibrium (SUE) methods have been developed , and are discussed in section 3.3.4.

Stochastic methods are based on the assumption that a driver minimises their perceived cost, or chooses the alternative that gives the highest utility.

Utility functions U_k may be expressed as the sum of a deterministic component V_k and a random error component ξ_k, where k is a member of the set of alternatives.

i.e.

$$U_k = V_k + \xi_k \qquad \forall k \tag{3.8}$$

The probability that an alternative is chosen is the same as the probability that that alternative has highest utility in the choice set. Whilst not being entirely exhaustive (Sheffi, 1985), the most commonly used stochastic method models assume either a Normal distribution (probit models), or the Gumbel distribution (logit models), for the drivers' perception error ξ_k.

3.3.1 Logit

The logit model is based on the use of the logistic function, which is a choice function used to choose between two or many alternatives.

It may be written:

$$p_i = \frac{\exp(-\theta.C_i)}{\sum_j \exp(-\theta.C_j)} \tag{3.9}$$

where p_i is the probability of choosing alternative i, C_i is the cost associated with route i and θ is a dispersion parameter; the lower the value of θ, the higher the level of uncertainty, conversely a high value of θ would correspond to drivers having an accurate view of actual route costs, i.e. the deterministic case.

The use of such a choice model to produce a stochastic assignment requires route costs to be known. These may be determined by path enumeration methods, but as such methods can be computationally prohibitive, methods which do not require path enumeration such as Dial's STOCH algorithm may be used (Dial 1971).

Dial's method consists of a scanning algorithm; it does not assign flows/choice probabilities to all feasible OD paths, but first applies an efficiency criteria, which identifies 'reasonable' paths, which essentially dis-allows looping paths. The algorithm essentially first assigns link 'Likelihoods' which are proportional to the logit model probabilities, in a forward pass it then assigns link 'weights' and in a backward pass link flows. This algorithm may be refined to make it more efficient using a new definition of 'reasonable path' (Sheffi 1985), the result being a single pass algorithm which requires similar computational power to an All or Nothing assignment.

The logit formulation has the advantage of mathematical tractability, and will be used in this thesis for illustrative purposes. Logit based loadings however have a significant disadvantage in that they do not account for overlapping paths in a satisfactory manner. For example three completely distinct paths would have flow assigned in the same way as a single path together with two paths including a significant overlap. If each path had around equal cost, then each path would be assigned around one third of the traffic irrespective of any overlap. In addition the logit method assigns traffic based on an absolute difference in cost (time), for example a five minute difference in journey time will produce the same route choice proportions whether the difference relates to route times of 5 and 10 minutes or route times of 200 and 205 minutes. In the first case one route takes twice as long as another, whilst in the second, the five minute difference may well not be perceived as 'any difference at all'. It would seem reasonable to require a model to account for the difference in journey time in relation to the total journey time when assigning traffic.

3.3.2 Probit

The probit model assumes that the random error term is normally distributed, and that the joint density function of these errors is Multivariate Normal (MVN).

Thus the probability distribution of cost for each link is Normal, with mean μ usually being the value of the free-flow cost, and variance σ^2 assumed to be proportional to the mean.

$$\beta = \text{Variance/Mean} = \sigma^2/\mu \quad \Rightarrow \quad \sigma^2 = \mu\beta \qquad 3.10$$

$$c_a \sim N\left(c_a(x_a), \beta c_a(x_a)\right) \qquad \forall a \qquad 3.11$$

The probit model solves the problem of overlapping paths by the use of correlations between the path cost perception errors, but does none the less have associated drawbacks.

In the case of only two alternatives, it is reasonably straightforward to calculate the probabilities for travel on each link, but this is not the case if there are more alternatives.

There are various methods of solution for the many alternative case: one is numerical integration of a multiple integral. This method had been considered to be computationally prohibitive for situations with more than about four or five alternative routes, however recent work (Rosa and Maher, 2002a) shows that feasible numerical integration approaches now exist which can be used for networks with up to around twenty alternative routes. A second alternative utilises 'Clark's Method' (Clark, 1961), where a successive approximation method is used, where the maximum of two normally distributed random variables is approximated by another Normal variable, and the

Stochastic Assignment Method SAM (Maher and Hughes, 1997a) is based on this. This method is applied iteratively, but does create a heuristic solution.

3.3.3 Monte Carlo

Alternatively a Monte Carlo simulation may be used, whereby a random value representing the perceived travel time of a link, is sampled from the density function for that link, and an All-or-Nothing assignment is carried out based on the set of sampled perceived travel times across all network links. The process of sampling and assignment is repeated (multiple times) and averaged to give the final flow pattern. The major drawbacks of Monte Carlo probit methods are due to the issue of repeatability and to long computation times. For example if this method were used to compare different highway scheme proposals, it may be unclear how much of a projected improvement may be due to random effects. In practice the method may have to be carried out a very large number of times to obtain a sufficient level of accuracy and stability, and would therefore be costly from a computational perspective.

3.3.4 Stochastic User Equilibrium SUE

Stochastic user equilibrium (SUE) assignment is based on the premise that each driver will act to minimise their *perceived* route cost, which follows a distribution such as those given in the logit or probit models. Thus deterministic user equilibrium can be viewed as being a special case of the SUE problem, where the 'distribution' of costs has zero variability. In SUE models, both congestion and driver perception is modelled, i.e. aspects of UE deterministic assignment and stochastic assignment methods are combined. In SUE modelling, the perceived costs are not modelled solely as random variables, but have also a dependency on flow, in that the *mean travel time* is taken to be a function of flow.

Probit SUE has generally been seen as a difficult problem, with Monte Carlo methods, or path enumeration being relied upon. Sheffi (Sheffi and Powell, 1982) formulates an unconstrained minimisation problem, for which the solution is the SUE flow pattern.

$$Z_{SUE}(\mathbf{x}) = \sum_a x_a c_a(x_a) - \sum_a \int_0^{x_a} c_a(x)dx - \sum_{rs} q_{rs} S_{rs}(\mathbf{x}) \qquad 3.12$$

(Where S_{rs} is the satisfaction function, i.e. the mean perceived travel cost between OD pair *rs*.) However the solution of this does not prove to be straightforward. MSA (Method of Successive averages) has also been applied by Sheffi but suffers from slow convergence in congested networks.

Logit SUE has also utilised MSA, but again with a slow convergence rate. Difficulties also exist in the efficiency criterion of the logit model, as different links may be 'inefficient' on different iterations. Recent work (Maher 1998), increases the speed of convergence by use of an optimal step length as opposed to MSA, and fixes the 'efficient' links early in the process.

A probit stochastic model has however been developed, that is numerical rather than Monte Carlo (Maher and Hughes, 1995); it is referred to as SAM (stochastic assignment method). SAM is formulated first as a pure stochastic assignment method, which does not require path enumeration. The algorithm is based on a process of scanning and merging, which will not be explained in detail here. The forward pass performs calculations based on Clark's approximation (Clark, 1961), giving the minimum of two normal random variables as normal. A backwards pass, in the reverse order to the merging is used to load the traffic. Rosa and Maher (2002a) propose an alternative to Clark within a path based algorithm, approximating multivariate normal integrals directly. Some difficulties arise in 'deadlock' and 'looping' whereby if a network contains loops, progress halts in the merging stage as no nodes are 'complete'. SAM deals with this by assigning no flow to the loops in 'two link' loops, and by estimating traffic proportions on higher order link loops. SAM is then combined with capacity restraint using BPR (or alternative) type cost flow relations, a vector of link flows will define a vector of link costs (as per UE), a stochastic loading can then be carried out using the link cost vector as the mean link costs. The flows obtained by such a loading may then be used for the next iteration of the algorithm. SAM may be used in a FORTRAN program NEWSAN (© Maher), which has been utilised in later chapters to calculate stochastic assignment flow patterns; NEWSAN allows for logit, SAM-probit, or monte-carlo stochastic assignment.

3.4 System Optimisation:

3.4.1 Deterministic Case

The aggregate effect of individual drivers acting to minimise their own travel costs does not minimise the total travel cost (or time) in the network (TNTC), which would theoretically be best for the system as a whole (Wardrop's second principle). A flow pattern which does give a minimal TNTC is known as a System Optimal (SO) flow pattern in the case of deterministic assignment. This may be formalised in the non-linear program below (Sheffi, 1985).

$$Z_{SO} = \sum_a x_a c_a(x_a) \qquad 3.13$$

subject to

$$\sum_k f_k^{rs} = q_{rs} \quad \forall r,s$$

$$f_k^{rs} \geq 0 \quad \forall k,r,s \qquad 3.14$$

The SO solution may be obtained by carrying out a UE type assignment, but by using marginal link-costs rather than unit link-costs in the assignment algorithm. Marginal cost functions may be obtained from the BPR function as below:

$$m_a = \frac{d}{dx_a}(x_a c_a(x_a)) = c_a + x_a \frac{dc_a}{dx_a} \qquad 3.15$$

where $m_a(x_a)$ is the marginal cost on a link, x_a is the link flow and $c_a(x_a)$ is the link cost as per the BPR function. It can be shown that the marginal cost function may be written in BPR format as;

$$m_a(x_a) = c_a(0)[1 + k(1+p)\left(\frac{x_a}{X_a}\right)^p \qquad 3.16$$

This can be illustrated using the same 2 link network as before, the marginal cost functions being as given in figure 3.3 below.

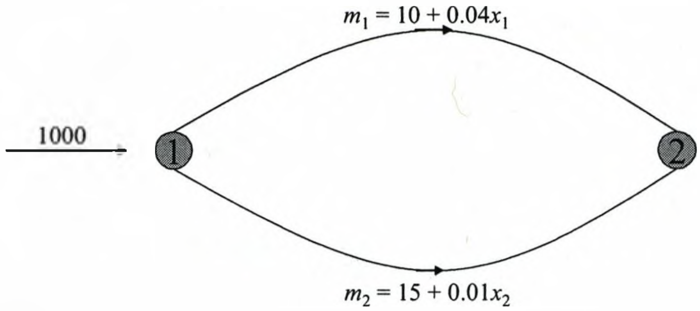


Figure 3.3: 2-link network with marginal cost flow functions

For the SO case the system is solved so that the marginal costs between OD pairs are equal, hence in this case, $m_1 = m_2$

This produces a flow pattern of: $x_1 = 300$ $x_2 = 700$

where the marginal path costs are equal: $m_1 = m_2 = 22$.

The original unit cost functions are then used to calculate the actual (unequal) link costs, giving $c_1 = 16$ $c_2 = 18.5$,

and consequently the minimum value of the total network cost; TNTC = 17750, which is clearly less than the UE value of TNTC = 18000. [In the case of this 2-link network, this result may also be derived by direct algebraic solution of the mathematical program (3.12) as x_2 may be written explicitly as $(1000 - x_1)$, in general however it is more efficient to use the marginal cost functions in a UE assignment.]

3.4.2 Stochastic Case

Whilst the SO in the deterministic case is well known, system optimisation in the stochastic case is rather less clear. Yang (1999), Penchina (2002, 2004), Yildirim and Hearn (2005) have assumed a utility maximising framework for Optimisation in the Stochastic case, i.e. where economic benefit is maximised under MSCP-tolling. The flow pattern so achieved is termed here the Stochastic Social Optimum (SSO) (Maher et al 2005). In system terms this solution corresponds to the flow pattern where the perceived total network cost is minimised (PTNTC), but this is generally not the same flow solution as where the 'actual' total network travel cost (TNTC) is minimised. Thus the 'desired flow pattern' in the stochastic case must be considered, and this issue is discussed in chapters 4 and 5, which examine potential 'desired flow patterns'.

3.5 Elastic Demand:

The assignment models previously described have assumed a fixed demand matrix, it seems reasonable however to assume that if costs increase (by perhaps monetary tolling or road narrowing etc.) then some trips will be suppressed and demand will reduce and conversely if costs decrease (by perhaps network improvement) suppressed trips will be released and demand will increase. Thus demand should be viewed as elastic rather than fixed. The SACTRA report on "Trunk Roads and the Generation of Traffic" (DfT, 1994) looks at the effect of induced traffic related to network improvement and concludes that demand elasticities should be incorporated into modelling methodology. [The concept of 'predict and provide' is essentially that "solving congestion by road building is like solving obesity by buying larger trousers" (anon).] A report by MVA(1997) examines the converse effect of traffic suppression in the case of capacity reduction. It is reasonable therefore to wish to extend any modelling methodology to incorporate elastic demand and this can be done in a simple way by allowing the trips between any OD pair to vary with the inverse of travel costs. If D(.) is a decreasing function, this may be written;

$$q_{rs} = D_{rs}(u_{rs}) \quad \forall r,s \qquad 3.17$$

Where u_{rs} is the minimum travel cost between r and s.

3.5.1 Deterministic assignment with elastic demand

The inclusion of elastic demand in deterministic assignment models is well understood and has been incorporated into commercial assignment software (e.g. SATEASY, SATURN (1993). The UE problem may be easily extended to UEED (user equilibrium with elastic demand), by the inclusion of an inverse demand term in the standard objective function (Beckman et al, 1956).

$$Z_{UEED} = \sum_a \int_0^{x_a} c_a(\omega)d\omega - \sum_{rs} \int_0^{q_{rs}} D_{rs}^{-1}(\omega)d\omega \qquad 3.18$$

This may be reduced to solution by a standard UE algorithm by the addition of a dummy pseudo-link between each OD pair, to which suppressed traffic is assigned. The unit-cost function for such links are derived from the inverse demand function (Sheffi, 1985).

3.5.2 Stochastic assignment with elastic demand

As will be further discussed in chapter 7, the inclusion of elastic demand in stochastic assignment models is not as straightforward as in deterministic. Leurent (1994) presents a solution for logit-based SUE by use of a dual algorithm, and a more general solution by simultaneously minimising the two objective functions, (3.19 and 3.20), for SUE and ED was derived by Maher and Hughes (1998).

$$Z_{SUE}(\mathbf{x}) = \sum_a x_a c_a(x_a) - \sum_a \int_0^{x_a} c_a(x)dx - \sum_{rs} q_{rs} S_{rs}(\mathbf{x}) \qquad 3.19$$

$$Z_{ED}(\mathbf{x}, q) = \sum_{rs} q_{rs} S_{rs}(\mathbf{x}) - \sum_{rs} \int_{q_0}^{q_{rs}} D_{rs}^{-1}(q)dq \qquad 3.20$$

Where rs comprise the network OD pairs, q_{rs} is the flow between OD pair rs, S_{rs} is the satisfaction for users on rs and $D_{rs}^{-1}(.)$ is the inverse demand function. Maher and Hughes (1998) further proposed the Balanced Demand Algorithm (BDA) where at each iteration the demands and network costs are maintained in balance. Maher et al (1999) presented a single objective function for SUEED;

$$Z_{SUEED} = \sum_a x_a c_a(x_a) - \sum_a \int_0^{x_a} c_a(x)dx + \sum_{rs} D_{rs}^{-1}(q_{rs}) D_{rs}(S_{rs}(\mathbf{c}(\mathbf{x})))$$

$$- \sum_{rs} S_{rs}(\mathbf{c}(\mathbf{x})) D_{rs}(S_{rs}(\mathbf{c}(\mathbf{x}))) + \sum_{rs} \int_0^{q_{rs}} D_{rs}^{-1}(q)dq - \sum_{rs} q_{rs} D_{rs}^{-1}(q_{rs}) \qquad 3.21$$

which can be minimised unconstrained as network conditions (flow non-negativity and path-flow sum consistence) will be given by the elastic demand functions. An equivalent and more compact program was presented by Rosa and Maher (2002b) and is derived from a modification of the Sheffi and Powell (1982) SUE objective function.

$$Z_{SUEED}(\mathbf{x}) = \sum_a x_a c_a(x_a) - \sum_a \int_0^{x_a} c_a(x)dx - \sum_{rs} \int_{S_{0rs}}^{S_{rs}(\mathbf{x})} D_{rs}\ (S_{rs}(\mathbf{c}(\mathbf{x})))dS_{rs} \qquad 3.22$$

An alternative approach based on derivatives was presented by Yang (1997).

3.6 Tolling

There has been much debate relating to the theory and practice of road user charging or road tolling, as discussed in Chapter 2. The modelling of tolling has likewise been much considered and methodologies are divided into 1st best and 2nd best optimisation. 1st best models tend to refer to link or path based tolling schemes, which in theory should potentially result in optimisation of the system (SO in the deterministic case), or economic benefit maximisation (marginal social cost pricing: MSCP). In this section, 1st best methods are discussed for the deterministic case in (3.6.1), marginal social cost pricing is illustrated in (3.6.2), alternative feasible toll sets are discussed in (3.6.3) and the minimal revenue toll problem is defined in (3.6.4). The concept of the minimal revenue toll problem is of particular interest and tolling to achieve this in the stochastic case is the focus of this thesis. In the deterministic case existing solution methodologies are reviewed (3.6.4.1), and the methods of Bergendorff and Dial I and II are presented (3.6.4.2-3.6.4.4). Tolling under stochastic assignment is introduced (3.6.5) and finally 2nd best methods, such as cordon charging or sub-optimal link based schemes are discussed (3.6.6).

3.6.1 1st Best Optimisation: Deterministic case

The SO flow pattern is the flow pattern where the total network travel cost is minimised in the case of deterministic assignment, but it is not a stable flow pattern. If the link costs were 16 and 18.5 as in the 2-link example in section 3.4.1, the flow pattern would not be maintained as drivers would change route to seek to minimise their personal cost, and so would revert to the UE flow pattern. The SO pattern is therefore the flow pattern that would be desirable, rather than the pattern which would naturally occur (Sheffi, 1985).

Tolling is a method to make the SO flow pattern stable, by equalising the 'total' cost for travel along a path by placing additional charges (tolls) on links, so that a stable equilibrium is obtained, and the SO flow pattern is achieved under a UE assignment.

Thus whilst it is theoretically possible to determine the minimal TNTC by SO assignment, some additional measure is required to obtain this flow pattern in practise. So if a System Optimal assignment is required it is necessary to modify the cost flow functions so that the 'selfish' driver is forced to act 'altruistically'. This can be achieved by imposing additional costs (i.e. tolls) on network links.

3.6.2 Marginal Social Cost Price Tolls

The classic economics solution to obtaining a SO flow pattern is to impose Marginal Social Cost Price tolls (MSCP), whereby a toll equal to the difference between the marginal social cost (that is imposed upon the network by the driver) and marginal private cost (that the driver experiences) is levied on each link (e.g. Pigou, 1920; Smith, 1979; Sheffi, 1985,). Thus making the total cost for the link equal to $m_a(x_a)$ as;

$$t_a(x_a) = m_a(x_a) - c_a(x_a) \qquad 3.23$$

where $t_a(x_a)$ is the MSCP toll for the link x_a.

The example two link network would thus have the cost functions as given in figure 3.4

where $t_1 = 22 - 16 = 6$

and $t_2 = 22 - 18.5 = 3.5$

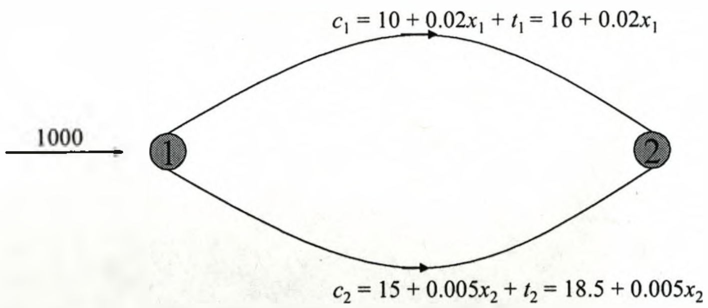


Figure 3.4: MSCP-tolls

Solving for UE equating the new cost functions and maintaining the demand such that

$$x_1 + x_2 = 1000,$$

gives the solution flow pattern;

$$x_1 = 300 \qquad x_2 = 700 \qquad (c_1 = c_2 = 22)$$

which is the same as that obtained under SO as required.

The total network cost is 22000, but this figure includes the toll revenue (4250), so that TNTC = 17750, the required minimum value (excluding the toll revenue).

MSCP tolls however, whilst theoretically forcing the user to pay for their 'total' cost, tend to be in practise rather high, and may be problematic to implement from a political perspective (Dial, 1999; Newbery and Santos, 2003, Wong et al 2003). There is however some evidence that efficient marginal social cost-based pricing may be more acceptable to the general public than policy makers might suppose, if such schemes are presented as part of an overall package of measures, (Sikow-Magny, 2003)

3.6.3 Other Feasible Toll Sets

Tolls which produce the SO flow pattern are however non-unique, for example once a valid toll vector (such as that for MSCP tolls) is obtained to give an SO flow pattern for a network, adding or subtracting a constant vector will still result in the toll vector giving SO flows. This can be illustrated as in the case of the 2 link example.

The cost functions including the MSCP tolls are;

$$c_1 = 16 + 0.02x_1 \qquad c_2 = 18.5 + 0.005x_2$$

The addition of a constant vector $a_i = a$ will give

$$c_1 = 16 + 0.02x_1 + a \qquad C_2 = 18.5 + 0.005x_2 + a$$

which will clearly cancel out in the solution for UE, giving the same flow pattern as before.

Thus the possibility exists for further optimisation, i.e. additional criteria could be imposed so that a unique toll set could be obtained to satisfy certain conditions.

One possibility is to maximise revenue, which might seem to be politically interesting. However under the modelling assumptions for simple deterministic assignment, demand is taken as constant (although tolling in the case of elastic demand is of interest later (Hearn and Yildirim 1999)). Thus in the case of constant demand, there would not be a toll set which maximised revenue as there is no upper bound. In reality of course, very high tolls would discourage more drivers from using the network, very low tolls would not, and so a toll set to maximise revenue would, intuitively exist at some intermediate value.

If it is not possible under the original modelling assumptions to maximise revenue, then minimal revenue tolls become of interest. ‘Minimal Revenue’ tolls assume that all tolls are strictly non-negative; if this condition were relaxed it would be possible theoretically to implement ‘zero revenue’ tolls or Robin Hood tolls (Hearn and Ramana, 1998). Thus a driver could receive credits for using an ‘unpopular’ longer route, or would incur debits for using a ‘popular’ fast route. In a way this concept has similarities to those in the governments recent document ‘Paying for Road Use’ (CfIT, 2002), where tolling charges are proposed within a ‘fiscally neutral’ scheme. This could be considered as maximising political acceptability.

3.6.4 Minimal Revenue Tolls

Minimal revenue tolls minimise the network revenue that would be raised by imposing the tolls, assuming that all tolls are strictly non-negative. Thus the toll problem may be formulated as:

$$Minimise\sum_{a}\left[(x_a(c_a).(t_a)\right] \qquad 3.24$$

where x_a are SO link flows and t_a are link tolls, subject to $t_a \geq 0 \;\; \forall a$, TNTC maintaining the SO value and all used paths having common costs.

For example in the two link case where the SO flow pattern is desired it can be seen that;

$$\text{Toll Revenue} = 300t_1 + 700t_2$$

and if the link toll is set to be equal to the MSCP toll plus a constant a then

$$t_1 = 6 + a \qquad t_2 = 3.5 + a$$

and thus

$$\text{Toll Revenue} = 4250 + 1000a$$

If it is required that $t_i \geq 0 \;\; \forall_i$ as for minimal revenue tolls, it can be seen that the smallest possible value for a is 3.5.

Thus $\qquad t_1 = 2.5 \qquad t_2 = 0$

and $\qquad$ Toll Revenue = 750

Minimal revenue tolls show a reduction from MSCP tolls of around one order of magnitude on test toy networks studied, and so in themselves may be more politically acceptable. They also have the useful property of assigning zero tolls to many links (and it may be possible to maintain zero tolled routes through the network which is beneficial from an equity viewpoint). Zero tolled links would further be potentially logistically beneficial depending on what technology was used for toll collection, for example if on street beacons were to be used, fewer beacons would be required than for MSCP tolls which tend to toll every link.

However if non-negativity constraints are not imposed, as for 'Robin Hood' tolls, the total Toll Revenue may be made zero where, in the two link example

$a = -4.25$ and $t_1 = 1.75$ $t_2 = -0.75$.

Such a scheme would therefore be fiscally neutral.

Thus for the simple two link example, tolls can be summarised in figure 3.5 below:

	MSCP	**Minimal Revenue**	**Robin Hood**
t_1	6	2.5	1.75
t_2	3.5	0	-0.75
Toll revenue	4250	750	0

Table 3.1: Toll comparison

Existing methods for the solution of the minimal revenue toll problem in the case of deterministic assignment, and a discussion on the limitations of existing methods are now discussed.

3.6.4.1: Minimal Revenue tolls: Existing Solution Methodologies

Various work has particularly focused on the solution of the minimal revenue toll problem in the case of deterministic assignment with fixed demand.

Bergendorff et al (1997) are concerned with the tolling methodologies which make user equilibrium assignment give system optimal flows. Much of the paper is concerned with the technical definitions and characterisation of valid toll set vectors, but also outlines the linear programming formulation to solve for minimal revenue toll sets for some specific toy networks. Such linear programming methods however require a full path enumeration matrix, which is computationally burdensome. Figure 3.5 illustrates that even for a simple grid network, with single OD pair, and no looping paths, the number of paths increases exponentially (for a grid of side 10, there are approx 185,000 paths).

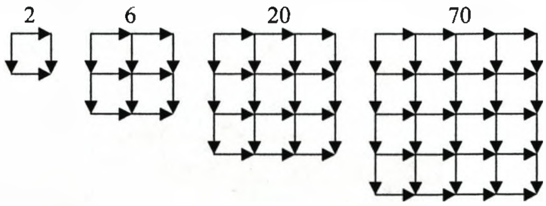


Figure 3.5: Number of paths from NW corner to SE corner of grids.

Dial (1999) presents his algorithm for the solution of minimal revenue tolls for single origin networks. The algorithm is particularly straightforward and easily replicable manually for small networks. That it is limited though to networks with demand from a single origin (although more destinations are possible), is a severe drawback, and resulted in the companion paper, which is Part II of the single origin paper.

Dial (2000) continues from the 1999 paper to solve for the general multiple origin case. Dial however departs from the straightforward algorithm from Part I and instead reformulates the single origin case as a linear program, to which a change of variable is applied. However this reformulation results in the beneficial feature of the algorithm in Part I to produce zero toll paths for each OD pair, to be lost. It would be preferable for a solution method to determine such paths if technically possible. This method also requires the network to be acyclic, which is unlikely on a real network, and may lead to the necessity of applying some sort of efficiency criteria.

3.6.4.2: Bergendorff et al: Linear programming method

The aim of Bergendorff's paper was not only to determine minimal revenue toll sets, but to more generally provide full mathematical characterisation of the set of all possible toll sets which result in System Optimal flow in the network. Thus the MSCP toll set and the Min Rev toll set would be possible subsets of the set of feasible toll sets. Once this characterisation is obtained it is mathematically possible to solve for toll sets relating to specific criteria of which the two sets above are only possibilities. The paper shows that under standard traffic assignment assumptions, linear programming methods may be used to determine a valid toll set.

Bergendorff demonstrates the formulation by use of toy networks, two of which have been specifically used; the Braess Network, and the Nine-Node network.

The nine-node network has been used in further papers (eg Dial 2000) and is therefore used as an example here. The network is illustrated in figure 3.6.

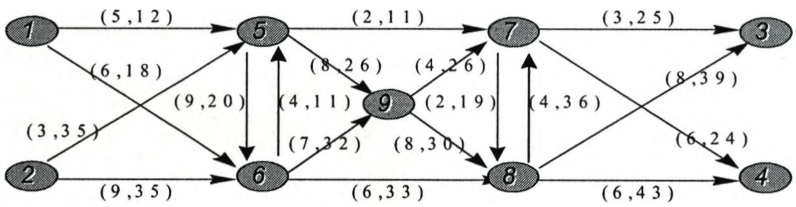


Figure 3.6: Bergendorff's 9-Node Network.

OD PAIR:	[1,3]	[1,4]	[2,3]	[2,4]
DEMAND:	10	20	30	40

The BPR cost functions for the network are given as:

$$c_a(x_a) = c_a(0)(1 + 0.15(x_a/Y_a)^4)$$

Where c_a is the unit link-cost, x_a is the link-flow, and $c_a(0)$ and Y_a are constants given on the links of figure 3.6, the link free flow costs and capacities.

Prior to commencing the linear programming methodology for the solution of a minimal revenue toll set, it is necessary to determine the system optimal flow set, which can be obtained easily from standard assignment packages. (eg. SATURN, WARDROP © Maher). For this example, the UE and SO flow patterns and their associated TNTCs are given in table 3.2, in each case 9000 iterations of Frank-Wolfe have been used. (The flows are given to the nearest whole number in the table, but full decimal values were used in any calculations).

Link	**UE flows**	**SO flows**
1-5	8	9
5-7	28	21
7-3	38	30
5-6	0	0
6-5	0	0
7-8	0	0
8-7	0	0
1-6	22	21
2-5	47	38
5-9	28	26
9-7	28	30
6-9	0	13
9-8	0	10
7-4	17	21
8-3	2	10
2-6	23	32
6-8	44	39
8-4	43	39
TNTC	2455	2254

Table 3.2: UE and SO flows

For the solution of the minimal revenue toll problem by linear programming a four stage program is used, as is detailed below.

Firstly a set of linear inequalities based on a path enumeration matrix are obtained, where the total path cost combined with the sum of the link tolls along that path must be equal to or greater than the common path cost (ρ) for that OD pair.

Thus for the 'first' OD pair [1,3]:

$$\rho_{13} \leq A^{13}.(\mathbf{c} + \mathbf{t}) \qquad 3.25$$

where A^{13} is the 24x18 path enumeration matrix for the OD pair [1,3],

c is the 18x1 column vector of link costs under SO, and

t is the 18x1 column vector of link tolls required.

ρ_{13} is a single value (the common OD travel cost), which is determined as part of the linear programming solution.

The complete inequality part of the linear program may be expressed in the form:

$$\mathrm{A}.\mathbf{x} \leq \mathbf{b} \qquad 3.26$$

where A is the negative of the 96x18 full path enumeration matrix (A^{f}) for all OD pairs, cocatenated with a 96x4 matrix so that the ρ_{ij} are included in the solution vector **x** as required. **x** is the solution vector, consisting of the 18x1 link toll vector, cocatenated with the 4x1 common path cost vector. **b** is the 96x1 path cost vector ($\mathrm{A}^{\mathrm{f}}.\mathbf{c}$).

Secondly, a single linear equality is required such that the TNTC is equal to the sum of the SO total network cost and the sum of the products of the link tolls and link SO flows, which maintains the SO flow pattern.

$$\mathrm{TNTC}_{\mathrm{SO}} + \sum_i (f_i, t_i) = \sum_r \sum_s D_{rs} \rho_{rs} \qquad 3.27$$

where f_i and t_i are the SO unit link flows and tolls and D_{rs} are the OD demands.

For the linear program solver this may be rewritten as

$$\mathrm{A}_{\mathrm{eq}}.\,\mathbf{x} = \mathbf{b}_{\mathrm{eq}} \qquad 3.28$$

where **x** is the solution vector as in (4.2), A_{eq} is a 1x22 row vector consisting of the SO link flows cocatenated with the negatives of the OD demands, and $\mathbf{b}_{\mathrm{eq}}$ is the negative of the TNTC at SO.

Thirdly there are non-negativity constraints requiring the link tolls to be strictly non-negative,

$$t_i \geq 0 \quad \forall_i \qquad 3.29$$

and as the common path costs ρ_{rs} should also be strictly non-negative, the non-negativity constraints may be included in the linear program by setting the lower bound for the solution vector **x** to be zeros.

Finally the objective function to be minimised is the sum of the products of the link tolls with the link SO flows;

$$\text{minimise } \Sigma_{i=1:12} (f_i.t_i) \qquad 3.30$$

This may be expressed in the linear program as

$$\text{minimise } \mathbf{fx} \qquad 3.31$$

where **x** again is the solution vector, and **f** is the 1x22 row vector, consisting of the 18 link flows, and four zeros, so that the common path costs are not included directly in the minimisation.

Thus the program returns the values for the 18 link tolls (figure 3.7), and the 4 common path costs for the OD pairs. (It should be noted that unused paths may have an equal or higher path cost value, in relation to the used paths which must have a common cost).

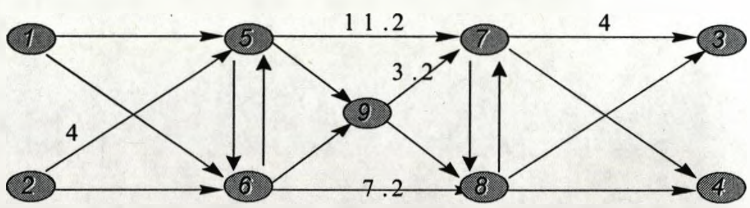


Figure 3.7: Min Rev Toll set from Bergendorff

The total revenue that would be raised by using MSCP tolls on this network is 1490 cost units (to 3sf) where as Bergendorff's toll set for Minimal revenue tolls gives a total revenue of 888 cost units (to 3sf), representing a 40% reduction.

The major drawback of this method is that it requires full path enumeration which for larger networks is impractical with regard to the computational requirement; the small test network given above requires a 96 x 22 path enumeration matrix, and the Headingley Network (often used in the literature as a 'real' network. e.g. Bai et al, 2004; Bekhor and Toledo, 2005) has 29 Origin/Destinations, and thus a potential 406 OD pairs, with such pairs typically having in excess of 100 non-looping feasible paths. It may be possible however to usefully adapt this method by applying partial (efficient) path enumeration techniques, as the number of 'sensible' paths for any OD pair may be around 10-15. (Ben Akiva et al, 1984; Cascetta et al, 1996, 1997; Bekhor and Toledo, 2005). Recent work (Bai et al, 2004), has extended Bergendorff et al's results to solve the Min-Rev problem for networks which are too large for standard linear programming techniques; decomposition techniques are used and commercial software CPLEX is utilised so that this problem may be solved for realistically sized networks.

3.6.4.3: Dial: Single origin algorithm for Minimal Revenue tolls

Dial's paper gives the formulation for his algorithm to solve for minimal revenue toll sets for single origin networks. It assumes that all trips have a common value of time and that user-optimising path choices are made.

Dial's method of solving for minimal revenue toll sets is a four stage process. Firstly from the set of all network links it extracts those with non zero flow under SO assignment and sorts nodes topologically (i.e. by numbering the nodes in order of fewest paths from the origin to that node as shown in figure 3.8 where the origin is node 0). In this example all links had non-zero flows. The 'bold' links represent 'Turnpike' links, which have limited residual capacity due to assumed through traffic, and the remaining links are referred to as Arterial links.

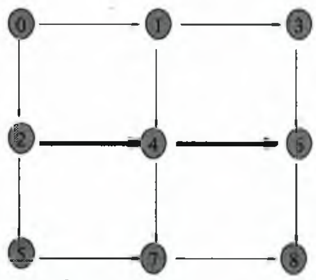


Figure 3.8: Dial's grid network

In the subgraph of flow bearing links, a maximum time (or cost) path tree is built, from the single origin. This can be seen in figure 3.10. A further subgraph of the network is then found called a bush, which contains in addition all links with positive 'potential difference' that is those where the cost of a route along that link, would be less than the cost of a route (with the same start and end nodes) which is contained in the max cost/time tree. This is illustrated in figure 3.11. Finally tolls are levied so that all used path costs are equal to the max used path cost.

Dial demonstrates this process by use of a nine node toy grid network as given in figure 3.9, with the link costs given (scaled by a value of time to replicate the results given in Dial's paper), and a single OD pair [0,8] with demand 1000. The cost flow relations used are in a standard BPR form and are given below:

$c_i(x_i) = 36/11(1 + 0.15(x_i/200)^4)$ for the 'Turnpike links' 2-4 and 4-6,

$c_i(x_i) = 9(1 + 0.15(x_i/400)^4)$ for the remaining 'Arterial links',

where x_i is the flow on link i, and the value of time used $0.10/min.

[So at the SO flow pattern the flow on link 0-1 is 500, so the link cost c_1 = 12.3 mins which when scaled with the VOT=$0.1/min gives a link cost of $1.23.]

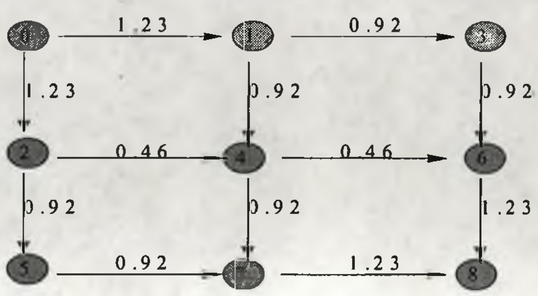


Figure 3.9: Dial's Grid Network: network data.

All links have non zero flow under SO and so the first part of Dial's algorithm is no different to the above. The max time tree is then constructed (fig 3.10) with the maximum time to each node from the origin (node 0) detailed on the diagram. This is not a unique max-time tree, as there are equally maximum paths for this example, but such paths will have zero tolls levied later (as can be seen in figure 3.11; links 5-7 and 7-8 could have been included in the max-time tree in place of links 4-7 and 6-8) so this is not an important issue.

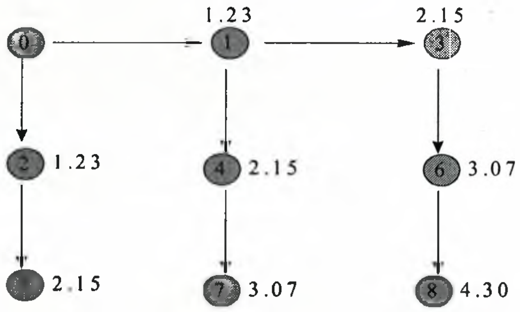


Figure 3.10: Max-Time Tree.

It is easy to see from the above diagram that the use of the two links 2-4 and 4-6 would result in cheaper (quicker) paths than those given by the max time tree. Tolls must therefore be levied upon these links to give equal cost along each possible path.

The Maximum time bush is then constructed as given in figure 3.11, showing the links added with dashed lines with the toll required above.

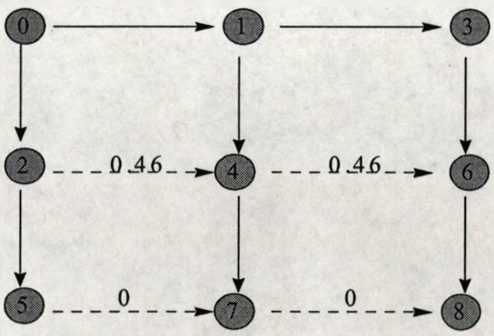


Figure 3.11: Max Time Bush.

Dial's algorithm is thus easily used to produce the minimal revenue toll set with an expected computer running time proportional to the number of nodes (Dial, 1999). The procedure used also ensures that there is at least one zero tolled route for each OD pair, which is desirable as such a tolling scheme would be more socially acceptable. Although each path has the same time/cost, whether a toll is paid or a longer route is taken, the possibility of being able to use a toll free route would be attractive to many users. Dial also claims stability for such a toll set, such that it would not be sensitive to changes in demand. This is obviously a useful characteristic as it would not be practically sensible to have to frequently carry out time consuming demand studies. The main weakness of Dial's algorithm however is that it applies only to single origin networks, and thus it has limited usefulness. Penchina (2004) does however comment that the algorithm is equally applicable to multiple origin networks if there is only a single destination, thus doubling the potential usefulness of this algorithm.

3.6.4.4: DIAL: Multiple origin algorithm for Minimal Revenue tolls

Dial's multiple origin algorithm is demonstrated on Bergendorff's 9-node network as shown in figure 3.12, with 2 origins, 2 destinations and 4 OD pairs.

This algorithm breaks the problem into single origin subproblems, and solves for link tolls using Dial's first algorithm. Dial's first algorithm begins by removing links with zero flow under SO assignment, resulting in the network given in figure 3.12, where the links are labelled with the link cost (to 1dp) under SO assignment.

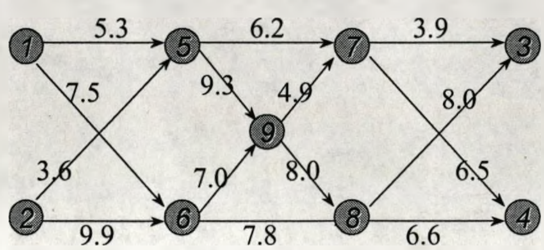


Figure 3.12: 9-node network: unit-link costs under SO

The problem is then broken into single origin subproblems, i.e. building max cost path trees for the network first including origin 1 and omitting origin 2, and second vice versa. Tolls are then imposed to give equal path costs for each OD pair, as given in figure 3.13.

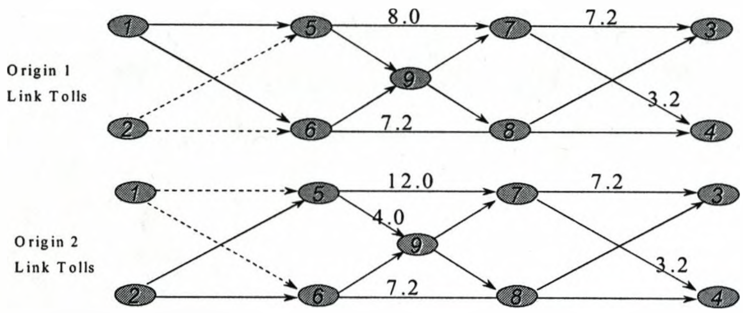
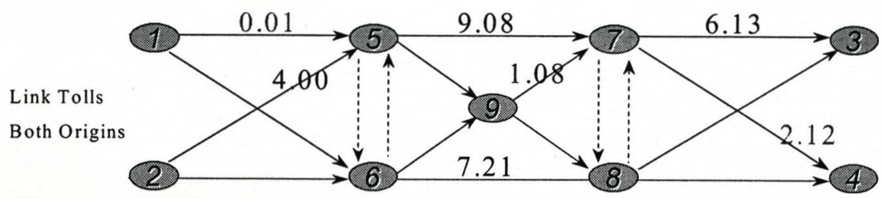


Figure 3.13: Link tolls for each origin under single origin Dial.

It can be seen that although some of the link tolls correspond, the full set of link tolls do not match and it is then required that the link tolls should match. This is achieved by



cycling through all origins and solving subproblems for Maximum Cost Flow (MCF). These subproblems are independent single commodity max cost flow problems, for which solution algorithms such as Dantzig's simplex algorithm, exist (Dial, 2000). The node potentials are changed until the link tolls are equalised across origins (within a specified error epsilon of the average toll). The algorithm initially uses the original node potentials as from the initial max time path solutions, and in solving for MCF the potentials may be increased by the utilisation of backward arcs, which is permissible whilst a positive arc sum constraint is honoured.

Figure 3.14: Solution toll set

The combined solution toll set is given in figure 3.14; it can be seen that it requires more links in total to be tolled than in Bergendorff's linear programming solution (additional links highlighted), although it can be shown that these two solution sets are equally optimal (solutions to the minimal revenue tolling problem being non-unique allows for the possibility of further optimisation with respect to additional constraints). The solution given above is that obtained by the particular solver used by Dial. Use of a different software package, using differing solvers could well produce a different equally optimal solution, such as that given by Bergendorff et al.

The main drawback of this method however is that it requires networks to be acyclic, which would produce difficulties in practice. The network given here was initially not acyclic, and became so only when those links with zero flows under the SO solution were removed. This would not necessarily be the case for all demand matrices. Further whilst zero toll paths exist for each OD pair in test network given, this is not a general feature of the algorithm and would be desirable in practice. The method described by Dial, given above has been used on a network where all OD flow is in a single direction. This is not a limitation of this algorithm however and it is equally applicable to more general networks such the Chen grid network (Chen and Alfa, 1991), although the results are not reproduced here.

3.6.5 Tolling Stochastic Case:

As discussed in 3.4.2, whilst the SO in the deterministic case is well known, system optimisation in the stochastic case is rather less clear. It is possible to attempt to produce optimal toll sets which achieve either the deterministic SO flow pattern where TNTC is minimised (Smith et al, 1994) or to toll to achieve economic benefit maximisation where PTNTC is minimised at the SSO solution. In the former case the question of whether meaningful link-tolls to achieve a deterministic SO flow pattern exist is in question, and this is discussed in Chapter 4. In the latter case MSCP tolls will produce the desired optima (Yang, 1999), but there is the further question of seeking Minimal Revenue tolls in such a case. Yildirim and Hearn (2005) extend their deterministic linear programming methods (Bergendorff et al, 1997; Hearn and Ramana, 1998) to calculate minimal-revenue tolls for SSO in the stochastic case, but it is of interest to examine link-based methods to achieve such tolls. Tolling to achieve the SSO is the subject of Chapter 5.

It is noted in Yang (1999) that MSCP-tolling under logit-based SUE does not necessarily result in a reduction in TNTC, and under certain circumstances TNTC can be higher at the SSO solution that at the SUE solution. It is questionable therefore whether such a tolling paradigm would be sensible; if the objective was to maximise utility then it could be, but if the objective was to reduce congestion by efficient re-routing, then it would appear not to be. Thus it is of interest to examine tolling options to achieve both SO and SSO.

3.6.6 Tolling with Elastic Demand

The case of tolling to achieve SO with elastic demand (SOED) in the deterministic case has been examined in an extension paper to Bergendorff et al (1997), (Hearn and Yildirim, 1999), where the linear programming method for the solution of minimal revenue tolls is extended. Yildirim and Hearn (2005) further extended this work to examine tolling to achieve the SSO with elastic demand (SSOED) in the stochastic case. Sumalee et al (2006) utilised sequential quadratic programming to solve for optimal link-tolls under probit-based SUEED. Penchina (2004) comments on the applicability of using Dial's single origin algorithm for minimal revenue tolls (Dial, 1999) in the case of elastic demand. It is noted that for either the deterministic case (SOED) or the stochastic case (SSOED), that optimal tolls are unique with respect to total revenue, and are essentially MSCP tolls. Thus the revenue required from the users to achieve such a flow pattern would be constant, and the minimal revenue toll problem would not 'exist' in the elastic demand case. It is possible however to relax the condition for economic benefit maximisation and seek 'optimal' flow patterns at a variety of demand values. This is the subject of Chapter 6 (seeking an SO flow pattern under SUEED) and of Chapter 7 (seeking an SSO flow pattern under SUEED).

3.6.7 Sub Optimal Tolling: 2^{nd} best models

All the tolling schemes discussed previously in this section are 'optimal' schemes. That is the toll sets determined will all force a UE assignment to produce the SO flow pattern or an SUE assignment to produce an SSO (or SO) flow pattern. .

Such schemes have not been implemented in practise however as link or path-based tolling schemes would require substantial investment in infrastructure, in order to charge vehicles for travel along particular links or paths. Thus operational tolling schemes to date have been cordon based. Such schemes do not seek to optimise traffic flow within a network, but rather to reduce congestion by discouraging travel and thus reducing demand. They also generally seek to raise revenue. In general various fixed scenarios for cordon placement and charging levels are compared, and a decision on which to implement made by comparing projected reduction in traffic, with projected raising of revenue, and a politically acceptable balance is determined, but it is possible to mathematically optimise for second-best tolls under certain network restrictions, namely where not all links in a congested network may be tolled (e.g. Verhoef 2002, Zhang and Yang, 2004; Hearn and Lawphongpanich, 2003). If the links tolled are required to form a cordon, the location of such a cordon will have a significant impact on the benefits that may be derived (May et al, 2002).

Most extant schemes have been the result of judgemental cordons based on the actual location of convenient 'ring roads, but genetic algorithms have been developed to design and generate optimal cordons where revenue is to be maximised under assumptions of elastic demand (Sumalee, 2004a, 2004b). Such optimal cordons have then been compared with judgmental cordons (Sumalee et al, 2005), where it is shown for the Edinburgh case study that their optimal cordons can potentially double the benefits of the best judgmentally produced single cordon. Using an optimal double cordon would potentially be around three times as effective as the proposed Edinburgh judgemental double cordon, and the optimal double cordon roughly doubles the benefits of the optimal single cordon. Thus it may be observed that second best optimality, as opposed to judgmental pricing may be highly beneficial in terms of raising revenue. Whilst this work implicitly considered the potential beneficial effects that might be achieved by re-routing within the cordon area, it did not allow for a precise amount of link by link fine tuning. The work in this thesis concentrates on optimising for efficient network flow, rather than optimising for the maximisation of potential revenue and is not based on cordon assumptions.

3.7 Summary

As may be seen from the literature, the minimal-revenue toll problem has been most considered in the deterministic case. Work in the stochastic case has primarily focussed on linear programming methods to achieve economic benefit maximisation, tolling to achieve the deterministic SO has been considered but not by the use of link-based methods.

This thesis aims to examine the possibility of tolling to achieve the deterministic SO under SUE by exact path-based methods and to approach the SO by using new heuristic link-based methods to produce low-revenue toll sets. In the case of the SSO this thesis examines the minimal revenue toll problem and again proposes a link-based heuristic to derive toll sets rather than utilising linear programming based methods.

When the fixed demand condition is relaxed and elastic demand is permitted, the literature notes that economic benefit maximisation occurs only at one particular demand value and so toll sets to produce either the SOED in the deterministic case, or SSOED in the stochastic case will have a fixed total revenue, so that the minimal-revenue toll problem would appear not to exist. This thesis considers relaxing the condition for the total demand to be that which maximises economic benefit and instead seeks to derive low-revenue toll sets which produce either SO or SSO flow patterns for particular (sub-optimal) values of demand.

CHAPTER 4

TOLLING TO ACHIEVE THE DETERMINISTIC SO UNDER SUE

4.1 Introduction

Chapter 3 has discussed the case for optimal tolling schemes under the premise of deterministic assignment. It is not however generally realistic to assume that all the drivers in a population will have perfect network knowledge and make decisions in a perfectly rational (utility maximising) manner. Stochastic assignment methods are accepted to be a better model of reality as they allow for drivers having different perceptions of travel costs. It is therefore of interest to examine tolling under the premise of stochastic assignment methods.

As discussed in chapter 3 there are a variety of possible tolling schemes which vary in the possible effects that they can impose on a network. First best road pricing is used to describe tolling schemes which produce the System Optimal flow pattern within a network whereas second best schemes normally refer to schemes which produce a range of possible desired effects, but do not achieve network optimisation (such as cordon schemes). However while the System Optimal is a well known concept in deterministic assignment, the nature of network optimisation within a stochastic environment has not been so well established. Section 4.2 discusses possibilities for 'desired' flow patterns within the stochastic environment that may be achieved by imposing tolling schemes and introduces the possibility of tolling to achieve either the deterministic SO flow pattern, or the SSO flow pattern (which is defined in chapter 5). This chapter then focuses on investigating tolling schemes to achieve the deterministic SO under principles of SUE. Section 4.3 investigates exact methods based on path-tolls, and discusses why it may not be possible to achieve the 'true' SO by link-tolls. Section 4.4 investigates the possibility of sub-optimal low revenue link based tolls that may be derived so that a flow pattern approaching the 'true' SO may be obtained, and presents a heuristic to derive such tolls. A discussion on the balance of revenue required against network cost reduction is given. Section 4.5 compares the path-based and link-based results, and section 4.6 illustrates the concept of the derived low revenue tolls using a simplified Edinburgh Network. Section 4.7 presents the chapter conclusions.

4.2 Question of desired flow patterns

In the case of deterministic assignment, it is assumed that drivers acting under their natural inclination, will result in the UE flow pattern. Tolls may then be imposed to 'force' a UE assignment to result in an alternative desired flow pattern. The desired flow pattern that has been examined is the SO flow pattern, when TNTC is minimised. Tolls that result in such a flow pattern being created are non-unique, the classical economics solution being MSCP tolls, but other constraints may be imposed such as toll revenue being minimised, resulting in min-rev tolls, or fiscal neutrality being obtained, resulting in Robin Hood tolls (where the non-negativity condition normally imposed upon toll sets is relaxed).

In examining the case of Stochastic user equilibrium there exist various possibilities for the desired flow pattern to be created. As will be demonstrated shortly, the classical economics solution of replacing cost flow functions with marginal cost flow functions, does not result in the total network cost being minimised. Thus tolls which are analogous to MSCP tolls in the deterministic case, do not give the system optimal solution (Yang, 1999). They may be considered to produce instead a 'Stochastic System Optimal', which is the subject of Chapter 5.

However if the true system optimal flow pattern is desired, it may be possible to derive tolls which are unrelated to marginal social cost pricing. It is not obvious if such tolls exist, or under which conditions they may exist, and if they are found to exist, if they are unique. If toll sets exist which are not unique, then as for the case in UE, it may be possible to impose additional constraints, and search for (for example) minimal revenue tolls.

These possible areas of interest are examined initially by use of simple toy networks, a two-link and a five-link network with a single OD pair, and a restricted version of Bergendorff's nine-node network (Bergendorff et al, 1997), with two origins and 2 destinations.

Whilst it is recognised that logit SUE assignment is not the most generally realistic method of SUE assignment, it is used here for illustrative purposes due to the mathematical tractability of the formulation. Additionally whilst it is known that MSA (method of successive averages) is not the fastest convergence method, it is used in the following examples for the sake of simplicity.

4.3 Toll sets to create SO flows under SUE: Path-Based

A methodology for tolling in a general network is derived from examining a set of toy-networks of increasing complexity. The toy-networks are used first to examine different network possibilities with a single OD pair. The most simple case (4.3.1) is a 2-link network where each path consists of a single link. This is then extended to a 5-link network (4.3.2) where each path consists of more than a single link but with a structure so that either link data or path data uniquely defines the other for both costs and flows. Multiple OD pairs are then introduced in a 9-node network (4.3.3) where link costs may not be uniquely determined from path costs. The principles are then formalised and generalised (4.3.4) and issues with such a path-based methodology are discussed. In the numerical examples the following notation will be used; (C_i , X_i , T_i : path-costs, path-flows, path-tolls,), (c_i , x_i , t_i : link-costs, link-flows, link-tolls).

4.3.1 Two-Link Network to illustrate theoretical feasibility of such tolls

In attempting to replicate the true deterministic SO flow pattern under SUE assignment, a path-based methodology was investigated primarily; the two link network in figure 4.1 below being used as an initial formative example. In this example each link is also a path, so $C_i = c_i$, $X_i = x_i$ and $T_i = t_i$ $\forall i$.

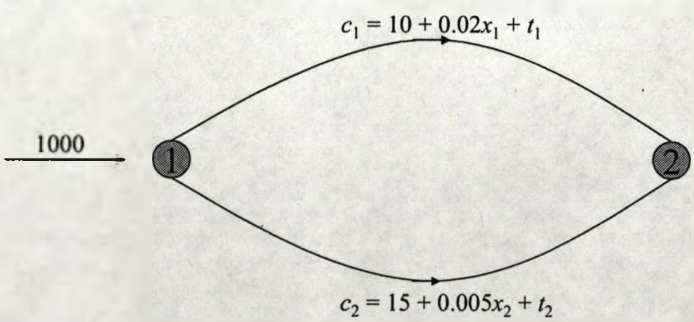


Figure 4.1: 2-link network

The logit choice function splits the demand by considering the costs on each of the viable paths. In this specific 2-link example link-costs/tolls are also path-costs/tolls and vice versa; in general path-costs/tolls will be the sum of link-costs/tolls comprising that path. As in this simple case the flow on path 2 is simply [1000 – flow on path 1], it is convenient to consider the solution for path 1 only.

The probability of assigning flow to path 1 under logit SUE is:

$$p_1 = \frac{1}{1+\exp\theta(C_1 - C_2)} \qquad 4.1$$

As $C_2 = c_2$ may be expressed in terms of x_1 and t_2, and $T_i = t_i$; $X_i = x_i$, it can be seen that;

$$C_1 - C_2 = 0.025X_1 - 10 + (T_1 - T_2) \qquad 4.2$$

Thus for simplicity the single toll difference $T = T_1 - T_2$ may be determined.

After n iterations (when convergence is assumed to have occurred), the flow may be expressed as;

$$X_1^{n+1} = \frac{1000}{1+\exp\theta(0.025X_1^{n} - 10 + T)} \qquad 4.3$$

Now, suppose that the required path-flow pattern is that at the deterministic SO solution;

$$X_1 = 300 \qquad X_2 = 700$$

thus in the limiting case as $n \to \infty$ and $X_1^n \to 300$;

$$\frac{1000}{1+\exp\theta(T-2.5)} = 300 \qquad 4.4$$

and so;

$$T = \frac{1}{\theta}\ln\left(\frac{7}{3}\right) + 2.5 \qquad 4.5$$

Therefore as θ varies the toll difference T varies as in the graph in figure 4.2.

It is clear that explicit tolls to achieve the deterministic SO flow pattern are achievable under logit-based SUE for the 2-link network above and that there are an infinite number of valid tolls for which there must be a specific constant toll difference. For minimal revenue tolls it is clear, that if the tolls are strictly non-negative, and $T_1 > T_2$ that revenue will be minimised when $T_2 = 0$, and thus when $T_1 = T$ (path-toll difference). Other valid toll sets could be determined with respect to different constraints. It can be seen that the minimal revenue link-toll set (t , 0) as theta varies, tends towards the minimal revenue link-toll set for the deterministic case (2.5, 0), as $\theta \rightarrow \infty$ as would be expected.

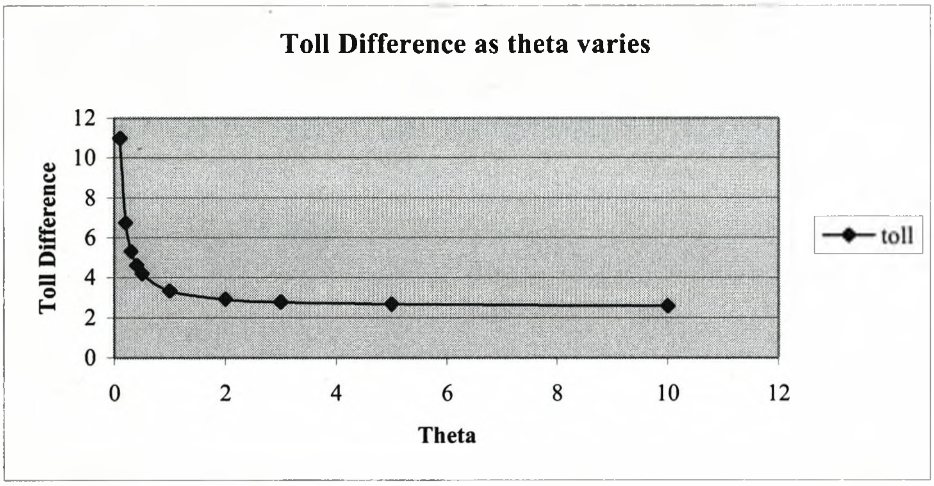


Figure 4.2: Toll difference on 2-link network as theta varies

4.3.2 Five –Link Network to illustrate existence of link tolls

The 2-link network presented clearly demonstrates the feasibility of calculating desired path-tolls, although in that example the 'paths' were also 'links and it does not illustrate the difference between path or link-based solutions. It is necessary therefore to examine a network where paths consist of more than a single link.

In general path-costs are summative along a link, as are path-tolls. Link-flows are the summation of path-flows which traverse that link. Thus if path-flows are known it is easy to calculate link-flows, but it is not necessarily possible to derive a unique set of path-flows if only link-flows are known. Similarly if cost (or toll) data is known for links, it is easy to calculate path-costs but if path-costs are known it is not always possible to uniquely determine link-costs. The 5-link network shown in figure 4.3 is of special structure in that link-flows may be uniquely determined from path flows, thus whilst it illustrates a more general case than the 2-link network in that its paths consist of more than a single link, it avoids certain problems that may occur in a general network (such as the 9-node network used in section 4.3.3).

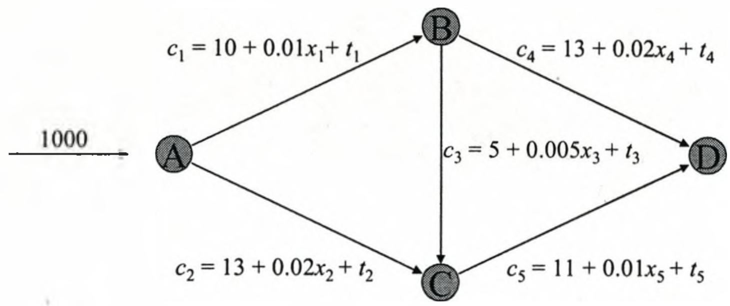


Figure 4.3: 5-link network: unit-cost functions

In this case there are 3 possible paths, [ABD], [ABCD], and [ACD] as shown in figure 4.4.

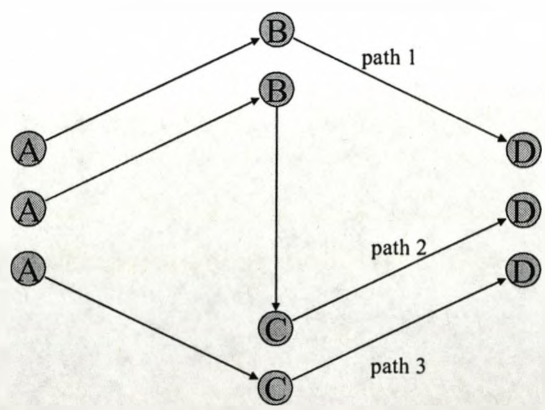


Figure 4.4: 5-link network: feasible paths

The path-costs are the summation of link-costs along that path and may be easily calculated;

Path-Cost [ABD]: $C_1 = c_1 + c_4$

Path-Cost [ABCD]: $C_2 = c_1 + c_3 + c_5$

Path-Cost [ACD]: $C_3 = c_2 + c_5$

Link flows are a summation of path-flows which traverse that link and so if the path-flows are known the link-flows may easily be deduced.

$$x_1 = X_1 + X_2$$

$$x_2 = X_3$$

$$x_3 = X_2$$

$$x_4 = X_1$$

$$x_5 = X_2 + X_3$$

Thus the special structure of this network may be easily seen; as link 2 is only included in path 3, link 3 is only included in path 2 and link 4 is only included in path 1. Thus path-flow data may in this case be uniquely derived from link-flows; but whilst this is straightforward for this network, it is not always the case as will be illustrated in section 4.4.3.

Path information was required here, as the Stochastic assignment was carried out using the logistic function directly. It would have been possible, of course to use Dial's Stoch algorithm, which removes the need to deduce path data when performing the stochastic assignment, but the direct application of the logistic function is used in the toll-difference calculation and so was used here for simplicity. The resulting flow patterns and TNTC for UE, SO and SUE ($\theta = 0.1$) are given in the table below (all entries to nwn).

	x_1	x_2	x_3	x_4	x_5	TNTC
UE	579	421	125	454	546	37875
SO	602	378	188	415	585	37793
SUE θ = 0.1	645	355	272	373	627	37933

Table 4.1: Comparative flow patterns for 5-link network

As in the two-link example, suppose that it is required that an SUE assignment using the original unit-cost functions with the addition of a toll, should produce the SO flow pattern that is obtained under deterministic assignment which minimises the TNTC.

Thus tolls should be determined that produce the SO solution given in Table 4.1. This flow pattern gives the link costs in table 4.2.

	c_1	c_2	c_3	c_4	c_5
Link Costs	16.02 + t_1	20.96 + t_2	5.94 + t_3	21.29 + t_4	16.85 + t_5

Table 4.2: Link costs for 5-link network

However, to apply the logistic function, the costs and flows are required for the paths rather than the links, paths 1, 2 and 3, being as previously described. Thus at convergence to the deterministic SO flow pattern as required, with OD demand D (=1000), then:

$$X_i = D\frac{\exp-\theta(C_i+T_i)}{\sum_j \exp-\theta(C_j+T_j)} \tag{4.6}$$

The required path data is given below (to 2dp):

	1	2	3
C_i	37.31	38.81	37.81
X_i	414.58	187.50	397.92

Table 4.3: Path costs and flows at SO for 5-link network

As more than 2 paths exist, it is not possible to express the logistic function in terms of a single path, and the three equations below are obtained.

$$414.58 = 1000\exp\text{-}\theta(37.31 + T_1) / S \tag{4.7}$$

$$187.50 = 1000\exp\text{-}\theta(38.81 + T_2) / S \tag{4.8}$$

$$397.92 = 1000\exp\text{-}\theta(37.81 + T_3) / S \tag{4.9}$$

where;

$$S = \exp\text{-}\theta(37.31 + T_1) + \exp\text{-}\theta(38.81 + T_2) + \exp\text{-}\theta(37.81 + T_3) \tag{4.10}$$

It can be seen that the 'toll difference' between pairs of path tolls, may be found by the division of pairs of equations, thus;

$$T_j - T_i = \frac{1}{\theta}\ln\left(\frac{X_i}{X_j}\right) + (C_i - C_j) \tag{4.11}$$

Using $\theta = 0.1$, the toll differences can be found to be as follows (to 3dp);

$T_2 - T_1 = 6.435$	$\Rightarrow$	$T_2 > T_1$
$T_3 - T_1 = -0.090$	$\Rightarrow$	$T_1 > T_3$
$T_3 - T_2 = -6.525$	$\Rightarrow$	$T_2 > T_3$

Therefore $T_2 > T_1 > T_3$

Assuming tolls must be non-negative, and seeking minimal revenue tolls, the smallest toll path may be set as zero.

From the network it may be seen that:

$$T_1 = t_1 + t_4$$

$$T_2 = t_1 + t_3 + t_5$$

$$T_3 = t_2 + t_5$$

and so setting $T_3 = 0$ implies that $t_2 = t_5 = 0$

and $T_1 = t_1 + t_4 = 0.090$

$$T_2 = t_1 + t_3 = 6.525$$

The system is thus under-constrained, and one valid (equally optimal) link-toll vector may be obtained when $t_1 = 0$.

So for $\theta = 0.1$ Link Tolls = (0, 0, 6.525, 0.090, 0).

Clearly the tolls will vary with theta, and looking at the toll difference equation (4.11), it can be seen that the value of theta may affect whether the toll difference is positive or negative, and thus will affect the relevant order of magnitude of the path-tolls, and consequently the smallest path-toll (to be set as zero) may be on a different path, as illustrated in table 4.4.

θ	$T_2 - T_1$	$T_3 - T_1$	$T_3 - T_2$	Order of magnitude
0.1	6.434952	-0.08969	-6.52464	
0.2	2.467476	-0.29484	-2.76232	
0.3	1.144984	-0.36323	-1.50821	$T_2 > T_1 > T_3$
0.4	0.483738	-0.39742	-0.88116	
0.5	0.08699	-0.41794	-0.50493	
0.6	-0.17751	-0.43161	-0.25411	
0.7	-0.36644	-0.44138	-0.07495	$T_2 > T_1 > T_3$
0.8	-0.50813	-0.44871	0.05942	
0.9	-0.61834	-0.45441	0.163929	
1	-0.7065	-0.45897	0.247536	$T_1 > T_3 > T_2$
2	-1.10325	-0.47948	0.623768	
3	-1.2355	-0.48632	0.749179	

Table 4.4: Order of magnitude of toll differences

The difference between the tolls on paths 1 and 2 is zero where $\theta = 0.53$, and for paths 2 and 3 where $\theta = 0.75$. Due to the simplicity of the network, it is possible to always have zero tolls on links 1 and 5, giving the remaining tolls as;

$$t_2 = T_3 \qquad t_3 = T_2 \qquad \text{and} \qquad t_4 = T_1$$

The relative magnitudes of the non-zero tolls as theta varies are illustrated in figure 4.5.

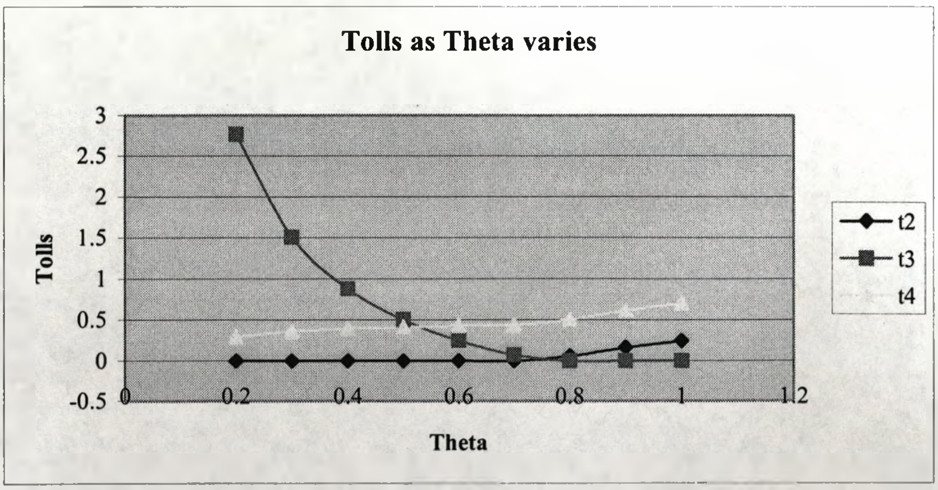


Figure 4.5: Relative orders of magnitude of non-zero tolls

As theta increases ($\theta \to \infty$), it can indeed be seen that the toll set approaches the minimal revenue toll set in the deterministic case, (0, 1, 0, 1.5, 0) (figure 4.6).

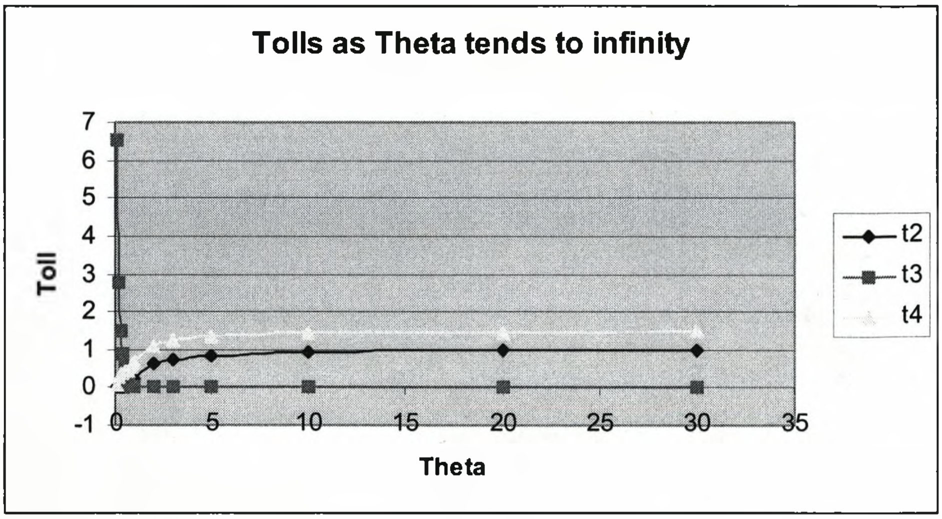


Figure 4.6: Tolls tending towards values for the deterministic case

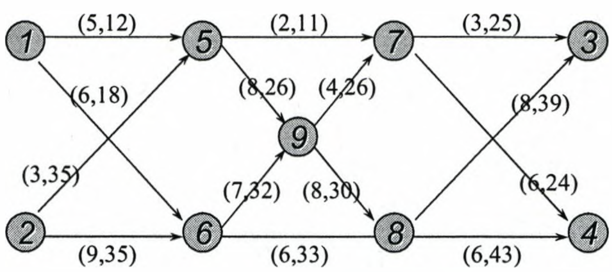
In the 2-link example, there was a single toll difference equation which meant that although there were infinitely many valid toll sets which would induce the deterministic SO flow pattern under a logit-based SUE assignment, there was a unique minimal-revenue toll set for any value of theta. The 5-link example illustrates two more general network features that were not illustrated by the 2-link network. Firstly when there are more than two feasible paths, instead of a single toll difference there will be a set of simultaneous toll difference equations. [In general if there are n paths between any OD pair, there will be $\frac{1}{2}(n(n-1))$ toll-difference equations.] From these equations it will be possible to determine an order of magnitude for the path-tolls; assuming all tolls must be non-negative the smallest path-toll may be set to zero and the remaining path-tolls determined. Secondly if each path consists of more than a single link there may be equally optimal ways to derive a set of valid link-tolls from the minimal-revenue path-tolls and secondary conditions such as maximising the number of untolled links may be imposed. Due to the special structure of this 5-link network, it was easy to derive consistent link-tolls from the path-tolls; this was due to the network feature that path-flows could be uniquely determined from link-flows, but this is not a general network property as will be illustrated in the 9-node network example in the following section.

In both the 2-link and 5-link examples, it was demonstrated that as the stochastic assignment tended to the deterministic (i.e. as $\theta \rightarrow \infty$) the minimal-revenue toll sets derived from this method were the same as those which may be determined in the wholly deterministic case. This is obviously to be expected and provides confirmation of the correctness of this method.

4.3.3 Bergendorff's Nine-Node Network (9-node, 2 origins, 2 destinations)

The path-based logit methodology to derive tolls was successful when applied to the previous two small networks. It is necessary however to test the procedure on a larger test-network with a more general structure and multiple OD pairs. The 9-node network with 2 origins and 2 destinations shown in figure 4.7 is consequently used. This network has been frequently used in the literature (Bergendorff et al (1997), Dial (2000); a restricted version is used here (with 4 vertical links (5↔6, 7↔8) carrying zero flow removed) to render the network a-cyclic (as for Dial's multiple origin algorithm), and thus to limit the path enumeration matrix so that 24 viable paths are obtained (six paths between each of the four OD pairs).

The link cost functions are of BPR type as shown where ($c_a^{(0)}$,Y_a) for each link are given on the diagram, x_a is link flow, c_a is link cost, $c_a^{(0)}$ is free flow link cost and Y_a is link capacity. The path-enumeration matrix is given in figure 4.8.



OD Pair: [1,3] [1,4] [2,3] [2,4]
Demand: 10 20 30 40
$c_a(x_a) = c_a^{(0)}(1 + 0.15(x_a/Y_a)^4)$

Figure 4.7: Nine-node network diagram, showing OD demand and link cost-flow relations.

A =

(1,5)	(5,7)	(7,3)	(1,6)	(2,5)	(5,9)	(9,7)	(6,9)	(9,8)	(7,4)	(8,3)	(2,6)	(6,8)	(8,4)	
1	1	1	0	0	0	0	0	0	0	0	0	0	0	
1	0	1	0	0	1	1	0	0	0	0	0	0	0	
1	0	0	0	0	1	0	0	1	0	1	0	0	0	
0	0	1	1	0	0	1	1	0	0	0	0	0	0	[1,3]
0	0	0	1	0	0	0	1	1	0	1	0	0	0	
0	0	0	1	0	0	0	0	0	0	1	0	1	0	
1	1	0	0	0	0	0	0	0	1	0	0	0	0	
1	0	0	0	0	1	1	0	0	1	0	0	0	0	
1	0	0	0	0	1	0	0	1	0	0	0	0	1	
0	0	0	1	0	0	1	1	0	1	0	0	0	0	[1,4]
0	0	0	1	0	0	0	1	1	0	0	0	0	1	
0	0	0	1	0	0	0	0	0	0	0	0	1	1	
0	1	1	0	1	0	0	0	0	0	0	0	0	0	
0	0	1	0	1	1	1	0	0	0	0	0	0	0	
0	0	0	0	1	1	0	0	1	0	1	0	0	0	
0	0	1	0	0	0	1	1	0	0	0	1	0	0	[2,3]
0	0	0	0	0	0	0	1	1	0	1	1	0	0	
0	0	0	0	0	0	0	0	0	0	1	1	1	0	
0	1	0	0	1	0	0	0	0	1	0	0	0	0	
0	0	0	0	1	1	1	0	0	1	0	0	0	0	
0	0	0	0	1	1	0	0	1	0	0	0	0	1	
0	0	0	0	0	0	1	1	0	1	0	1	0	0	[2,4]
0	0	0	0	0	0	0	1	1	0	0	1	0	1	
0	0	0	0	0	0	0	0	0	0	0	1	1	1	

Figure 4.8: Path enumeration matrix: 9-node network

As in section 4.3.2 if it is desired that an SUE assignment using the original unit-cost functions with the addition of a toll should produce the deterministic SO flow pattern, where the TNTC is minimised, then using logit-based SUE assignment this may be formulated as in (4.6) and (4.11).

$$X_i = D\frac{\exp - \theta(C_i + T_i)}{\sum_j \exp - \theta(C_j + T_j)} \qquad 4.6$$

$$T_j - T_i = \frac{1}{\theta}\ln\left(\frac{X_i}{X_j}\right) + (C_i - C_j) \qquad 4.11$$

A resulting order of magnitude of path-tolls (for each OD pair) may be deduced and requiring tolls to be non-negative and seeking minimal revenue tolls, the smallest toll-path (for each OD pair) may be set to zero, and the remaining path tolls calculated as in the previous 5-link example. In the 9-node case there are six feasible paths between each OD pair which would result in a set of 15 simultaneous toll-difference equations for each OD pair.

However the use of the logistic function to determine toll-difference equations requires all path-flows to be non-zero, as zero flows would clearly result in infinite tolls (4.6).

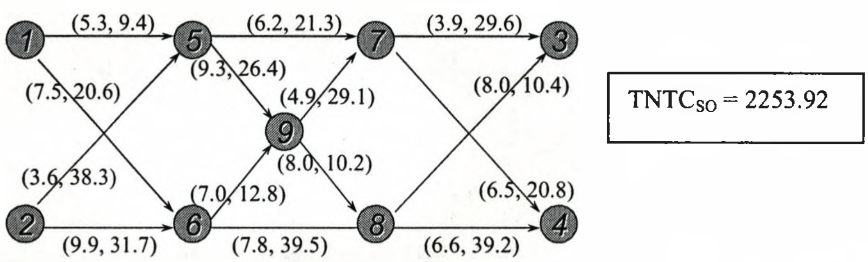


Figure 4.9: SO solution: 9-node network: (c_a, x_a): c_a = link-costs; x_a = link-flows.

The SO link-costs and link-flows for the 9-node example are given in figure 4.9. Whilst normal deterministic assignment algorithms give link-flows as an output, path flows (although not necessarily unique) can easily be produced if required (by storing the minimum cost paths for each OD pair at each iteration). Although all links can be seen to carry non-zero flow at SO (in this example) this is not necessarily true of all feasible paths; in a Wardropian SO assignment (using 100 iterations of MSA) three paths were completely unused, as can be seen in the SO path-based solution given in table 4.5.

OD pair	[1,3]						[1,4]					
C	15	23	31	23	31	23	18	26	29	26	29	22
X	2.6	1.6	**0**	3.2	**0**	2.9	3.1	1.2	1.2	1.7	1.3	11.4
OD pair	[2,3]						[2,4]					
C	14	22	29	26	33	26	16	24	28	28	32	24
X	7.7	20.4	0.1	4.4	**0**	7.4	8.2	5.2	6.7	1.3	0.9	17.8

Table 4.5: 9-node network: path-costs and path-flows at SO

Thus it would seem to only be possible to derive toll sets that will make a logit-based Stochastic User Equilibrium assignment method produce the Wardropian SO flow pattern in those networks where all paths have non-zero flow at SO. There are consequently difficulties to be encountered when dealing with complex networks, as pointed out by Smith et al (1994), where generally there will exist technically feasible paths which have zero flow at SO. Thus it may be necessary in general to derive toll sets which produce a solution which is close to the SO solution rather than being exact.

In the case of Stochastic assignment, some flow, however small, will theoretically be allocated to every feasible path between OD pairs. In the case of logit-based SUE as the variability parameter θ tends to infinity the stochastic assignment tends to the deterministic and in the case of probit-based SUE as the dispersion parameter β tends to zero the stochastic assignment tends to the deterministic. Thus if marginal-cost functions are used in the place of unit-cost functions in an SUE assignment the resulting flow pattern will tend towards the deterministic SO flow pattern as θ increases or as β tends to zero. Each iteration of such a stochastic assignment would assign non-zero flows to every path (although they may still be very small), and so the problem of zero flow paths could be overcome by using a stochastic 'MSCP' assignment with large value of θ (or β close to zero) as the desired flow pattern rather than the exact SO. In the case of logit-based assignment this means that it is possible to find a finite value of

θ

such that it is possible to achieve a value for TNTC which is as close as may be desired to the $TNTC_{SO}$. The toll difference equation (4.11) could then be used to derive path tolls. (As illustrated in Appendix A.1) If the true SO solution had contained zero-flow paths, those paths would be allocated very little flow under the method proposed above and as the stochastic assignment tended to the deterministic these path flows would tend to zero. Thus the larger the value of θ used, the closer the relevant path flows would be to zero and this would consequently result in path tolls of increasing magnitude as θ increases.

In the nine-node network a desired path flow set was obtained using logit-based SUE assignment with $\theta = 15$; for which the TNTC = 2254.0 , which is very close to the 'true' SO value.

A viable toll set is given below for $\theta = 0.1$.

OD pair	[1,3]						[1,4]					
T =	12	9	56	0	39	2	15	16	18	16	12	0
OD pair	[2,3]						[2,4]					
T =	11	1	45	5	39	0	16	11	7	28	19	0

Table 4.6: Path toll set: Approximate SO

The path tolls correspond to the paths given in the path-link incidence matrix A with 6 paths for each OD pair.

Thus it would appear achievable to create viable path toll sets, which will create a flow pattern approaching the true SO flow pattern as closely as is desired under logit SUE. However in the limiting case ($\theta \rightarrow \infty$), some tolls will tend to infinity, which is clearly not desirable, and so an appropriate degree of closeness to the SO solution would need to be determined.

In addition it may be possible to exclude paths with very small flows from the tolling calculations, as if such paths are of such high cost that they are never used in a true SO solution, they may be excluded from a set of 'sensible paths', which may be used instead of the full set of feasible paths. Path based assignment methods which have been developed (Rosa, 2001), are based on partial (efficient) path enumeration techniques.

A further and possibly more significant difficulty with this method is that although a viable path toll set can be determined, it is not possible to derive a consistent link based toll set from it. The inconsistency may be demonstrated by combining sets of paths as is given in Figure 4.10.

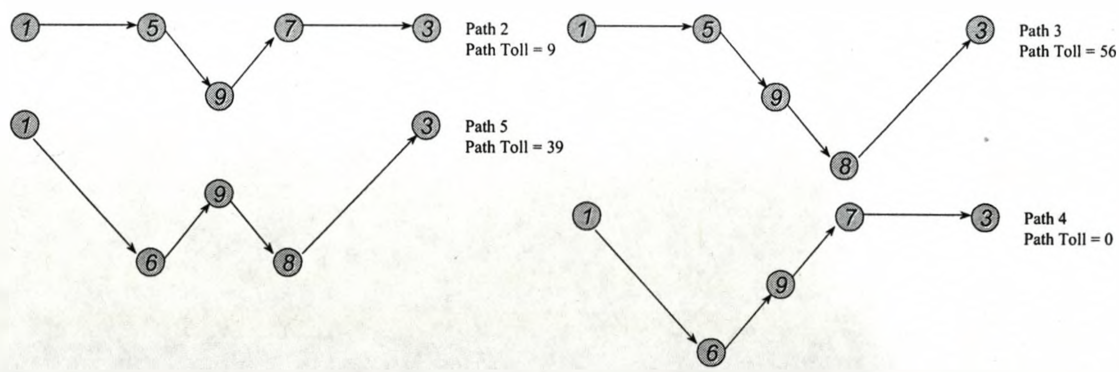


Figure 4.10: Path combinations from Bergendorff's nine-node network.

Paths 2 and 5 together contain the same links as paths 3 and 4 together, therefore for consistent link tolls the total toll on paths 2 and 5 (total=48) should equal the total toll on paths 3 and 4 (total=56). This is clearly not the case (similar examples of inconsistency may also be demonstrated), so consistent link tolls may not be determined.

Thus it must be considered whether a path based tolling methodology would be sensible for implementation, even if developing efficient path based assignment methods could be utilised. A tolling solution where the same link has a different cost depending on the overall route travelled, would intuitively not appear to be equitable or practicable. Further, the technology required to implement a path based tolling scheme, would require vehicle tracking, which although technically feasible would result in concern over privacy issues.

This section has extended the path-difference methodology presented in sections 4.3.1 and 4.3.2 to a more general network with overlapping paths between multiple OD pairs. In a larger network the path enumeration matrix would clearly be much larger and the number of difference equations for an OD pair would be greater. There are no particular limitations to this methodology that would make it ungeneralisable to larger networks. The nine-node network was acyclic, but providing some assumption regarding limiting looping paths was made path-toll sets could be calculated. This method relies only on being able to enumerate paths between OD pairs. Formalisation of this methodology is presented in section 4.3.4.

4.3.4: Formalisation

The methodology illustrated in the toll-set solutions for the toy-networks given may be formalised as below where the issue of infinite tolls is dealt with more precisely:

If it is desired that an SUE assignment using the original cost-flow functions with the addition of a toll, should produce the SO flow pattern that is obtained under deterministic assignment, where the TNTC is minimised, then using logit-based SUE this may be formulated as below:

$$X_i = D\frac{\exp-\theta(C_i + T_i)}{\sum_j \exp-\theta(C_j + T_j)} \tag{4.6}$$

where D is the OD demand, C_i and X_i are the path costs and flows at the SO solution which may be found using deterministic assignment methods, and T_i are the desired path tolls to be determined. The 'toll difference' between pairs of path tolls for each OD pair may then be found by the division of pairs of equations, thus:

$$T_j - T_i = \frac{1}{\theta}\ln(X_i / X_j) + (C_i - C_j) \tag{4.11}$$

A resulting order of magnitude of path tolls (for each OD pair) may be deduced, and assuming tolls to be non-negative, and seeking minimal revenue tolls, the smallest toll path (for each OD pair) may be set as zero, and the remaining path tolls calculated.

It would appear that the use of the logistic function to determine path differences requires all path flows to be non-zero, as zero flows would clearly result in infinite tolls (4.6). (It should also be noted here that this problem would also apply in the case of the probit model). There are consequently difficulties to be encountered when dealing with networks where there technically feasible paths which have zero flow at SO exist.

It is however possible to divide the set of feasible paths into two sets, Ω^0 for zero-flow paths and Ω^1 for non-zero flow paths as defined below;

Let $\Omega^0 = \{k : X_k = 0\}$ and $\Omega^1 = \{i : X_i > 0\}$

Then $\forall\, ij \in \Omega^1$, let T_i and T_j satisfy (4.6) and $\forall\, k \in \Omega^0$, let $T_k = M$

Then, the path flows, $X_i(T)$, associated with the above tolls are:

$$X_i(T) = D\frac{\exp-\theta(C_i + T_i)}{\sum_{i\in\Omega^1}\exp-\theta(C_i + T_i) + \sum_{i\in\Omega^0}\exp-\theta(C_i + M)}, \quad \forall\, i \in \Omega^1 \tag{4.12}$$

$$X_k(T) = D\frac{\exp-\theta(C_i+M)}{\sum_{i\in\Omega^1}\exp-\theta(C_i+T_i)+\sum_{i\in\Omega^0}\exp-\theta(C_i+M)},\ \forall k\in\Omega^0 \qquad 4.13$$

It is clear that $X_i(T)\to X_i$ and $X_k(T)\to 0$ as $M\to\infty$.

Hence for any $\varepsilon>0$, there exists a sufficiently large M such that $X_i - X_i(T)\le\varepsilon$, $\forall i\in\Omega^1$, and $X_k(T)\le\varepsilon$, $\forall k\in\Omega^0$.

Thus it is possible to determine viable path toll sets, which will create a flow pattern approaching the true SO flow pattern, as closely as is desired under logit SUE. However in the limiting case ($\varepsilon\to 0$), M (the toll on zero-flow paths) will tend to infinity, and so an appropriate degree of closeness to the SO solution would need to be determined.

For Bergendorff's network as given in figure 4.7, the solution may be expressed more easily without needing to utilise a stochastic assignment which tends to the deterministic, the deterministic itself may be used assigning a toll M to the zero-flow paths.

If this is done, a viable toll set is given in table 4.7 for θ = 0.1. When M=50, TNTC = 2253.99. The path tolls corresponding to the paths are given in the path-link incidence matrix A (figure 4.8).

OD pair	[1,3]						[1,4]					
T =	12	9	M	0	M	2	15	16	18	16	12	0
OD pair	[2,3]						[2,4]					
T =	11	1	45	5	M	0	16	11	7	28	19	0

Table 4.7: Path tolls: Nine-node network

It may be easily seen however that this does not alter the fact that it is not possible to derive consistent link-tolls from these paths tolls. Whilst path-based assignment methods might feasibly become sufficiently efficient to derive path-tolls in this manner, the inability for these to be sensibly converted into link tolls is a significant disadvantage. Further, the path difference equations on which this section's methodology rely require the use of logit assignment, which does not model networks with overlapping paths as well as probit. If a network with only two links is used, it is possible to solve for tolls algebraically in the probit case, but this is not the case for any more complex network. Consequently this method whilst theoretically feasible is not considered to be of potential use for practical implementation and link-based heuristic methods are developed in the following sections.

4.4 Toll sets to create SO flows under SUE: Link-Based

A link-based methodology to derive tolls that would create a flow pattern approaching the SO would appear to be more desirable than path-based methods as discussed in section 4.3. It was assumed from the previous results, that link-based tolls might not be sufficient to replicate the desired SO flow pattern in the limiting case, but that good sub-optimality would be acceptable for practical purposes.

The objective is still to minimise the total network travel cost, and this is attempted by seeking a link flow pattern that approaches the flows obtained under deterministic SO assignment. Thus links where the flow is higher than that desired have link costs progressively increased by the addition of a toll until the desired flow pattern is approached. Thus the link-based methodology derived is based on a heuristic which seeks to match the flow on each link with the desired link flow at the SO solution. This heuristic does not guarantee a Minimal Revenue solution in a theoretical sense, but applies additional tolls to links in small increments so that at each stage the additional revenue required to produce a reduction in TNTC may be monitored and assessed. This will enable a traffic planner to potentially 'trade-off' a reduction in TNTC (approaching the SO solution) against an increase in revenue to be extracted from the users. After a certain point the relative benefit from imposing more tolls is outweighed by the relative disadvantage of imposing higher tolls for lesser additional effect. Such tolls may be thought of as 'low-revenue' tolls rather than 'minimal-revenue' tolls.

Step 1: Find the SO solution and let $\mathbf{F}_{SO}$, $\mathbf{C}_{SO}$ and $TNTC_{SO}$ denote the corresponding flow pattern, link cost, and total network travel cost.

Step 2: Link toll vector set to zero: $\mathbf{T}_0 = \mathbf{0}$

Step 3: Set $n = 0$

Step 4: Perform SUE assignment with current link tolls $\mathbf{T}_n$: $\mathbf{C}_n$ and $\mathbf{F}_n$ obtained

Step 5: Calculate: $P_j = (F_j^{(n)} - F_j^{(SO)})(\,|\,C_j^{(n)} - C_j^{(SO)}\,|\,)\ \forall j$

Step 6: Determine link j where $P(j)$ is greatest.

Step 7: *Perform iteration to calculate t_j s.t $F_j^{(n)} = F_j^{(SO)}$ to required degree of accuracy.

7a: Set $t_{j_0} = \left|C_{j_0} - C_j^{(SO)}\right|$ where C_{j_0} is the current cost on link j (as per step 4)

7b: Set m=1

7c: Perform SUE assignment, calculate $\left|C_{j_m} - C_j^{(SO)}\right|$

7d: Set $t_{j_m} = t_{j_{m-1}} + \left|C_{j_m} - C_j^{(SO)}\right|$

7e: Calculate P_{j_m} : Stop if sufficiently close to zero and let $t_j = t_{j_m}$, or set

$m = m + 1$ and repeat from step 7c.

Step 8: $\mathbf{T}_{n+1} = \mathbf{T}_n + \mathbf{t}$; where $t_i = t_j$ when $i = j$ and $t_i = 0$ otherwise

Step 9: Calculate TNTC: Stop if TNTC sufficiently close to TNTC_{SO} or set $n = n + 1$ and repeat from Step 4.

* The internal iteration in step 7 only regards the output flow for the single link where P_j is greatest as per step 6, and results in the link tolls shown in Table 4.7 (for nine-node network). A more efficient interpolation procedure could be refined for the internal iteration for use in larger networks but the potential benefits of allowing manual adjustments to be permitted at step 7 will be discussed later.

This method is illustrated using the previously used 9-node network (see Figure 4.7), with logit SUE where $\theta = 0.1$.

In Step 1 a deterministic SO assignment determines the desired link flow set, and the link costs are calculated for these flows. The minimum value of the TNTC is also recorded (for this example $TNTC_{SO} = 2253.9$). An initial toll vector is then set with all tolls being zero (Step 2). An SUE assignment is then completed, and the link costs, link flows and the TNTC compared with those desired.

The toll set is constructed in a step-wise process, where only a single link toll is considered in each iteration; thus the link to be tolled in that iteration must be chosen. Steps 4 and 5 determine which link is chosen: choosing simply the link where the flow was most in excess of the desired SO flow for that link would not take into account the relative costs, and so a product of flow and cost difference is used here, although this may be refined in future work. As only non-negative tolls are being imposed, the absolute value of the cost difference is used, so that the chosen link, where the value of the product is greatest has a flow strictly greater than that desired. Table 4.8 shows the stepwise construction of a toll set.

It can be seen from Table 4.8 and from the graph in Figure 4.11 below, that the first few iterations are by far the most significant, and no great benefit is gained from continuing to approach the $TNTC_{SO}$ for many iterations. Further if it is desirable to keep as many links toll free as possible, it is not then sensible to continue to add small tolls on additional links, to reduce the TNTC only by tiny amounts.

Iteration	0	1	2	3	4	5	6	7	8	9	10	11	12
t_1 (1-5)	-	-	-	-	-	-	-	-	-	-	**0.9**	0.9	0.9
t_2 (5-7)	-	**7.2**	7.2	7.2	7.2	7.2	7.2	**8**	8	**8.9**	8.9	8.9	8.9
t_3 (7-3)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_4 (1-6)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_5 (2-5)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_6 (5-9)	-	-	-	-	-	-	-	-	**0.6**	0.6	0.6	0.6	0.6
t_7 (9-7)	-	-	-	-	-	-	**1.4**	1.4	1.4	1.4	1.4	1.4	1.4
t_8 (6-9)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_9 (9-8)	-	-	-	**13**	13	13	13	13	13	13	13	**13.8**	13.8
t_{10} (7-4)	-	-	**7.9**	7.9	**12.9**	12.9	12.9	12.9	12.9	12.9	12.9	12.9	12.9
t_{11} (8-3)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_{12} (2-6)	-	-	-	-	-	**3.6**	3.6	3.6	3.6	3.6	3.6	3.6	**3.8**
t_{13} (6-8)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_{14} (8-4)	-	-	-	-	-	-	-	-	-	-	-	-	-
TNTC	2441	2385	2337	2285	2268	2262	2259	2258	2257	2256	2255	2255	2254
REV	0	154	307	449	568	705	746	759	777	797	813	819	822

Table 4.8: Iterative building of 'Optimising' toll set

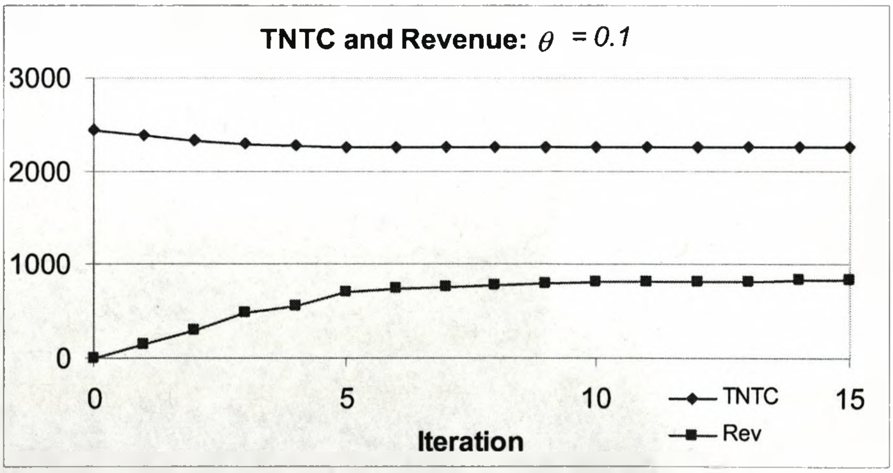


Figure 4.11: Total Toll Revenue required for reduction in TNTC

The link toll set resulting from the 12 iterations given above, is shown in Figure 4.12, where link width is proportional to the size of the link toll. The TNTC achieved after 12 iterations is only 0.02% greater than the Minimum TNTC. However if the process was stopped after only 4 iterations, the TNTC achieved is still only 0.6% greater than $TNTC_{SO}$ and 4 links that could be tolled, would remain toll-free. [$TNTC_{SUE}$ is 8.30% higher than the $TNTC_{SO}$, and if the SO flow pattern were achieved it would represent a decrease in total travel cost of 7.64%, whilst this might not seem a huge decrease surprisingly small decreases can result in much more favourable and free flowing driving conditions (Jones, 2003).]

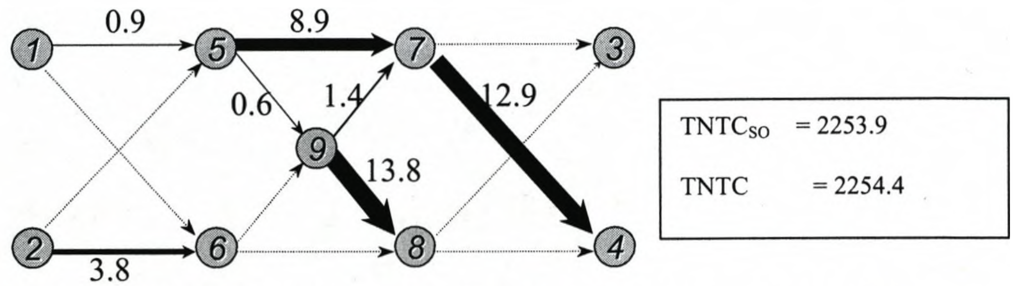


Figure 4.12: logit toll-set for Bergendorff's network (θ = 0.1) – 12 iterations

Whilst this methodology has been demonstrated using logit assignment, it may equally be used for other stochastic assignment models. Figure 4.13, shows the reduction in TNTC achieved for different values of the dispersion parameter θ using logit assignment, and for the variability parameter $\beta = 0.5$ using probit assignment. (The Stochastic Assignment Method SAM (Maher and Hughes, 1997a) was used here to obtain the probit results).

It must be noted that the method used does not result in the TNTC strictly decreasing at every iteration, although the overall trend is that it does reduce as the desired flow pattern is approached. The internal iteration at Step 7, has in these examples been used to reduce the flow on a particular link so that it is very close to the desired flow value for that link at SO. During this internal iteration process, at some point the value of the difference product P will be greatest for a new link, after this point, the overall TNTC may no longer decrease. It is possible to amend this internal iteration, so that the link toll is determined at the minimal value for the TNTC that can be achieved by just varying the toll on this link. However it appears in practice that as this will generally give a smaller toll being added at each iteration, that it causes a greater number of the main iterations to be required. Consequently, the objective at each internal iteration is that the flow difference on that link should be reduced to (approximately) zero.

In applying SUE assignment a choice of variability parameter must be made to calibrate the model with reality. In the numerical examples given a value of $\theta = 0.1$ for logit-SUE has generally been used but as shown in figure 4.13 whatever the value of the variability parameter the heuristic will produce a toll set which effectively reduces the TNTC to the desired SO value. Further for the nine-node example network there is no great difference in the number of iterations required to reach convergence.

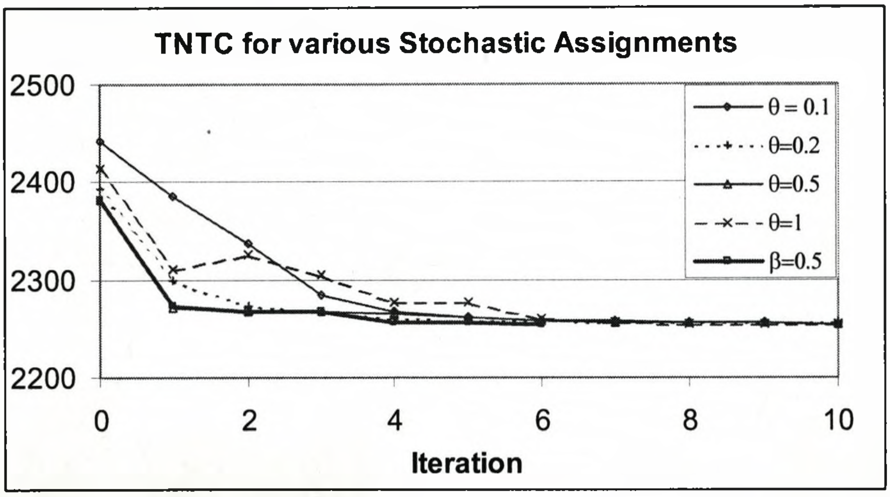


Figure 4.13: TNTC with increasing iterations for various Stochastic Assignments

It may be noted that if the logit and probit models are to be compared using the relation:

$$\text{Var}\,(U_k) = \pi^2/(6\theta^2) \qquad 4.14$$

(Cascetta, 1990), where U_k is the probit utility function as in equation (3.8), then despite the link variances in the probit case obviously being different for each link, an approximate correspondence can be found in this case between probit $\beta = 0.5$, and a logit sensitivity parameter of $\theta \approx 0.5$. It can be observed in Figure 4.13 above, that while all the graphs are quite similar in form the graphs for $\beta = 0.5$ and $\theta = 0.5$ predominantly coincide (that for $\beta = 0.5$ almost totally covering that for $\theta = 0.5$).

Section 4.4 has presented a heuristic to derive link tolls which produce a flow pattern close to the desired SO flow pattern under SUE assignment and which reduce the TNTC to a value close to the $TNTC_{SO}$. Its use has been demonstrated using the nine-node network introduced in section 4.3. For this small network it is observed that the $TNTC_{SO}$ may be very closely approached in a relatively small number of iterations and that this is not dependent on the chosen value for the variability parameter. Further this heuristic may be utilised alongside any SUE assignment method rather than being limited to logit-based SUE. In section 4.5 the results for this heuristic are compared to the results from the path-based methodology of section 4.3 and its applicability to larger networks is illustrated in section 4.6.

4.5 Comparison of link-tolls and path-tolls for Bergendorff's 9-node network.

Whilst the path-tolls derived in section 4.3.3 could not be used to derive consistent link-tolls, obviously if desired the converse would be possible. Whilst it is in no way intended that the heuristic presented in section 4.4 should be used to calculate path-tolls for the purpose of potentially implementing path-tolls it is nevertheless of interest to derive them for comparative purposes. As the toll set obtained from 12-iterations of the heuristic resulted in a solution which was very close to producing the deterministic SO flow pattern, these tolls will be used to create a set of link tolls.

OD pair	[1,3]						[1,4]					
T_{ex}	12	9	M	0	M	2	15	16	18	16	12	0
T_{heur}	9.8	2.9	15.3	1.4	13.8	0	22.7	15.8	15.3	14.3	13.8	0
OD pair	[2,3]						[2,4]					
T_{ex}	11	1	45	5	M	0	16	11	7	28	19	0
T_{heur}	8.9	2.0	14.4	5.2	17.6	3.8	21.8	14.9	14.4	5.2	17.6	3.8

Table 4.9: Comparison of 'exact' and heuristic path-toll sets

T_{ex} are the 'exact' tolls which achieve SO under the path-based methodology, T_{heur} are the approximate tolls which produce a solution close to SO under the link-based methodology. It may be seen that whilst not in any way an exact match, the paths with low or zero tolls roughly coincide and paths with higher tolls tend to be high under both methodologies. Owing to the tendency of 'exact tolls' to tend to infinity in the case of zero-flow paths at the SO solution, this different is not particularly surprising as the toll sets obtained will be very sensitive to small changes in the desired degree of accuracy to be achieved. The relative reduction in TNTC achieved by tolling is perhaps a better comparison than actually examining the corresponding path-tolls for these cases. In the 12-Iteration solution for the low-rev tolls, the TNTC was only 0.02% greater than the $TNTC_{SO}$ (the closeness of solution is illustrated in figure 4.14), while the revenue required to achieve the $TNTC_{SO}$ by the path based methodology was 586 (assuming the

SO solution is essentially an SSO solution with $\theta = 15$) and that required to be extracted from the user by using the link-based methodology with 12-Iterations was 822. Clearly whilst both methodologies create a flow pattern which is very close to the desired $TNTC_{SO}$ value, the use of the path-based methodology achieves this solution whilst extracting less cost from the user. Given the fact that it was unviable to create consistent link-tolls from those path tolls, it is not surprising that to force link-tolls to create the same effect requires a greater cost. However it should be noted that 4 Iterations of the heuristic produced a TNTC which was still only 0.6% higher than the $TNTC_{SO}$, at a revenue cost to the users of only 568, which is a very similar value to that from the path-based methodology. It would appear that 'forcing' the network into replicating the SO flow pattern accurately is only achievable at additional cost, and that the extra efficiency obtained would not be worthwhile. This issue is further discussed in Chapter 5.

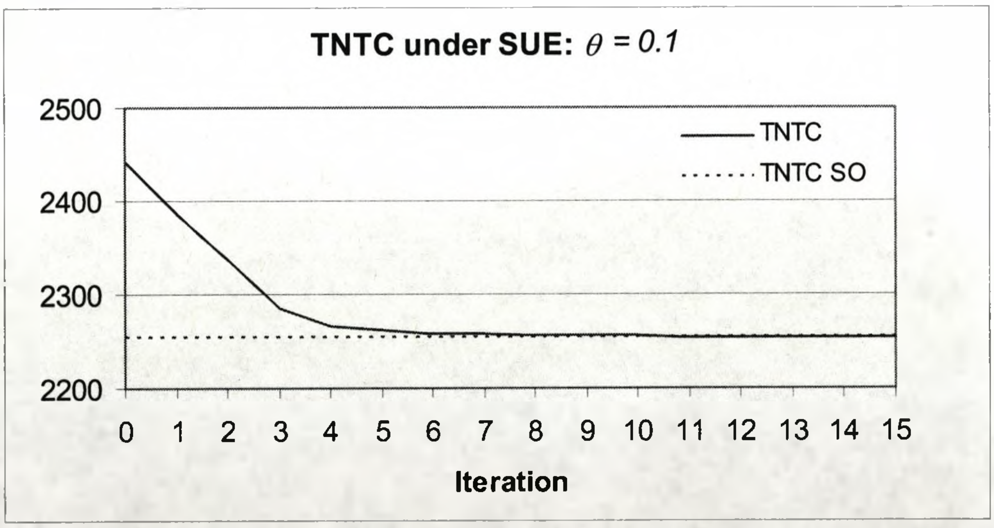


Figure 4.14: TNTC with increasing heuristic link-based tolls

4.6 Edinburgh Example

The proposed and rejected Edinburgh cordon charging scheme has been discussed in Chapter 2. It is not clear whether the negative referendum result was wholly due to the unpopularity of congestion charging in general and this scheme in particular, or if it was purely a negative consequence of holding a referendum for this sort of political decision. It is possible though, that many of the perceived negative features of this cordon scheme would have been less severe had a link-based tolling scheme been proposed.

The following numerical example gives an indication of how link-based schemes could be presented. The network used for illustrative purposes only, is a 13 node toy-network, based around a notion of an outer and inner ring (to mimic the cordons see figure 2.3); the cost flow functions used are synthetic and aim to roughly represent a pm peak (network data is available in Appendix A.2). Figure 4.15 shows the MSCP-toll set which would create the SSO flow pattern under SUE and figure 4.15 shows a Low-Rev toll set with 2-iterations using the SO heuristic. [It also serves as an example network which is more 'general'.]

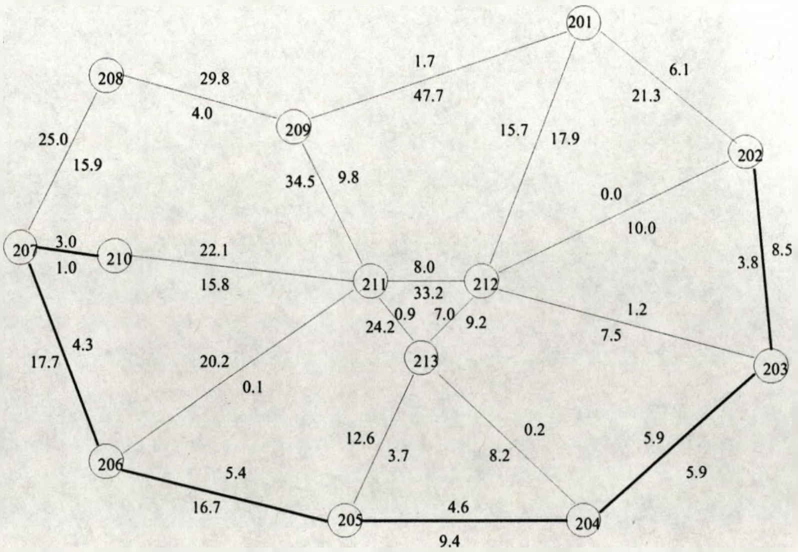


Figure 4.15: SSO: MSCP-tolls for Idealised Edinburgh Network

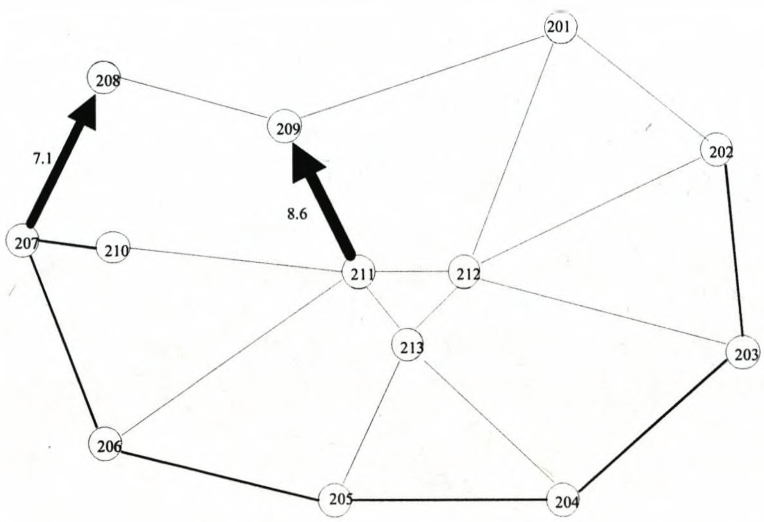


Figure 4.16: SO: Low-Rev tolls for Idealised Edinburgh Network

The heuristic concentrates the tolls on the links which are deemed to be 'worst' i.e. those with a product of link-cost and flow most greatly in excess of the desired SO value. The MSCP tolls reduce the TNTC by 3.1%, whereas the maximum achievable reduction at the theoretical SO is 10.5%. However tolling using the 'SO heuristic' produces a reduction in TNTC of 6.1% after only 2 full iterations, at a cost to the user of only 1.8% of the MSCP-toll value (MSCP-toll revenue = 757,300 ; Low-Rev-toll revenue = 13,500). Thus the Low-Rev tolls can potentially produce a significant improvement in network performance (given a fixed level of demand) whilst imposing less cost upon the user and tolling only a few links. Such a scheme with obvious equity benefits might be more politically acceptable than a cordon-based scheme.

In this rather more general example (i.e. the network includes 2 way links and is not acyclic) the heuristic produces double the reduction in TNTC than that achieved by MSCP tolls in only 2 iterations.

4.7 Conclusions

The series of example networks given, illustrate that it is possible to produce viable path toll sets that will make a Stochastic User Equilibrium assignment method produce the Wardropian SO flow pattern as closely as may be desired, but that it would not appear that it is always possible to create viable link-toll sets.

Whilst the problem of 'zero-flow' paths may be overcome by using a stochastic approximation to the converged SO flow pattern, the issue of very small flows may still remain, which would result in unacceptably large tolls being levied on popular paths. This may be approached by examining a subset of all feasible paths, to include for example 'reasonable' paths which are used more than a certain amount.

The fact however that the existence of viable path tolls does not imply viable link tolls is a major difficulty. It obviously makes seeking an efficient link based method for their precise determination somewhat futile, although analysis of more efficient path based methods might be possible.

The usefulness of calculating path tolls must however be considered. If such a system were to be implemented, it would not be sufficient merely to detect vehicles passing along each link, but to track their entire path through the network. Such a system would then make route cost evaluation difficult for the user, as if many paths were feasible, they would have to examine a large number of possibilities. This would assume that the driver would have a network knowledge akin to survey knowledge, which is unlikely, even with the aid of a route guidance system, they are more likely to display route knowledge.

Thus a heuristic to produce link-toll sets with the objective of reducing the TNTC has been developed. In the general case it would not appear to be desirable to attempt to replicate the SO flow pattern too closely as the value of adding increasingly large tolls to produce relatively little further reduction in TNTC must be considered. As in the case of the path-based methodology where tolls might tend to infinity, in the link-based case so might to tolls produced by the heuristic become unreasonably large. Thus the heuristic seeks to reduce the TNTC toward the objective of the $TNTC_{SO}$ but would be halted at an earlier point when it was decided that sufficient revenue had been imposed upon the user. Thus a balance between revenue extracted and TNTC reduction may be achieved.

The question of which flow pattern is to be desired must be considered further. Is it more desirable in the Stochastic case to minimise 'actual' or 'perceived' total network travel cost? It is of relevance to examine further the case for 'Stochastic System Optimal', and to look for minimal revenue tolls relating to the SSO as well as tolls to attempt to closely reach the SO. The resulting toll sets for SSO and SO may then be compared. Consideration of tolling to achieve the SSO is the subject of Chapter 5.

CHAPTER 5

TOLLING TO ACHIEVE THE STOCHASTIC SSO UNDER SUE

5.1 Introduction

In the case of stochastic user equilibrium, it could be argued that it is not the 'actual' or deterministic total travel cost which should be minimised, but rather the total 'perceived' network travel cost. In the case of deterministic assignment, it is well known that that the Total Network Travel Cost is minimised and the System Optimal flow pattern is obtained when cost flow functions are replaced by marginal cost flow functions. Recent work (Maher et al 2005), has shown that the analogous case is true under stochastic assignment, i.e. that when marginal social cost-flow functions are used in the place of link cost-flow functions, then perceived total network travel cost is minimised and the Stochastic System Optimal (SSO) flow pattern is obtained..

This chapter examines the possibility of tolling to achieve the SSO under an SUE assignment: section 5.2 examines marginal cost pricing under stochastic assignment and section 5.3 introduces the concept of minimising the total perceived network travel cost; section 5.4 gives the formal definition of the SSO, as derived in Maher et al (2005); section 5.4 looks at tolling to achieve the SSO and considers the determination of minimal revenue tolls by linear programming and presents an algebraic method to determine the same minimal revenue tolls by use of path enumeration; section 5.6 demonstrates how the heuristic presented in chapter 4 may be used to determine SSO-tolls; section 5.7 compares toll sets which achieve the SSO with those that 'achieve' the SO under SUE and section 5.8 gives some chapter conclusions.

5.2: SSO in the Stochastic Case: Marginal Social Cost Pricing

The stochastic case may be considered as being totally analogous to the deterministic case (as in figure 5.1), but where all link-cost functions are replaced by perceived link-cost functions as discussed in section 3.3. The Stochastic System Optimal (SSO) flow pattern would then be that where the Perceived Total Network Travel (PTNTC) costs should be minimised. If the stochastic case is completely analogous to the deterministic, then this flow pattern should be achieved from a stochastic assignment where unit link-cost flow functions are replaced by marginal social link-cost functions. However it was not immediately obvious if this were indeed the case, and this section examines some initial considerations.

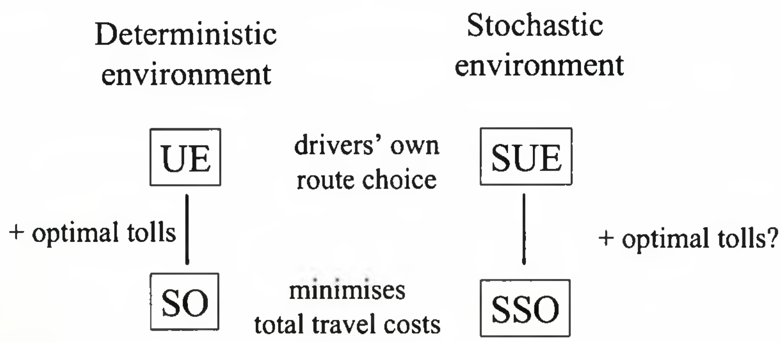


Figure 5.1: Marginal Social Cost Pricing under SUE

The outcome of applying MSCP-tolls under SUE was not initially assumed to result in any particular flow pattern, although this would be where economic utility would be maximised (Yang, 1999). Initial examination of flow patterns resulting from MSCP-tolls was conducted using small toy-networks as in chapter 4. The 2-link example network used previously is again used here. [Again each link is also a path, so $C_i = c_i$ and $X_i = x_i \ \forall i$.]

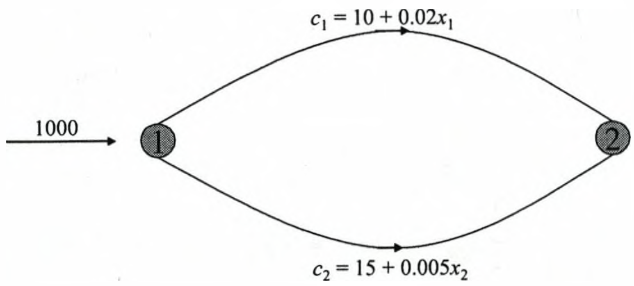


Figure 5.2: Two-link network

Using again the logit method of SUE, at each iteration of the loading process flow is assigned to path i with probability .

$$p_i = \frac{\exp-\theta(C_i)}{\sum_j \exp-\theta(C_j)} \tag{5.1}$$

In this 2-link example, as the links are also the paths, and $x_1 + x_2 = 1000$, the problem may be solved by considering only path 1 (the upper path), and it may be seen again that;

$$p_1 = \frac{1}{1+\exp\theta(0.0025X_1^{\,n} - 1)} \tag{5.2}$$

Using the demand of 1000, and assigning arbitrarily $\theta = 0.1$ it follows

$$X_1^{\,n+1} = \frac{1000}{1+\exp\theta(0.025X_1^{\,n} - 1)} \tag{5.3}$$

Thus using an initial value of $x_1 = X_1 = 400$ (the UE solution), the assignment converges to:

$X_1 = 462$ $X_2 = 538$

This results in TNTC = 18406 [UE: TNTC = 18000]

The same example may then be used to examine the effect of replacing the unit cost/flow functions with marginal cost/flow functions, and it follows that:

$m_1 = 10 + 0.04x_1$ $m_2 = 15 + 0.01x_2$ and $x_1 + x_2 = 1000$.

Therefore when the unit link-cost/flow functions are replaced by marginal link-cost/flow functions under SUE then the probability of assigning flow to path1 will be

$$p_1 = \frac{1}{1 + \exp\theta(m_1 - m_2)} \qquad 5.4$$

This may be solved as before, resulting in convergence for $\theta = 0.1$ at

$x_1 = 389.7$ $\qquad x_2 = 610.3$

which gives $\qquad$ TNTC = 17951 $\qquad$ (SO: TNTC = 17750)

This is summarised in table 5.1

	x_1	x_2	TNTC
UE	400	600	18000
SO	300	700	17750
SUE $\theta = 0.1$	462	538	18406
MSCP $\theta = 0.1$	390	610	17951

Table 5.1: Comparison of flow patterns: 2-link network

It is clear that the imposition of marginal social cost price tolling does not result in the 'actual' TNTC being minimised, which occurs at the SO flow pattern.

The SUE TNTC value, is higher than the UE value, the SUE solution tending to the UE solution as θ tends to infinity in the case of logit SUE (or $\beta \rightarrow 0$ for probit). Similarly the MSCP-tolling TNTC value is higher than the SO value, and this solution tends to SO as θ tends to infinity. As this is the case, it seems likely that MSCP tolling does indeed produce the SSO where perceived costs are minimised, as in the limiting case perceived costs are tending to 'actual' costs. In this case however, whilst not (except in the limit) producing a TNTC as small as that for SO, this solution does represent a reduction in TNTC from both the UE and SUE values.

The effect of TNTC as θ varies may be seen in figure 5.3 below. [NB as $\theta \rightarrow 0$, the path split tends to 50:50, and TNTC→18750 for SUE with both unit costs and marginal social costs.]

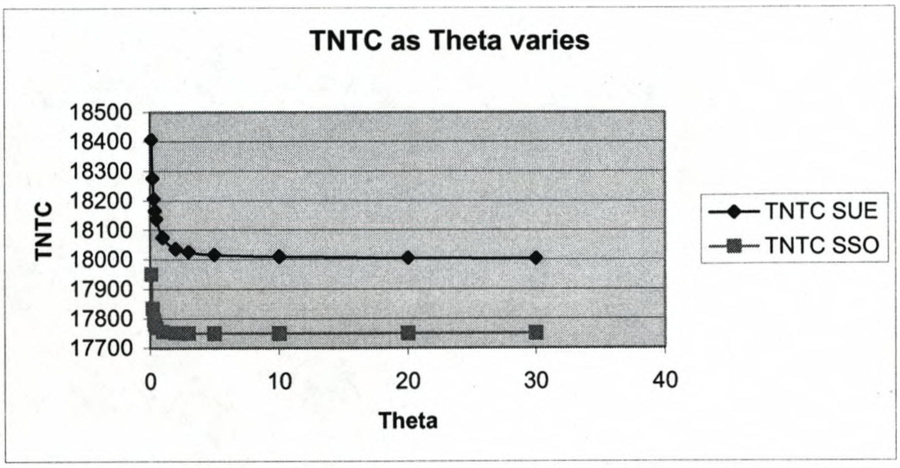


Figure 5.3: Stochastic tending to the deterministic

This section has illustrated (by the use of the simple two-link example) that in the stochastic case TNTC is not minimised by marginal social cost pricing. Minimisation of the Total Perceived Network Travel Cost (TPNTC) is the subject of section 5.3.

5.3 Examining Minimisation of Total Perceived Network Travel Cost

Whilst it is well understood how to minimise the TNTC under deterministic assignment, to understand the concept in the analogous stochastic case it is helpful to first examine a small discrete case. The principles of stochastic assignment (whichever method is being used) are based upon the drivers' perceptions of costs following a distribution (as illustrated in figure 5.4 using the normal distribution).

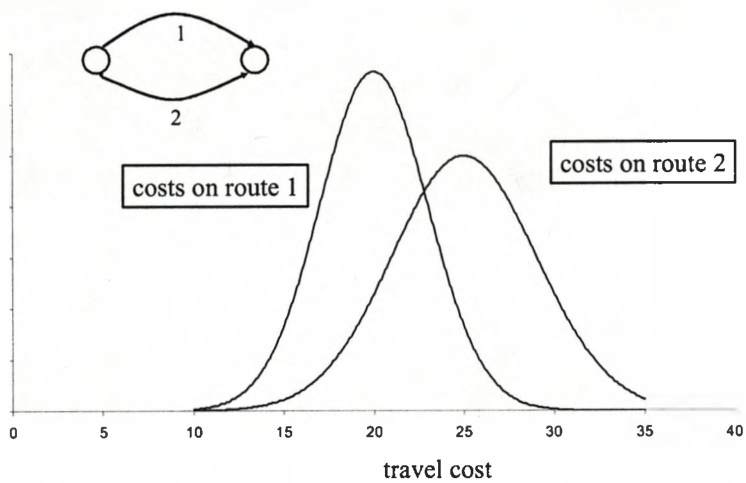


Figure 5.4: Distribution of costs under Stochastic Assignment for 2-link network

The travel costs on link 1 are assumed to have a lower 'actual' cost than those for link 2, but a narrower range of perception of those costs. Thus some drivers travelling on link 2 will perceive that their travel costs are smaller. If a sample of potential link travel times are picked from 2 such distributions, we may examine and compare the perceived costs on each link, as in figure 5.5.

Perceived costs

route 1	route 2
24	27
32	30
27	28
29	25
32	31
25	26
27	30

Perceived costs

route 1	route 2
24	27
32	30
27	28
29	25
32	31
25	26
27	30

Figure 5.5: Perceived costs and travel decisions on a two link network.

Drivers will each have their own perception of the travel costs on each of the links, and will then make their route choice decision based upon which route they perceive to be cheapest. Out of the sample of seven drivers in figure 5.5, 4 perceive that route 1 would be cheaper and 3 choose route 2 under Stochastic Assignment. For a different desired flow pattern, we might want to insist that only 2 drivers are allocated to route one and 5 to route 2. If we wished to create such a flow pattern we must create some decision rule whereby drivers are re-routed. It seems reasonable to rank them in relation to how great the difference in perceived costs for each link is for each motorist, as if a toll is to be imposed to create the desired flow pattern, the drivers who will re-route are the ones for whom that imposed toll has altered which route they perceive to be cheaper. The difference in perceived costs are ranked in figure 5.6. The two drivers who perceive route 1 to be most cheaper than route 2 will not be rerouted, but 2 drivers who perceive route 1 to be only just cheaper than route 2 could be persuaded to change their routing decision by imposing a toll (such that 1< toll <3) upon route 1, so that the 'desired' flow pattern of 5 drivers on route 2 could be achieved.

Perceived costs		
route 1	**route 2**	**diffs**
24	27	-3
27	30	-3
25	26	-1
27	28	-1
32	31	+1
32	31	+2
29	25	+4

Figure 5.6: Perceived cost differences

The 'desired flow pattern' above though was completely arbitrary and if it is desired to minimise perceived network travel cost then some way of obtaining this flow pattern must first be derived.

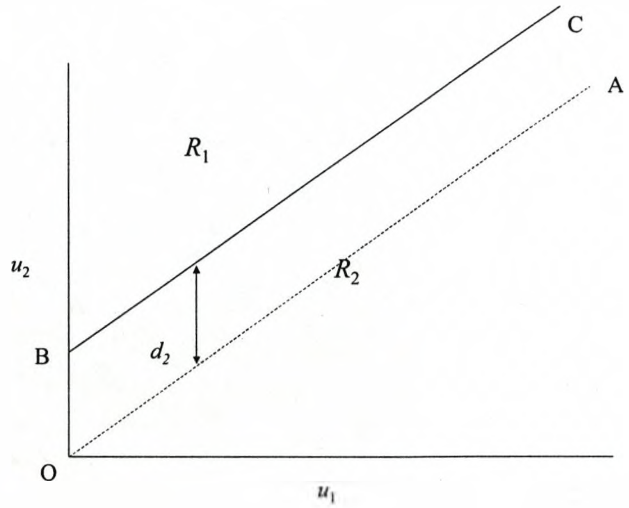


Figure 5.7: sample space of perception errors u_1, u_2 divided into regions R_1 and R_2 (Maher et al, 2005)

Considering a two-path network, with a single OD pair and fixed demand q, then the path-flows may be denoted h_1 and h_2 ($h_1 + h_2 = q$) and the perceived path costs u_1 and u_2, with a probability density function $f(u_1, u_2)$. Given the path flows it is necessary to derive a rule to assign drivers to paths in order to minimise total perceived costs, and this will require some drivers to travel by a path which is not their cheapest perceived path. In figure 5.7 the line OA divides the space into two regions; drivers whose perceived costs lie in region R_1 would be assigned to path 1 and those in region R_2 to path 2. The line BC has equation $u_2 = u_1 + d_2$ and the value of d_2 is that where the probability mass in R_1 is $p_1 = h_1/q$). Those drivers with perceived costs lying between lines BC and OA will be those who will be required to re-route away from their smallest perceived cost path for the good of the system.

It is required therefore to find d_2 such that;

$$\int_{R_1} f(u_1, u_2)\, du_1\, du_2 = p_1 = \frac{h_1}{q} \qquad 5.5$$

and at such an assignment the total perceived costs will be;

$$z(h_1,h_2)=q\left(\int_{R_1} u_1 f(u_1,u_2)\,du_1 du_2 + \int_{R_2} u_2 f(u_1,u_2)\,du_1 du_2\right) \qquad 5.6$$

It is shown in Maher et al (2005), that for paths costs c_1 and c_2, this may be written as;

$$z(h_1,h_2)=q\big(S(c_1,c_2-d_2)+d_2 p_2\big)=qS(c_1,c_2-d_2)+h_2 d_2 \qquad 5.7$$

Where S denotes the 'satisfaction' which in the logit case is given as;

$$S(c_1,c_2)=-\frac{1}{\theta}\log(\exp(-\theta c_1)+\exp(-\theta c_2)). \qquad 5.8$$

The SSO is then defined as the flow pattern which minimises the objective function for the perceived total network travel cost (PTNTC) in (5.7).

This may be illustrated numerically using the two link network as used in Chapter 4, and reproduced here in figure 5.8. Again logit SUE is used, with a value of the sensitivity parameter $\theta = 0.1$. Since u_1 and u_2 are Gumbel distributed with means c_1 and c_2, the proportion p_1 of drivers for whom $u_1 < u_2 - d_2$ is the same as the proportion for whom $u_1 < u_2$ when the means are c_1 and $c_2 - d_2$.

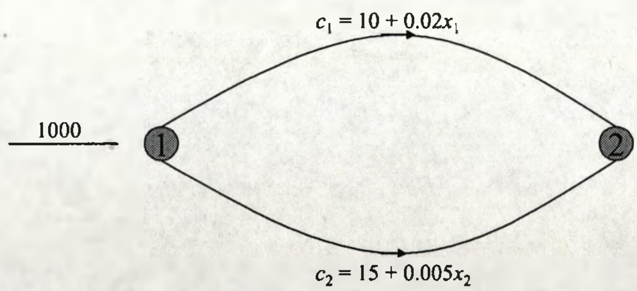


Figure 5.8: 2-link network

Hence;

$$p_1=\frac{\exp(-\theta c_1)}{\exp(-\theta c_1)+\exp(-\theta(c_2-d_2))}. \qquad 5.9$$

and so that the value of d_2 required to give the correct probability mass $p_1 = h_1/q$ is:

$$d_2 = -\frac{1}{\theta}\log\left(\frac{h_1}{h_2}\right) - c_1 + c_2 . \tag{5.10}$$

And so the SSO objective function in this case is;

$$z(h_1,h_2) = -\frac{q}{\theta}\log(\exp(-\theta c_1(h_1)) + \exp(-\theta(c_2(h_2) - d_2))$$

$$-\left(\frac{1}{\theta}\log\left(\frac{h_1}{h_2}\right) - c_1(h_1) + c_2(h_2)\right)h_2 \tag{5.11}$$

In this 2-link example, the value of this SSO objective function may be readily plotted against only h_1 (as $h_1 + h_2 = q$). For comparative purposes, the plots of the UE, SO and SUE objective functions against h_1 are also shown (figure 5.9). The positions of the minima show that the solution for UE is $\boldsymbol{h}$ = (400, 600), that for SO is $\boldsymbol{h}$ = (300, 700), that for SUE is $\boldsymbol{h}$ = (462, 538) and that for SSO is $\boldsymbol{h}$ = (390, 610). (This SSO solution being the same as the SUE solution that is obtained by replacing the unit cost-flow functions by the marginal social cost-flow functions as presented in section 5.2).

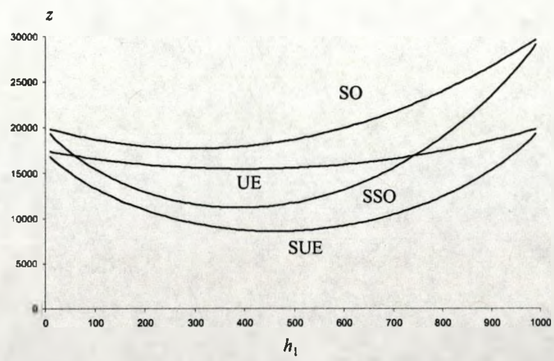


Figure 5.9: plots of z_{SSO}, z_{SO}, z_{UE} and z_{SUE} against h_1, the flow on path 1. (Maher et al, 2005).

It is further possible to plot the SSO objective function (figure 5.10) for several values of the sensitivity parameter θ (= 0.1, 0.25, 0.5, 1 and 2) and it can be seen that as θ increases, the plot of the SSO objective function steadily approaches that of the SO objective function, as would be expected, since the degree of variation in the perceived costs is steadily reducing towards zero. This also corresponds to applying marginal social cost functions for varying values of the sensitivity parameter.

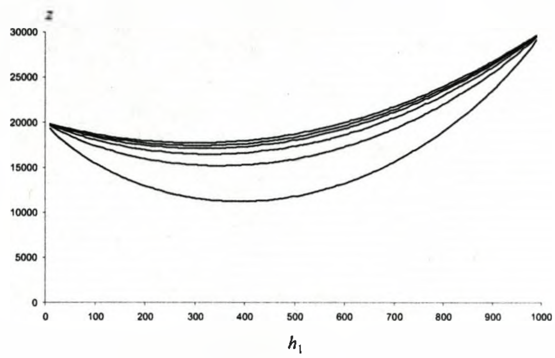


Figure 5.10: (from bottom to top) plots of z_{SSO} for values of θ = 0.1, 0.25, 0.5, 1 and 2, and z_{SO}, against h_1, the flow on path 1 (Maher et al, 2005).

Clearly in the general case, there are rather more possibilities than merely two paths to choose between, and while it was relatively clear in the two-link case to imagine which drivers would re-route to the SSO flow pattern under the imposition of a toll, this is not so apparent in the more general case. For simplicity, the discrete example as illustrated in figs 5.5-5.6 above, will be used again, but with the addition of an extra path, so that the choice is between 3 paths rather than just two. Where as in the 2-path case, if a driver is being rerouted, they must reroute to the only other available path, it is not so clear when there are more links which drivers would have to re-route, and then which alternative path they would re-route to. This may be seen in figure 5.11.

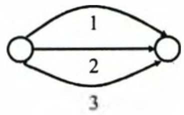


Figure 5.11: 3-link network

Perceived costs

	route 1	route 2	route 3
	28	34	31
	23	26	32
	29	27	35
	31	29	28
	23	27	32
	33	29	30
	31	27	34
numbers required	2	2	3

If the desired flow pattern was known to be 2 cars on each of routes 1 and 2, and 3 on route 3, it is not as obvious as in the 2-link case which drivers should re-route. It might be best for the drivers who have least difference in perceived cost to move straight to the most underloaded link, or it might be better for two drivers on route 2 to move to route 3, and a driver on route 1 to be rerouted to route 2.

So the question is how should users be assigned to routes so as to minimise the total perceived costs (or to achieve any desired flow pattern) whilst satisfying the constraint that the desired flow split should be achieved (correct column totals)? In addition each driver may only travel on one link (row totals must equal unity).

This discrete version of the problem may be seen to be a special case of the 'classical transportation problem' which is well known (e.g. Taha, 1976) and for which solution algorithms exist (e.g. stepping stone algorithm). In the above example there are N (=7) drivers who must choose between J (=3) routes, and in a basic solution to the problem exactly $(N + J - 1)$ of the NJ cells must be used (9 out of 21). In this special case all N drivers must choose exactly one route so that N cells (1 per row) will take a value of 1, the other $J - 1$ cells required to be in the basic solution taking a value of zero. The optimal solution is a basic solution such that no other cell could be brought into the basis (in exchange for a current basis cell) and reduce the value of the objective function. The objective in this case being the minimisation of the Total Perceived Travel Cost. The optimal solution (4-Iterations of stepping stone algorithm) is shown in figure 5.12.

Optimal Basic Solution

route 1	route 2	route 3
-	-	1
1	-	-
0	1	-
-	-	1
1	-	-
-	-	1
-	1	0
2	2	3

Perceived costs

route 1	route 2	route 3
28	34	31
23	26	32
29	27	35
31	29	28
23	27	32
33	29	30
31	27	34
2	2	3

Figure 5.12: 3-link network: Optimal basis for 2:2:3 split.

It is known that (for the general transportation problem) that for any iteration the following conditions are true for any basic solution;

there exist values α_i ($i = 1,\ldots\ldots N$) and β_j ($j = 1,\ldots\ldots J$), such that for any basic cell,

$u_{ij} = \alpha_i + \beta_j$ where (in this case) the u_{ij} are the perceived route costs,

and for each non basic cell,

$v_{ij} = u_{ij} - \alpha_i - \beta_j$ where at an optimal solution the v_{ij} are all non-negative.

At an optimal assignment in any row a user i must be assigned to a route j such that the route has a value of 1 in the basic solution. Thus $\alpha_i = u_{ij} - \beta_j$ and for any other path k, $v_{ik} \geq 0$ and so $u_{kj} - \alpha_i - \beta_k \geq 0$. So $u_{kj} - \beta_k \geq u_{ij} - \beta_j$ for all k paths in a row which are not in the optimal solution, hence the optimal solution is that where each user is assigned to that path where $(u_{ij} - \beta_j)$ has a minimum value. Extending from the 2-link example such that the β_j represent the d_j (and the u_j the c_j), the decision rule in this case will therefore be to calculate "reduced costs" $(c_1 - d_1)$, $(c_2 - d_2)$, $(c_3 - d_3)$ and assign drivers to the route that has minimum reduced cost. In this example the required values are found to be; $d_1 = 0$, $d_2 = -2$, $d_3 = +5$, and the solution is as given in figure 5.13.

It may be seen that only 2 drivers would need to be induced to move by tolling to create the desired 2:2:3 flow pattern and that if the table of reduced costs ($c_j - d_j$) is examined then it may be seen that the path assigned to each user corresponds to their minimum reduced cost (or equal minimum).

$(c_j - d_j)$

route 1	route 2	route 3
28	36	26
23	28	27
29	29	30
31	31	23
23	29	27
33	31	25
31	29	29

Perceived costs

route 1	route 2	route 3
28	34	31
23	26	32
29	27	35
31	29	28
23	27	32
33	29	30
31	27	34

Figure 5.13: 3-link network: Optimal 2:2:3 split.

This principle may be readily generalised in the discrete case to and number of paths and users (Maher et al, 2005). Since it is possible to make the discrete case as close as may be desired to the underlying continuous case, by making the number of rows (users) as large as desired, it may be deduced that the same result applies in the continuous case: that is, to find the optimal assignment of users to paths such that the proportions so assigned should be constrained to take the values h_j/q, we need to determine values d_j such that each user is assigned to that path j for which their value of $(u_j - d_j)$ is minimum.

5.4 Definition of SSO

In the general continuous case, the probability space for perceived path costs must be divided into regions as indicated in figure 5.14 as an extension of the 2-link continuous example.

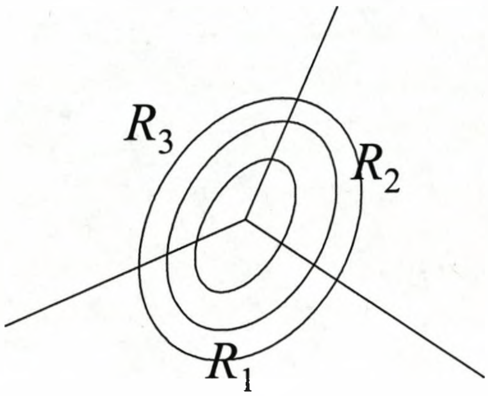


Figure 5.14: probability space divided into regions.

Given a (multivariate) distribution of drivers' perceived path costs $f(c_1, c_2, ..)$, in which the means depend on the path flows h_1, h_2,..etc, how should drivers be assigned to paths, by dividing the space into regions? Firstly the flows (h_1, h_2, ...) must be of a specified size to produce the SSO flow pattern, and then the flows must be optimised.

This may be achieved by extending the principles in the continuous case from those given for the 2-link example; the full derivation for which is given in Maher et al (2005), resulting in the following where the whole space of perceived costs $\boldsymbol{u}$ is to be partitioned.

Firstly the number of paths are increased, extending (5.5) such that:

$$\int_{R_j} f(u_1, u_2, ...)\, du_1\, du_2 ...... = p_j = \frac{h_j}{q} \qquad 5.12$$

where users are assigned to paths to minimise their perceived travel costs.

Extending (5.6) the total perceived cost is:

$$z(h_1, h_2, ...) = q \sum_j \int_{R_j} u_j f(u_1, u_2, ...) du_1 du_2 ..... \quad 5.13$$

which, setting $w_j = u_j - d_j$ and denoting by C_j the set of perceived path costs for which path j is the optimum gives:

$$z(h_1, h_2, ...) = q \sum_j \int_{C_j} (w_j + d_j) f(w_1, w_2 ...; c_1 - d_1, c_2 - d_2, ...) dw_1 dw_2$$

$$= q \sum_j \int_{C_j} w_j f(w_1, w_2, ...; c_1 - d_1, c_2 - d_2, ...) dw_1 dw_2 .. + q \sum_j d_j p_j \quad 5.14$$

$$= \; qS(c_1 - d_1, c_2 - d_2, ...) + \sum_j d_j h_j$$

then allowing for multiple OD-pairs r-s.

$$z_{SSO}(\mathbf{h}) = \sum_{rs} q_{rs} S^{rs}(\mathbf{c}^{rs} - \mathbf{d}^{rs}) + \sum_{rs} \sum_j d_j^{rs} h_j^{rs} \quad 5.15$$

It should be noted that in the derivation it can easily be seen that the condition for the SSO solution is that the mean travel cost on any path between a given OD pair, is the marginal cost for that path, as was expected to be the case.

The objective function for the special case of logit SUE is presented below with the well known SUE objective function for comparative purposes. For completeness the deterministic UE and SO objective functions are also included, and a pleasing symmetry may be observed.

$$z_{SSO}(\mathbf{h}) = \sum_{rs} \sum_j h_j^{rs} c_j^{rs}(\mathbf{h}) \; + \frac{1}{\theta} \sum_{rs} \sum_j h_j^{rs} \log h_j^{rs} \; . \quad 5.16$$

$$z_{SUE}(\mathbf{h}) = \sum_a \int_0^{x_a} t_a(\omega) d\omega + \frac{1}{\theta} \sum_{rs} \sum_j h_j^{rs} \log h_j^{rs} \; . \quad 5.17$$

$$z_{SO}(\mathbf{h}) = \sum_{rs} \sum_j h_j^{rs} c_j^{rs}(\mathbf{h}) = \sum_a x_a t_a(x_a) \, . \quad 5.18$$

$$z_{UE}(\mathbf{x}) = \sum_a \int_0^{x_a} t_a(\omega) d\omega \, . \quad 5.19$$

Yang (1999), comments that the 'true' SO may be restrictive in terms of an economic context and argues that it is necessary in determining a desired flow pattern to also require 'economic reasonableness' which should be (in Yang's opinion) based on externalities. Yang also shows that MSCP tolls are economically meaningful in an SUE context. Yang's paper is formulated upon logit-based stochastic assignment and presents known results from Williams (1977) showing that indirect utility may be expressed as the satisfaction function, which is related to consumer surplus. He further shows that direct utility of a representative consumer (for aggregate demand) may be expressed according to representative consumer theory (Oppenheim, 1995) as;

$$U = -\frac{1}{\theta}\sum_{rs}\sum_{j} h_j^{rs} \log h_j^{rs} + T_0 \qquad 5.20$$

where U is the direct utility function and T_0 a constant relating to the cost of non-travel items. The total social cost of all the travellers throughout the network is then;

$$TC = \sum_{rs}\sum_{j} h_j^{rs} c_j^{rs}(\mathbf{h}) \ . \qquad 5.21$$

and the net economic benefit would be described as the "traveller's benefit – total transportation cost" ($U - TC$), where for socially optimal pricing this quantity would be maximised. It may readily be seen that this will be equivalent to the objective function 5.16 being minimised (the constant being deleted). Thus in the logit case, the approach of Maher et al (2005) gives an equivalent result to the economic derivation in Yang (1999), but approaching the problem from the perspective of minimising total perceived travel cost rather than maximising economic benefit. Yang (1999) also shows that marginal social cost pricing provides a solution to the optimisation programme (in the logit case), which is also derived in Maher et al (2005) for Stochastic Social Optimisation under general Stochastic Assignment.

As it has been established (Maher et al, 2005) that at the SSO solution, a stochastic loading using the marginal path costs produces an auxiliary flow pattern that is identical to the current flow pattern, this means that a link-based objective function for the general, utility-maximising case may be written down.

Sheffi and Powell (1982) showed that the general SUE problem is equivalent to the unconstrained minimisation, with respect to the link flows **x** of the following objective function:

$$z_{SUE}(\mathbf{x}) = -\sum_a \int_0^{x_a} t_a(u)du + \sum_a x_a t_a(x_a) - \sum_{rs} q_{rs} S_{rs}[\mathbf{t}(\mathbf{x})]. \qquad 5.22$$

where t_a is the travel cost on link a, and S_{rs} is the "satisfaction" function, the expected value of the minimum perceived travel cost for users travelling between OD pair rs. For individuals the second two terms will represent consumer surplus (Connors et al, 2005) and the first term represents total benefits (Verhoef, 2002b) under the general utility maximising assumptions (underlying either probit or logit assignment). (Sheffi and Powell (1982) further showed that, under the conditions set out earlier for the link performance functions, the uniqueness of the SUE solution is guaranteed. Sheffi (1985) showed that the derivative of this function with respect to a link flow is given by:

$$\frac{\partial z_{SUE}}{\partial x_a} = (x_a - y_a)\frac{dt_a(x_a)}{dx_a} \qquad 5.23$$

It follows that at the SUE solution, the auxiliary flows $\{y_a\}$ are equal to the current flows $\{x_a\}$. That is, when a stochastic loading is carried out using mean link costs based on the current link flows, the resulting auxiliary flow pattern is identical to the current flow pattern:

$$y_a(\mathbf{t}(\mathbf{x})) = x_a \quad \forall\, a \qquad 5.24$$

Maher et al (2005) showed that at the SSO solution, if the marginal costs are used in place of the unit costs, the auxiliary flow pattern produced from a stochastic loading is equal to the current solution. It follows then that merely replacing the unit link costs t_a

in the Sheffi and Powell objective function for SUE by the marginal link costs $m_a = t_a + x_a\, dt_a/dx_a$ will give an objective function for SSO:

$$z_{SSO}(\mathbf{x}) = -\sum_a \int_0^{x_a} m_a(u)du + \sum_a x_a m_a(x_a) - \sum_{rs} q_{rs} S_{rs}[\mathbf{m}(\mathbf{x})] \quad 5.25$$

At the SSO solution, it is therefore the case that:

$$y_a(\mathbf{m}(\mathbf{x})) = x_a \quad \forall\, a \quad 5.26$$

The SSO solution thus maximising social economic benefit for the system, opposed to the SUE maximising economic benefit for the individual). Thus in the remainder of this chapter, marginal social costs are used in SUE solution algorithms to obtain SSO flow patterns.

This section has presented the formalisation of the SSO (as per Maher et al, 2005) as a generalisation of the discrete case illustrated in section 5.3. The SSO has been defined as the flow pattern where the Total Perceived Network Travel cost is minimised and it has been shown that this flow pattern will be created under a standard SUE assignment where unit-cost functions are replaced by marginal-cost functions. Thus the SSO in the stochastic case is completely analogous to the SO in the deterministic case.

5.5 Tolling 'Exact' Min-Rev tolls: Path Enumeration

A Stochastic Social Optimum (SSO) was formulated such that at the SSO solution, the total of the users' perceived costs is minimised. It was shown that under a general utility-maximising framework that includes the two most important cases of logit and probit loading, the augmented path costs at the SSO solution were the marginal social costs, and hence the relationship of the SSO solution to the SUE is the same as that of SO to UE. In particular, the SSO solution can be found by means of an SUE algorithm, by replacing the standard path costs by the marginal social path costs in the stochastic loading; and the same form of toll set that is optimal in the deterministic case is optimal in the stochastic case. Thus MSCP tolls may be easily found using existing link-based assignment methods.

The minimal revenue toll problem is thus similar to that in the deterministic case, differing though in that all used paths will clearly not have a common cost, and so existing techniques may not be utilised directly.

Whilst mathematical programming methods may be used to derive minimal revenue tolls in the stochastic case (Maher et al, 2005), such methods could be costly in computational terms, and further if several equally optimal solutions exist (as for the deterministic case) methods where the links to be left untolled may be considered throughout the solution procedure may be desirable. Initially the problem was examined algebraically by use of the previously used 2-link network.

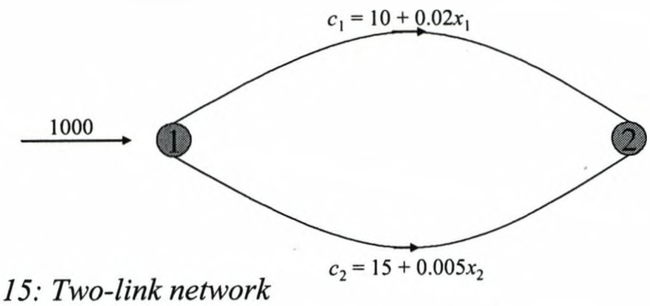


Figure 5.15: Two-link network

Again using logit SUE with $\theta = 0.1$, the SSO flow pattern may be obtained by use of marginal cost functions in the stochastic assignment.

$x_1 = 389.7$ $\qquad$ $x_2 = 610.3$

$m_1 = 10 + 0.04x_1$ $\qquad$ $m_2 = 15 + 0.01x_2$

$c_1 = 10 + 0.02x_1$ $\qquad$ $c_2 = 15 + 0.005x_2$

$t_1 = m_1 - c_1 = 0.02x_1 = 7.794$ $\qquad$ $t_2 = m_2 - c_2 = 0.005x_2 = 3.0515$

If the original cost flow functions are amended such that $c_i^{mscpt} = c_i + t_i$

$c_1^{mscpt} = 17.8 + 0.02x_1$ $\qquad$ $c_2^{mscpt} = 18.05 + 0.005x_2$

These cost flow functions will indeed produce the SSO flow pattern when an SUE assignment is performed. Alternatively the minimal revenue tolls in this case would be a single toll of 4.74 on link 1 which is easily observed due to the simple structure of this network. As θ varies, so clearly do the tolls, as $\theta \rightarrow$ infinity, and SUE $\rightarrow$ UE, the toll values for MSCP tolls and Min-Rev tolls tend to their respective values under a deterministic assignment.

In this two link example, as links are also paths it is trivial to calculate minimal revenue tolls. To examine some of the issues in attempting to derive such tolls a more complex network structure is required, and thus the simple 5 link example is now used again.

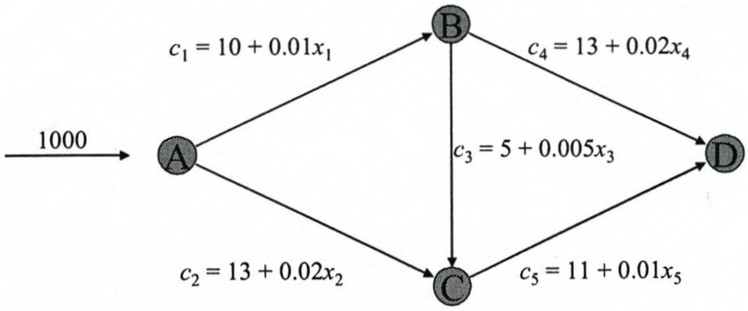


Figure 5.16: Five-link network

The path and link structure of this network has been stated in chapter 4, where the fact that link and path data could be easily derived given one or the other was noted.

Path information was required here as the Stochastic assignment was carried out using the logistic function directly. It would have been possible, of course to use Dial's Stoch algorithm, which removes the need to deduce path data, but path data is in fact utilised to derive the minimal revenue tolls in this section.

The resulting flow patterns and TNTC for UE, SO and SUE and SSO ($\theta = 0.1$) are given in table 5.2 (all entries to nwn).

	x_1	x_2	x_3	x_4	x_5	TNTC
UE	579	421	125	454	546	37875
SO	602	378	188	415	585	37793
SUE $\theta = 0.1$	645	355	272	373	627	37933
SSO $\theta = 0.1$	639	361	267	372	628	37919

Table 5.2: Comparative flow patterns

It can be seen that in this case not only does the SSO solution not reduce the TNTC to the SO value, it does not even reduce it below the UE value, and in fact the SSO flow pattern is not very different to that for SUE, making it questionable as to whether the imposition of MSCP tolls would be worthwhile. In general (Yang 1999), in fact it is not necessarily true that MSCP tolls applied under SUE will reduce the TNTC at all, in some cases the TNTC can actually be increased. Thus the reason for imposing tolls must be considered. If the objective is to cause drivers to pay an equitable amount for the externalities their travel causes then the imposition of MSCP tolls could be appropriate, regardless of the effect to 'actual' TNTC. It could also be considered that as it is the drivers 'perceived' costs that are used in the stochastic case, then it should also be the 'perceived total travel cost' which is to be minimised.

This questions the relevance of attempting to seek minimal revenue tolls for the SSO, if economic benefit maximisation would be the main purpose of attempting to achieve the SSO flow pattern. However it is of interest to attempt to achieve such tolls for comparative purposes.

The MSCP tolls that would force an SUE assignment to produce the 'SSO' flow in this case would be:

t_1	t_2	t_3	t_4	t_5
6.39	7.22	1.33	7.45	6.27

Table 5.3: MSCP link-tolls

In the 2-link example, it was trivial to derive minimal cost tolls, as the path tolls were compared and the lower one set to zero. In the 5-link network there are 3 distinct paths such that;

$$T_1 = t_1 + t_4$$

$$T_2 = t_1 + t_3 + t_5$$

$$T_3 = t_2 + t_5$$

so from table 5.3 the MSCP path tolls are;

$$T_1 = 13.84$$

$$T_2 = 13.96$$

$$T_3 = 13.49$$

Clearly toll path 3 has the smallest toll value and should be set to zero for minimal revenue tolls, so that;

Min-Rev $T_1 = 0.35 \quad \Rightarrow \quad t_1 + t_4 = 0.35$

Min-Rev $T_2 = 0.47 \quad \Rightarrow \quad t_1 + t_3 + t_5 = 0.47$

Min-Rev $T_3 = 0 \quad \Rightarrow \quad t_2 + t_5 = 0$

Clearly then, $t_2 = t_5 = 0$

$t_1 + t_3 = 0.47$

$t_1 + t_4 = 0.35$

Thus the system is underdetermined and there are a number of feasible equally optimal toll sets satisfying the above. If it is desired also to maximise the number of zero-tolled links, then there are two feasible Min-Rev link-toll solutions:

t_1	t_2	t_3	t_4	t_5
0	0	0.47	0.35	0
0.35	0	0.12	0	0

Table 5.4: Equally optimal Min-Rev tolls

In a larger network the number of possible equally optimal solutions whilst maximising zero-toll links might be greater, and the use of a standard linear programming solver may well not illustrate these as easily as direct methods.

Thus if path enumeration is feasible, as it is for the 2 and 5-link examples above and the 9-node network used previously, then Min-Rev toll sets may be determined from the MSCP toll sets. If a link-toll set exists (i.e. MSCP) the path-enumeration matrix may be used to obtain the corresponding path-toll set. A zero-toll tree should then be derived such that the path with the smallest MSCP toll for each OD-pair is given a zero-toll, and the other paths have their toll reduced by the same amount. The remaining link-tolls may be derived from the Min-Rev path toll set by matrix algebra, using a reduced path-enumeration matrix, where the zero-toll paths are removed. The zero-toll trees thus obtained will depend on the value of the sensitivity parameter in both logit and probit cases. This is further illustrated using the 9-node network.

<u>9-node example: Toll trees:</u>

Logistic assignment has again been used in this example, but the method used is equally applicable to other stochastic models. The MSCP-tolls for SSO with θ = 1 for Bergendorff's network are shown in figure 5.16

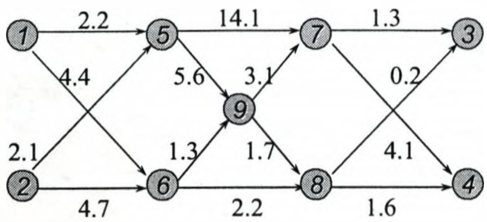


Figure 5.17: Bergendorff's 9-node network

The MSCP-link tolls and corresponding path tolls, and the related Min-Rev path tolls are shown in table 5.5. [Order of links and paths are defined as in the path enumeration matrix figure 4.8].   MSCP Revenue = 1185.5   Minimal Revenue = 401.8

Link	1-5	5-7	7-3	1-6	2-5	5-9	9-7	6-9	9-8	7-4	8-3	2-6	6-8	8-4
MSCP-t	2.2	14.1	1.3	4.4	2.1	5.6	3.1	1.3	1.7	4.1	0.2	4.8	2.2	1.6

OD pair	[1,3]						[1,4]					
MSCP-T	17.6	12.2	9.6	10.0	7.4	6.7	20.4	14.9	11.1	12.8	8.9	8.2
Min-REV-T	10.9	5.5	2.9	3.3	0.8	0.0	12.2	6.8	2.9	4.6	0.8	0.0
OD pair	[2,3]						[2,4]					
MSCP-T	17.5	12.0	9.4	10.4	7.8	7.1	20.2	14.8	10.9	13.1	9.3	8.5
Min-REV-T	10.4	5.0	2.4	3.3	0.8	0.0	11.7	6.2	2.4	4.6	0.8	0.0

Table 5.5: 9-node network SSO link and path tolls (θ = 0.1)

It may be seen that the zero-tolled path for each OD pair may be easily observed as that with the smallest MSCP path-toll. Zero-toll trees may consequently be constructed. [Data for different values of theta is included in Appendix A.3] The zero-toll trees for varying values of the parameter θ are shown in Figure 5.18; the zero-toll links being represented by the bold print arrows. As θ increases the driver's assumed perceived knowledge of network costs increases, so that as θ tends to infinity, the logit stochastic assignment tends towards a deterministic assignment. The final zero-toll tree (θ=5) is indeed the same as that obtained by deterministic methods.

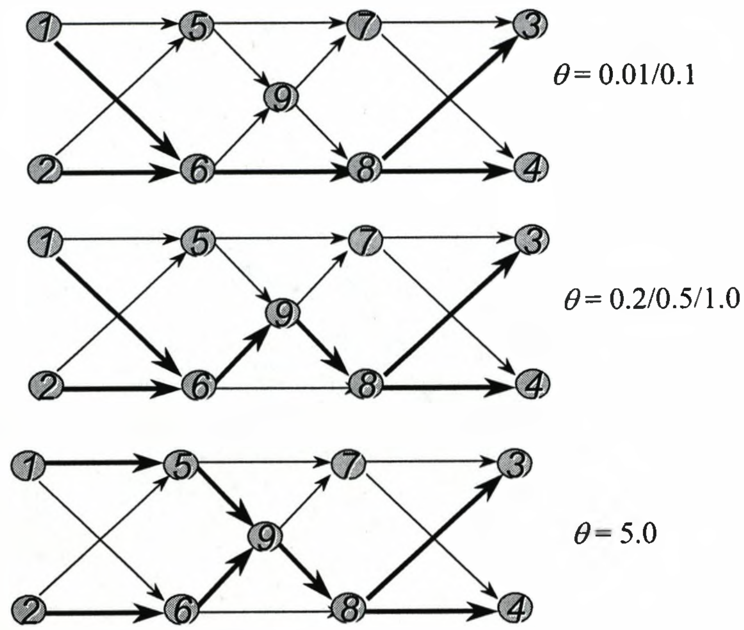


Figure 5.18 – Zero-toll trees as θ varies

In determining minimal revenue toll sets, if there are a greater number of viable paths than there are links (as will generally be the case), the minimal revenue path-toll vector (which is easily determined), together with the path enumeration matrix will result in an over-determined system, which is why a viable path-toll set will not necessarily yield consistent link-tolls (as demonstrated in section 4.3.3). However in this the SSO case, the system of equations in fact reduces to one which is underdetermined and so various equally optimal link-toll solutions exist and further optimisation may be possible if desired. In particular it is of interest to obtain as many links with zero-toll as possible, but even with this provision, in this example there were four equally optimal toll sets for each value of θ. A possible Min-Rev toll-set is given in Table 5.6 for various θ.

The links corresponding to the zero-toll trees are highlighted. Other zero-toll links may be observed (and are unshaded in figure 5.6), although it must be remembered that there are other equally optimal solutions which are not shown. Despite the existence of three distinct zero-toll trees for varying values of the sensitivity parameter, the change in individual link-toll values as θ varies appears to be reasonably smooth. For example as

θ increases from 0.1 to 0.2 the zero-toll tree replaces link 6-8 with links 6-9-8. The toll on link 6-8 whilst zero for θ = 0.01/0.1, then increases gradually as θ increases. Similarly the toll on link 6-9 decreases before being part of the zero-toll tree for larger values of θ. Thus while the change in toll-tree to exchange certain links with others must always be a discontinuous jump, the actual toll change along a link is gradually increasing or diminishing.

Toll \ θ	0.01	0.1	0.2	0.5	1	5	Deterministic
t1 (1-5)	12.8	2.9	1.0	0.2	0.1	0.0	0.0
t2 (5-7)	0.0	5.4	6.9	7.7	7.9	8.0	8
t3 (7-3)	0.1	2.6	3.7	5.1	5.9	6.9	7.2
t4 (1-6)	0.0	0.0	0.0	0.0	0.0	0.0	0
t5 (2-5)	6.0	2.4	2.2	2.9	3.4	3.9	4
t6 (5-9)	5.3	0.0	0.0	0.0	0.0	0.0	0.0
t7 (9-7)	0.0	0.0	0.0	0.0	0.0	0.0	0
t8 (6-9)	9.1	0.7	0.0	0.0	0.0	0.0	0.0
t9 (9-8)	0.0	0.0	0.0	0.0	0.0	0.0	0.0
t10 (7-4)	6.4	3.9	3.3	3.1	3.1	3.2	3.2
t11 (8-3)	0.0	0.0	0.0	0.0	0.0	0.0	0.0
t12 (2-6)	0.0	0.0	0.0	0.0	0.0	0.0	0.0
t13 (6-8)	0.0	0.0	1.9	4.4	5.6	6.8	7.2
t14 (8-4)	0.0	0.0	0.0	0.0	0.0	0.0	0

Table 5.6: Minimal Revenue Toll Sets as θ varies

This section has considered the determination of exact Minimal Revenue tolls in the case of tolling under stochastic user equilibrium methods to achieve the stochastic system optimal. Marginal social cost price (mscp) tolls are easy to determine and may be used as a starting point in the determination of minimal revenue tolls. If mscp link tolls are calculated using standard assignment techniques mscp tolls may be determined uniquely by path enumeration. The path with the smallest toll may be easily noted for each OD pair and set to zero for minimal revenue tolls (the others being determined by maintaining the difference between tolls as for mscp). Path-based minimal revenue tolls may then be easily found, but to determine link tolls some linear algebra or mathematical programming would be required.

5.6 Heuristic solution of SSO

Whilst for the 9-node toy-network that has been used for illustrative purposes may be readily solved for Min-Rev SSO tolls by either linear programming or path-enumeration based methods, link-based methods have generally been preferred to those which require full path enumeration and so if such tolls are to be derived then it would be preferable to utilise link-based methods. The heuristic presented in chapter 4 was developed to seek any desired flow pattern by tolling under SUE assignment. It would seem feasible then to seek low-rev tolls to achieve the SSO (or to closely approach the SSO) by use of the same heuristic, where flows and costs at SO are replaced with flows and costs at SSO. This has been illustrated using the same 9-node network as in the previous section and the iterative building of a toll set is given in table 5.7.

Iteration	0	1	2	3	4	5	6	7	8	9	10	11	12
t_1 (1-5)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_2 (5-7)	-	**8.8**	8.8	8.8	**10.8**	10.8	10.8	10.8	10.8	**11.1**	11.1	11.1	11.1
t_3 (7-3)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_4 (1-6)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_5 (2-5)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_6 (5-9)	-	-	-	1.6	1.6	**2.0**	2.0	2.0	2.0	2.0	2.0	**2.1**	2.1
t_7 (9-7)	-	-	**3.6**	3.6	3.6	3.6	3.6	3.6	3.6	3.6	3.6	3.6	3.6
t_8 (6-9)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_9 (9-8)	-	-	-	-	-	-	-	**0.6**	0.6	0.6	0.6	0.6	**0.7**
t_{10} (7-4)	-	-	-	-	-	-	0.7	0.7	0.7	0.7	**1.0**	1.0	1.0
t_{11} (8-3)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_{12} (2-6)	-	-	-	-	-	-	-	-	**0.4**	0.4	0.4	0.4	0.4
t_{13} (6-8)	-	-	-	-	-	-	-	-	-	-	-	-	-
t_{14} (8-4)	-	-	-	-	-	-	-	-	-	-	-	-	-
TNTC	2441	2385	2356	2347	2346	2344	2341	2336	2335	2336	2334	2334	2333
REV	0	179	287	331	363	374	388	404	419	424	430	432	435

Table 5.7: Iterative building of toll-set to achieve SSO

The minimum (SSO) value of the total perceived network travel cost (TPNTC) is 2332 units, and it may be observed that with the toll set derived from 12 iterations of the heuristic, the TNTC achieved is only 0.04% greater than the SSO value. The minimal revenue tolls (section 5.5) to achieve the SSO however only extract a total revenue of 402 units from the users where as in 12-iterations as shown above the total revenue is 435 units. It would seem sensible therefore to halt the procedure after fewer iterations;

7-iterations result in a revenue of 404 units which is similar to that achieved by the minimal revenue tolls, and such tolls reduce the TNTC to a value which is still only 0.17% higher than for $TNTC_{SSO}$. However the first 5 iterations of the heuristic resulted in only 3 links being tolled, the two additional links tolled using 7 iterations are only tolled by small amounts and it seems unnecessary to move from a desirable zero-toll state for such small toll values. The solution after 5-Iterations reduces the TNTC to a value which is still only 0.5% higher than $TNTC_{SSO}$ ($TNTC_{SUE}$ is 4.67% higher than $TNTC_{SSO}$, so the majority of the potential improvement in network efficiency would still be achieved) whilst the revenue extracted is 374 units which is lower than the minimal revenue value.

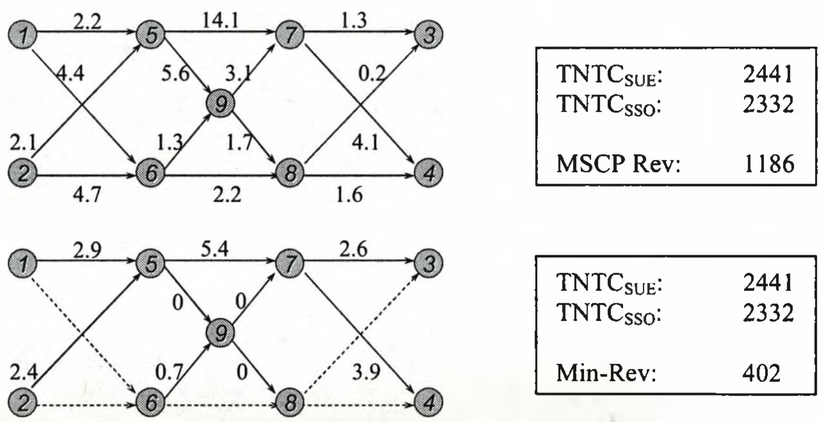


Figure 5.19: Nine-node network diagram, MSCP and Min-Rev tolls to achieve SSO.

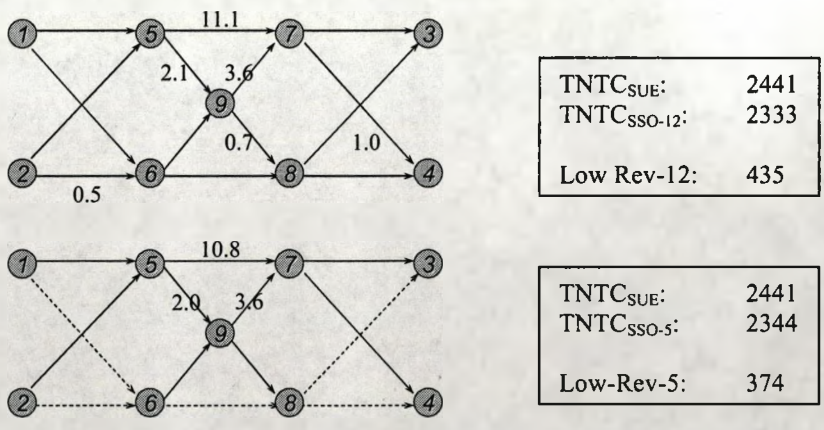


Figure 5.20: Nine-node network diagram, Heuristic tolls to achieve SSO.

As it seems desirable to stop the heuristic before the $TNTC_{SSO}$ is reached in order to achieve a sensible balance between revenue extracted, total network travel cost reduction and number of links tolled, this heuristic may be said to produce low-revenue tolls rather than the exact minimal revenue tolls. While the objective at each iteration is to reduce Total Network Travel Cost, full convergence results in a revenue to be collected which is greater than the minimal revenue value. It seems sensible therefore to instead reach a balance between the several desirable objectives.

It may be seen that while the low-rev link-tolls obtained by use of the heuristic are not placed on the same links as the algebraically derived minimal revenue tolls, either the same number of links are tolled in total, or rather less depending on where there heuristic is halted (although the 'worst' link (5-7) is the same in all cases). Further whilst if the heuristic is run to achieve very good agreement with the SSO solution the zero toll tree for min-rev tolls is not preserved, it is if the heuristic is halted at a 'sensible' time, as illustrated by the dashed links in figure 5.21. Clearly if such a heuristic is being utilised to produce low-revenue tolls to negate the need for either linear programming or path enumeration based methods, then the minimal-revenue solution will not generally be known, and which links would be untolled at this solution would not be known either. It would not be possible then to make a decision to halt the heuristic based on a comparison with the minimal revenue solution and a decision rule balancing the reduction in total network travel cost against the revenue required and the number of links which need to be tolled would have to be derived. A similar decision would have to be made in the case of tolling to achieve a value of the total network cost close to the minimum value at the deterministic system optimal solution (chapter 4). However if it is known that greater network benefits may be achieved by seeking tolls to produce a network flow pattern which closely approaches that at the deterministic system optimal, there seems to be little necessity in attempting to produce the stochastic system optimal flow pattern by tolling as it tends to have intermediate network benefit. A comparison with toll sets to replicate the 'true' System Optimal flow pattern under stochastic user equilibrium assignment is contained in section 5.7.

This section has utilised the heuristic developed in chapter 4 to determine toll sets that aim to reduce the total network cost to a value close to the Stochastic System Optimal value. The heuristic produces toll sets which can approach the desired flow pattern closely but at the cost of exceeding the revenue required to do this using the exact minimal revenue tolls discussed in section 5.5. It is recommended therefore that the heuristic should be halted at an earlier point when closeness to the desired flow pattern and reduction in network cost may be balanced against the amount of revenue to be collected. This has the further advantage of also requiring fewer links to be tolled, as if the heuristic puts a very small toll on a link which produces little additional benefit it can easily be halted just prior to tolling that link. The tolls derived from the heuristic are termed low revenue tolls.

5.7 Comparison of link-based tolls: SO vs SSO

In chapter 4 the primary objective was to seek toll sets that produced network flow patterns that were close to that obtained at the deterministic system optimal where total network cost is minimised. Exact path-based methods were found to produce toll sets that could not create viable link tolls sets and so an approximate low revenue link toll heuristic was developed. This chapter has examined toll sets which exactly replicate the stochastic system optimal (SSO) flow pattern (marginal social cost price tolls and minimal revenue tolls) and has also adapted the heuristic from chapter 4 to produce low revenue link based tolls that aim to approach the SSO rather than replicate it precisely. This section compares first the exact tolls to achieve the SSO and then the low-revenue heuristic toll sets with low revenue tolls derived from the heuristic seeking the deterministic SO.

In comparison with the SSO min-rev toll set shown in figure 5.21, it may be seen that the low-rev toll solution after 12 iterations of the heuristic (figure 5.22) results in a reduction in TNTC (compared to the SUE cost of 2441) of 7.64% (the true SO would result in a reduction of 7.66%), where as the SSO min-rev tolls produced a reduction of 4.47%. The SSO min-rev tolls however required less revenue to be extracted from the user, 402 as opposed to 822 units. For comparative purposes reductions in TNTC, revenue required, and an index I, (where I = percentage reduction in TNTC per 1000 units of revenue), are summarised in table 5.8 for SSO with MSCP-tolls and Min-Rev tolls, and for SO with both 4 and 12 Iterations of low-rev tolls.

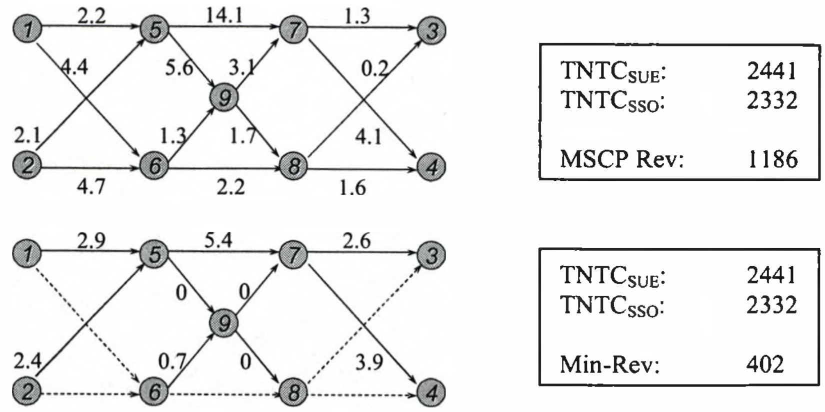


Figure 5.21: Nine-node network diagram, MSCP and Min-Rev tolls to achieve SSO.

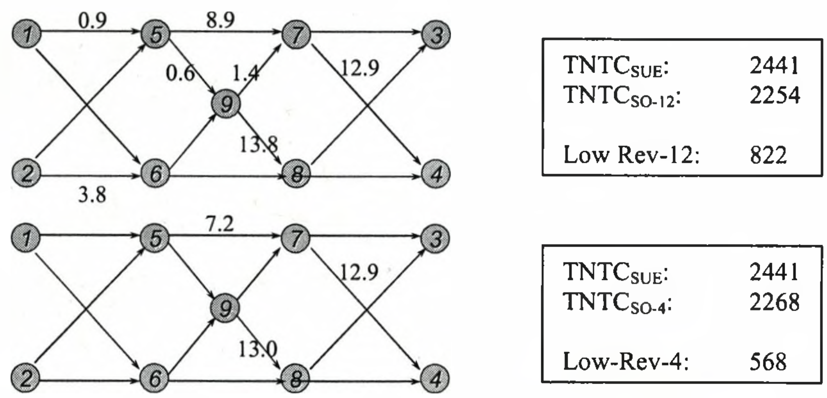


Figure 5.22: Nine-node network diagram, Low-Rev tolls to achieve SO.

	SSO: MSCP-tolls	SSO: Min-Rev-tolls	SO: Low-Rev-tolls 4-Iterations	SO: Low-Rev-tolls 12-Iterations
% Reduction TNTC	4.47	4.47	7.09	7.64
Revenue	1186	402	568	822
Index: I	3.77	11.11	12.48	9.29
Links Tolled	14	6	3	7

Table 5.8: Comparison of SSO (MSCP and Min-Rev) and SO toll sets

It can be readily seen that the extra reduction in TNTC from using a 12-iteration toll set, does not seem justifiable in terms of revenue required. The value of I is not much greater for the SSO: min-rev tolls than the SO: low-rev tolls with 4-Iterations, but the required number of links to be tolled is clearly least under the SO low-rev tolls with 4 iterations. Consequently in this example it would appear that the low-rev tolls seeking the deterministic SO with a small number of iterations gives the most desirable results. This comparison is not however necessarily generalisable, but the benefit of such a heuristic to derive tolls lies in the ability to halt the process when the desired number of links to be tolled is reached. It also tolls the 'worst' links first, i.e. those where the product of cost and flow is most different from that at the SO solution.

	SSO: Low-Rev-tolls: 5 Iterations	SSO: Low-Rev-tolls: 12 Iterations	SO: Low-Rev-tolls 4-Iterations	SO: Low-Rev-tolls 12-Iterations
% Reduction TNTC	3.97	4.42	7.09	7.64
Revenue	374	435	568	822
Index: I	10.62	10.17	12.48	9.29
Links Tolled	3	6	3	7

Table 5.9: Comparison of SSO (heuristic) and SO toll sets

In terms of comparing tolls to approach the SSO using the heuristic and tolls to produce the SO, it can be seen in table 5.9 that the network benefits for the latter appear to be significantly greater. The possible percentage reductions in TNTC are far more attractive and whilst less cost is imposed upon the user under SSO tolling, the Index for reduction in cost with respect to revenue indicates that the SO tolls after 4-iterations is the best solution in this case, (with no more links requiring a toll than for the SSO low-rev 5-iteration solution). It may be noted that whilst the only link to be universally tolled under each toll set is the 'worst' link (5-7), there is in fact a similarity in which links are tolled under the heuristic taken to 12-iterations (near full convergence), whether the SSO or the SO is the target 'desired flow pattern'.

For the illustrative network presented in this section it appears that the tolls to approach the deterministic System Optimal after 4 iterations of the heuristic produce more beneficial results than any toll set derived to approach the Stochastic System Optimal.

5.8. Conclusions

This chapter has introduced the concept of the Stochastic System Optimal where total perceived network travel cost is minimised rather than 'actual' total network travel cost. Marginal social cost price tolling may be used to achieve the SSO flow pattern under an SUE assignment, and the same network flows can be achieved by Minimal Revenue tolling as in the deterministic case.

In chapter 4, in attempting to approach the 'true' SO flow pattern through tolling, the algebraic logit formulation derived path-tolls which could not then be separated into consistent link tolls. Also the issue of small (or zero) path flows encountered at the Deterministic System Optimal solution would, in the logit case, result in unreasonably large tolls on some routes. Further, algebraic methods are untenable in the probit case, and so such methods were not felt to be desirable and instead an iterative heuristic link-based method was derived. For the toy-network used here for illustration the desired deterministic SO flow pattern where TNTC was minimised could be closely approached within a small number of iterations. A sensible trade off between the cost imposed upon the drivers to achieve the reduction in TNTC, and the actual reduction obtained would need to be established for practical purposes. In additional it may be desirable to require certain links to be zero-tolled, and this could be easily included in this type of process.

In attempting to achieve the Stochastic Social Optimum flow pattern, by use of minimal-revenue tolling, the marginal social cost price tolls (known to create the desired flows) were used as a starting point. Path enumeration was then required to use these to derive minimal revenue path-based tolls and from these, link-based tolls. The minimal-revenue toll problem in this case is analogous to that for deterministic assignment, but with the stochastic nature of the assignment removing the condition for all used paths to have a common cost. It was possible to use the iterative heuristic method to derive low-rev SSO tolls, but these were not found to give greatly attractive network benefits when compared to those for the SO.

The desired flow pattern to be achieved in the stochastic case remains though an issue to be resolved. Is it more desirable in the stochastic case to minimise ‘real’ or ‘perceived’ costs throughout a network? The illustrative example presented in this chapter would suggest that it is more beneficial to the network to seek tolls to reduce the total network cost as far as possible, i.e. to seek the deterministic System Optimal. From the perspective of the user however there might be greater perceived benefit if the perceived costs were to be minimised across the network and the Stochastic System Optimal used as the objective.

CHAPTER 6

ACHIEVING SO IN THE STOCHASTIC CASE WITH ELASTIC DEMAND

6.1 Introduction

Chapter 4 developed methodologies to examine the minimal revenue toll problem in the case of Stochastic User Equilibrium and looked for toll-sets that would achieve the deterministic SO solution. Chapter 5 discussed the definition of a Stochastic Social Optimum (Maher et al, 2005) and looked at tolling methodologies to achieve this flow pattern.

This work has however been based on the assumption of a fixed demand stochastic equilibrium model. It is clear that imposing tolls on a network will directly affect demand as well as being able to influence route choice and this chapter investigates tolling under Stochastic User Equilibrium with elastic demand (SUEED).

Elastic demand may be readily included in stochastic equilibrium models (Maher et al, 1999) and MSCP tolls may be easily derived by using marginal cost functions in an SUEED algorithm. It has also been shown that in the deterministic case with elastic demand, a System Optimal with Elastic Demand (SOED) may be defined in terms of economic benefit maximisation (Hearn and Yildirim, 2002) such that all valid toll sets generate the same toll revenue (Larsson and Patriksson, 1998; Hearn and Yildirim, 2002). This Chapter investigates the existence of lower revenue toll sets that do not maximise economic benefit but can produce SO flow patterns for different demand values. The heuristic (presented in chapter 4) that was used to derive toll sets that would induce a System Optimal flow pattern under an SUE assignment was based on the premise of fixed demand. In this chapter it is extended to include elastic demand; further iteration is required to account for the change in the 'desired flow pattern' as each link toll is increased.

Under deterministic assignment the System Optimal (SO) Solution where the Total Network Travel Cost (TNTC) is minimised is well established as being the 'desired' flow pattern i.e. that which would give the most beneficial distribution of network flows. In the case of economic benefit maximisation, marginal social cost price tolls (mscp) may be applied to network links so that the SO is achieved. Under stochastic assignment though, the previous chapters have demonstrated that the desired flow pattern is not immediately obvious. Two possible desired flow patterns have been suggested in the stochastic case: the SSO (Maher et al 2005), where economic benefit is maximised and the Total *Perceived* Network Travel Cost (TPNTC) is minimised, and the 'True SO' where Total Network Travel Cost is minimised, i.e. the same SO flow pattern as in the deterministic case.

This chapter aims to examine tolling to achieve the SO under stochastic assignment methods when elastic demand is permitted. Section 6.2 discusses and reviews tolling to achieve the SO under deterministic assignment with elastic demand. An SOED formulation is given (Hearn and Yildirim, 2002) under which there is a lack of opportunity for additional optimisation to reduce the total revenue required to be collected from the users. The possibility for meaningful reduced revenue toll sets is discussed. Section 6.3 examines the possibility of achieving a 'deterministic SO solution' under SUEED, and extends the methodology presented in Chapter 4 (for the fixed demand) case to allow for elastic demand. Section 6.4 discusses the existence of marginal social cost price tolling and section 6.5 gives chapter conclusions. Illustrative numerical results for toy networks are included.

6.2 System Optimisation under deterministic assignment with elastic demand.

The assumption that travel demand is fixed is not in general realistic and it is important to consider both the conditions within the network and the effect that they are likely to have on demand. Network improvements designed to lessen the effects of traffic congestion will be likely to also release *suppressed trips*, so that any decrease in TNTC predicted under a fixed demand model is likely to be overly optimistic. The general acceptance of the theory of suppressed traffic has led planners in recent years (DETR, 1998) to move away from the concept of *predict and provide* and to consider alternative measures that will instead act to further suppress demand rather than to release the suppressed traffic on to the network, tolling being one such measure.

In modelling elastic demand a simple elasticity model is generally used, where the trips between an OD pair vary inversely with the cost of the trip, so that:

$$q_{rs} = D_{rs}(u_{rs}) \quad \forall r,s \qquad 6.1$$

where q_{rs} is the trip rate, D_{rs} is the demand function (decreasing) and u_{rs} is the minimum travel cost (minimum perceived travel cost in the stochastic case), for trips between any OD pair *r-s* (Sheffi, 1985). The UE problem is thus extended to UEED by adding a term to the objective function:

$$Z_{UEED} = \sum_a \int_0^{x_a} c_a(\omega)d\omega - \sum_{rs} \int_0^{q_{rs}} D_{rs}^{-1}(\omega)d\omega \qquad 6.2$$

and may be solved by adding an extra *pseudo-link* between each OD pair *r-s*, with cost-flow function equal to $D_{rs}^{-1}(q_{rs})$ and solving this extended network as a simple UE problem (ibid).

Hearn and Yildirim (2002), define the Elastic Demand Social Optimal Problem as:

$$Z_{SOED} = \sum_a x_a c_a(x_a) - \sum_{rs} \int_0^{q_{rs}} D_{rs}^{-1}(\omega)d\omega \qquad 6.3$$

where the objective function Z_{SOED} is to be minimised. This is equivalent to maximising net user benefit, i.e. the difference between total user benefit $[\sum_{rs} \int_0^{q_{rs}} D_{rs}^{-1}(\omega)d\omega]$, and total system cost $[\sum_a x_a c_a(x_a)]$. It can be seen that this objective function is analogous in form to that for the UEED problem. It has been shown that in the case of deterministic assignment with elastic demand, that to achieve the above SOED flow pattern (with welfare maximising demand) from a UEED assignment with tolls, that all such toll-sets will generate the same revenue and that this revenue is that obtained from imposing MSCP tolls (Larsson and Patriksson, 1998; Hearn and Yildirim, 2002). This is illustrated in this section using the simple 2-link example network.

This property is essentially due to the fact that the solution to (6.3) is an optimal flow pattern associated with an optimal demand matrix. This results in there not being a 'minimal revenue toll problem' to solve, as the revenue achieved by mscp tolling will be the same as that achieved by any toll set which produces the demand/flow pattern associated with minimising the objective function Z_{SOED}. Imposing a lower revenue toll set would result in more suppressed traffic moving onto the network and the demand matrix differing from the optimal demand matrix (demand flows generally increasing) so that the objective function would no longer be minimised.

It is not however true that MSCP tolls are the only such set of link tolls, as some further optimisation is possible in regard to which links are tolled whilst path tolls are fixed at the MSCP values.

The elastic demand case is illustrated in comparison to the fixed demand case for UE, SO, UEED and SOED, and graphs for each objective function are shown in figures 6.1-6.6.

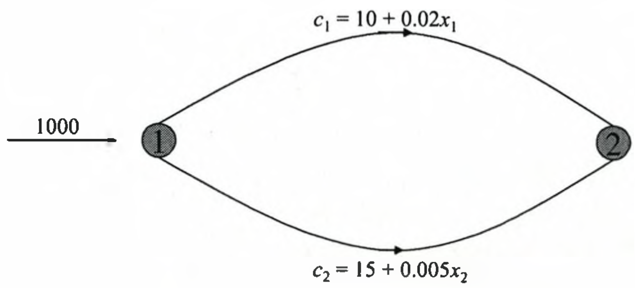


Figure 6.1: Two-link network: Fixed Demand

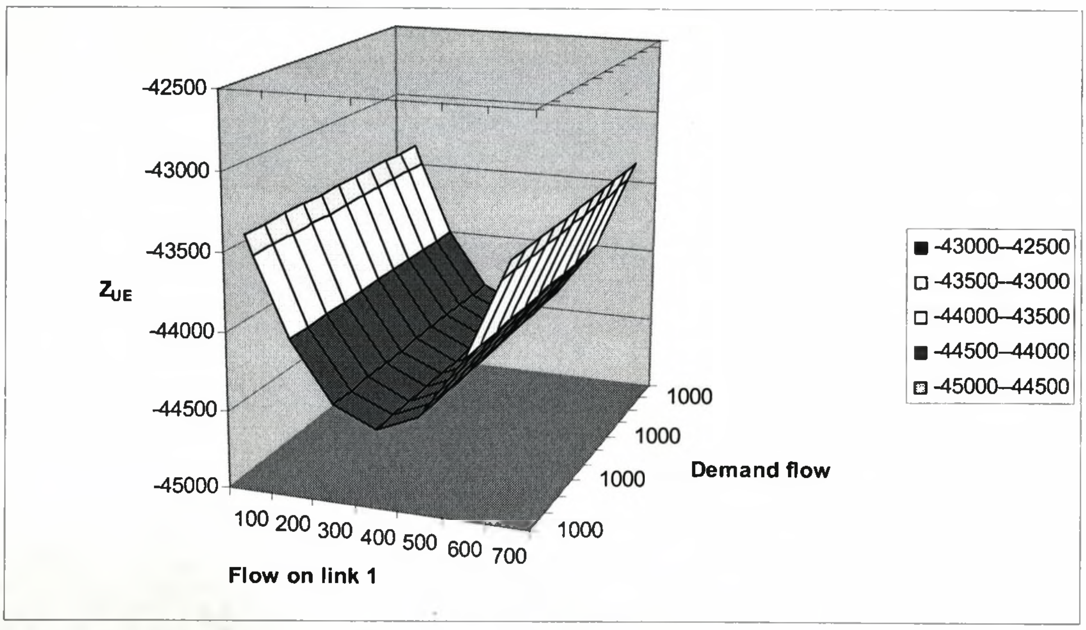


Figure 6.2: Objective Function for Deterministic UE with Fixed Demand

In the fixed demand case, varying the toll level has no effect on the demand value (as it is fixed), thus the same desired SO flow pattern may be achieved by either MSCP tolls or by a range of equally optimal tolls as has been previously discussed. In the two link example given, the flow pattern solutions are: UE (400, 600), SO (300, 700), SUE (462, 538) and SSO (390, 610) as illustrated.

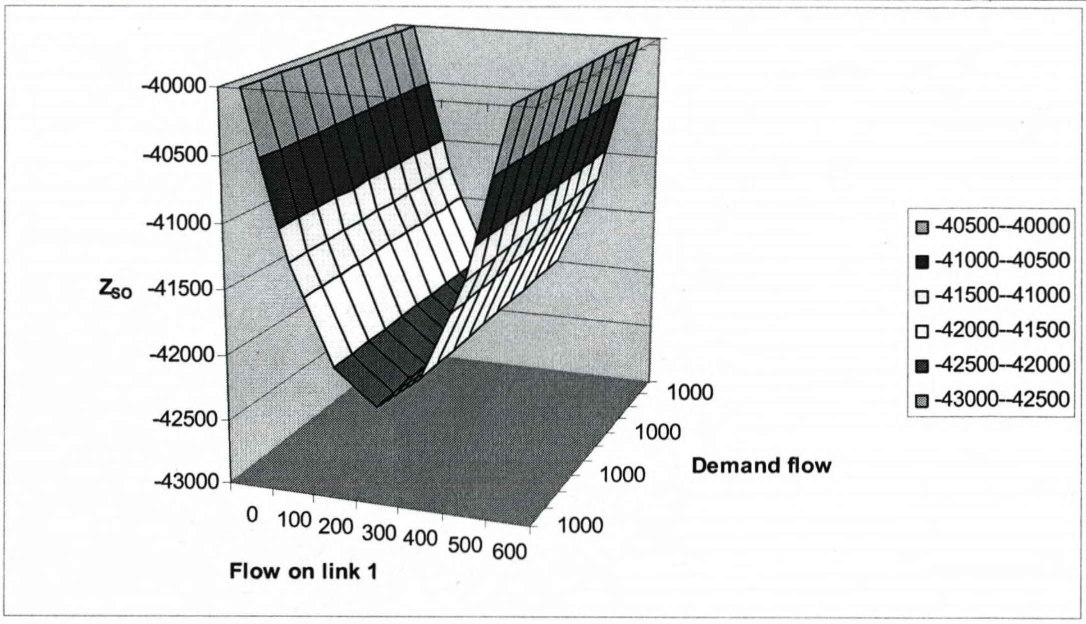


Figure 6.3: Objective Function for Deterministic SO with Fixed Demand

For SOED the two-link example has a 'pseudo-link' added which is assumed to carry additional flow (figure 6.4).

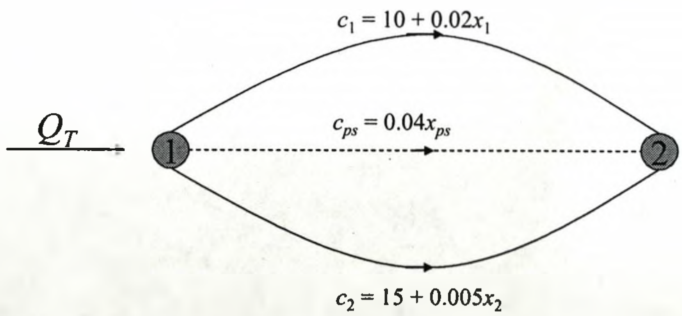


Figure 6.4: 2-link network: Elastic Demand

The cost function on the pseudo-link has been obtained by using a demand function $Q = 2000 - 25C$ and a value of Q_T = 2000, where Q is the 'actual' flow and Q_T is the total theoretically available flow including the suppressed demand. The form of the UEED and SOED objective functions (6.2) and (6.3) for this network are shown in figures 6.5 and 6.6.

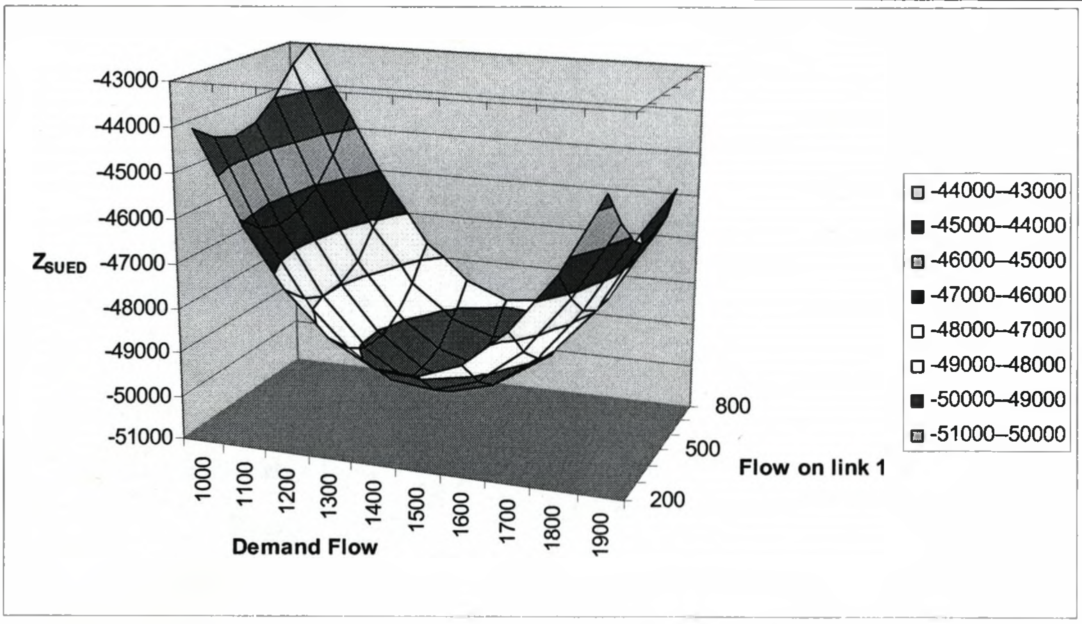


Figure 6.5: Objective Function for User Equilibrium with Elastic Demand

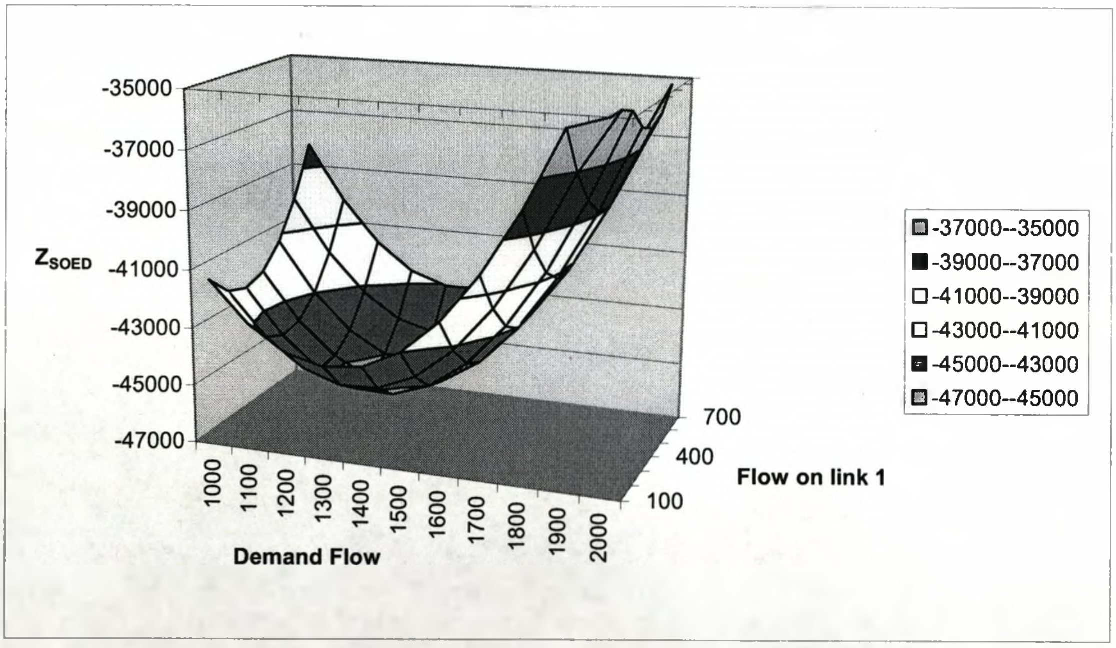


Figure 6.6: Objective Function for Social Optimum with Elastic Demand

It may be seen that the objective function for SOED is minimised at (x_1, x_2, Q) = (375, 1000, 1375), where Z_{SOED} = -45625, and at no other values for the link flows and demand. [The objective function for UEED is minimised at (x_1, x_2, Q) = (500, 1000, 1500), where Z_{SOED} = -50,000.]

If the SOED solution is to be found from a UEED assignment with appropriate tolls, then it may be seen that any reduction in the total toll revenue from that which produces the optimal solution above, will result in lower link costs. A reduction in link costs will then produce a higher value of demand Q which will no longer be optimal.

The SOED solution for the 2-link network is given in table 6.1; where x_i are link flows, c_i are link costs, m_i are marginal link costs and $mscp\ t_i$ are link marginal social cost price tolls:

Link	x_i	m_i	c_i	$mscp\ t_i$
1	375	25	17.5	7.5
2	1000	25	20.0	5.0

Table 6.1: SOED solution under UEED

$Q = 1375$ is the SOED optimal total flow. Total revenue = 7812.5, TNTC = 26563.

In the fixed demand case it would be possible to derive minimal revenue tolls $(t_1, t_2) =$ (2.5, 0), but in the elastic demand case as defined above it may be easily seen that such a toll reduction would reduce the common path cost at UE and more of the suppressed traffic would be allowed onto the network, increasing the total flow Q from the optimal value as found above.

It is possible however that a traffic planner might not wish to impose such high revenue tolls as MSCP tolls upon the user, and may not wish to suppress traffic to the point where net user benefit is maximised. If economic benefit maximisation is not an absolute objective, it is still reasonable to wish to achieve optimal routing within a network for any particular demand level, i.e. the SO flow pattern for a particular fixed demand.

In the example above, if the toll set (t_1, t_2) = (2.5, 0) is used as an initial toll set, it is possible to recalculate the network demand to achieve an SO flow solution for that particular value of the demand. If this is done then the converged solution is as given in table 6.2 below, where x_i, m_i, c_i and *Min-Rev* t_i are the link flows, marginal costs, link costs and link minimal revenue tolls ($t_1 = c_2 - c_1$, $t_2 = 0$).

Link	x_i	m_i	c_i	*Min-Rev* t_i
1	397.7	25.9	17.95	2.5
2	1090.9	25.9	20.45	0

Table 6.2: An SO solution under UEED

The total Network Demand Q = 1488.6, Total Revenue = 994.3 and TNTC = 29455.

Z_{SOED} = -45315

It can be seen that the above is a valid SO solution under UEED, although the network demand is higher than that obtained at the SOED solution and Z_{SOED} is no longer minimised (minimal value = -45625). It is possible therefore to obtain a system optimising flow pattern associated with a particular demand under UEED where the total revenue is less than that which is obtained under mscp tolling. However this solution will not maximise net user benefit. This principle of attempting to achieve a valid ‘optimal’ flow pattern, whilst allowing the demand to vary as particular tolls are imposed is the basis of the extension of the SUE heuristic to achieve an SO flow pattern under SUEED which is presented in section 6.3.

Whilst the remainder of this chapter will focus on stochastic assignment with elastic demand the principle of tolling to achieve a particular flow pattern whilst allowing demand to be elastic has some essential features that are common to both the deterministic and stochastic cases. Any change in network costs will, under the principles of elastic demand, result in a change in demand; in particular any increase in network cost due to the imposition of a toll will result in a degree of traffic suppression. If the demand is not highly elastic then the traffic suppression effect may be small, however if the demand has a high elasticity then the traffic suppression effect of tolling will be very much more significant. It is important therefore to ensure that elasticities are carefully estimated. In addition to the traffic suppression (which would occur if any charge were levied on the network) the tolls that are being considered in this thesis aim also to produce a re-routing effect, so that in addition to reduced network flow there will also be some level of optimality in the routing. For any chosen demand value there must exist a corresponding System Optimal flow pattern where network costs are minimised for that value of demand, however every additional cost increment will suppress more traffic so the 'desired' flow pattern will change as tolls are added incrementally onto the network. The re-routing effect (in addition to the traffic suppression effect) is where link-based tolls can potentially be more efficient in the reduction of congestion than cordon-based tolls. These principles are illustrated in the stochastic case in section 6.3.

6.3 Achieving the SO by tolling under stochastic assignment with elastic demand.

The SUEED problem has been solved for general networks using link based methods by Leurent (1994) for logit-based SUEED by use of a dual algorithm and by Maher et al (1999) for probit-based SUEED by the minimisation of a single objective function, also by Rosa and Maher (2002b) by the minimisation of a more compact single objective function. In the case of SUEED the elastic demand q_{rs} is assumed to depend only on the expected perceived minimum travel cost S_{rs}, (the satisfaction function) for OD pair *r-s*, such that;

$$q_{rs} = D_{rs}(S_{rs}) \quad \forall r,s \qquad 6.4$$

There are several possible forms for the demand function (Maher et al, 1999). The *Power Law* form will be used here for illustrative purposes,

$$D_{rs}(S_{rs}) = D_0\left(\frac{S_0}{S_{rs}}\right)^e \qquad 6.5$$

where D_0 and S_0 are base values for the demand and satisfaction, and *e* is the fixed elasticity. The SUEED solution occurs where the Satisfaction value obtained from the inverse demand function associated with (6.4) above is equal to the Satisfaction value obtained from a stochastic loading. Using the balanced demand algorithm (BDA) (Maher et al, 1999) the demands are maintained in balance with the flows throughout. In the logit-based case the Satisfaction function may be written (Williams, 1977):

$$S_{rs} = -\frac{1}{\theta}\ln(\sum_i \exp(-\theta(C_i))) \qquad 6.6$$

In the fixed demand case discussed in Chapter 4 a heuristic was developed to derive link tolls such that the deterministic SO flow pattern was closely approached under Stochastic assignment methods. This section develops that heuristic to allow for elastic demand. It was first necessary to consider which deterministic SO flow pattern should be 'the desired flow pattern' when the demand was no longer fixed. It is clear that for any particular SUEED assignment, there would be an associated OD demand matrix that would represent the balanced flows on the network. It would then be of interest to look at the deterministic SO flow pattern for that particular OD matrix, i.e. the optimal network flow pattern for that particular amount of traffic. The network links would then be compared to assess how different the SUEED flow pattern was from the desired SO flow pattern for that particular demand level. However any attempt to try to create the SO flow pattern under SUEED by the addition of link tolls would result in demand being suppressed, the OD matrix changing, and the desired deterministic SO flow pattern being altered. This process is illustrated by the schematic in figure 6.7.

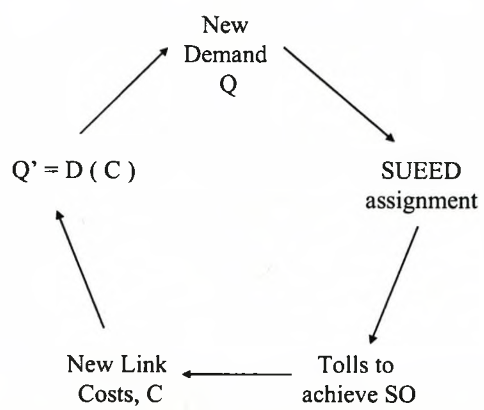


Figure 6.7: Tolling cycle under elastic demand

For each value of the demand it can be assumed that traffic will be routed through a network following the principles of an SUEED assignment. If a toll is then imposed to attempt to create a known and desired flow pattern (such as the SO) the addition of the toll will effect the travel cost on each link which will cause additional traffic suppression according to the demand function. This will result in a new value of of demand Q and also a new target SO flow pattern associated with such a demand value. The cycle would then have to repeat until convergence was achieved where the converged toll set created an SO flow distribution for a particular value of demand Q.

The existing heuristic has been developed to account for this. When the SUEED flow pattern was compared to the desired SO (for a particular OD matrix), the 'worst' link would still be determined and a toll applied to that link such that the link flows and costs for that particular link, matched those at that SO. The addition of a toll would then affect the demand level and the demand functions would be used to find the new demand level for affected ODs and the OD matrix would be updated. No further links would then be tolled until the new desired SO flow pattern had been recalculated by use of the updated OD matrix. The process would then be repeated until the TNTC under SUEED closely matched that of the deterministic SO with the same OD matrix. This heuristic procedure is given below:

Step 1: Link toll vector set to zero: $\mathbf{T}_0 = \mathbf{0}$

Step 2: Set $n = 0$

Step 3: Perform SUEED with current link tolls $\mathbf{T}_n$ and obtain OD matrix, $\mathbf{OD}_n$, link costs $\mathbf{C}_n$ and link flows $\mathbf{F}_n$

Step 4: Find the deterministic SO solution $\mathbf{SO}_n$ using $\mathbf{OD}_n$ and determine link flows $\mathbf{F}^{(SO_n)}$, link costs $\mathbf{C}^{(SO_n)}$, and the total network travel cost $\mathrm{TNTC_{SO}}^n$.

Step 5: Calculate: $P_j = \left(F_j^{(n)} - F_j^{(SO_n)}\right)\left(\left|C_j^{(n)} - C_j^{(SO_n)}\right|\right)$

Step 6: Determine link j where P_j is greatest.

Step 7: Perform iteration to calculate t_j s.t $F_j^{(n)} = F_j^{(SO_n)}$ to required degree of accuracy.

7a: Set $t_{j_0} = \left|C_{j_0} - C_j^{(SO_n)}\right|$ where C_{j_0} is the current cost on link j (as per step 4)

7b: Set m=1

7c: Perform SUEED assignment, calculate $\left|C_{j_m} - C_j^{(SO_n)}\right|$

7d: Set $t_{j_m} = t_{j_{m-1}} + \left|C_{j_m} - C_j^{(SO_n)}\right|$

7e: Calculate P_{j_m} : Stop if sufficiently close to zero and let $t_j = t_{j_m}$, or set

$m = m + 1$ and repeat from step 7c.

Step 8: $\mathbf{T}_{n+1} = \mathbf{T}_n + \mathbf{t}$; where $t_i = t_j$ when $i = j$ and $t_i = 0$ otherwise

Step 9: Calculate $\text{TNTC}_{\text{SUEED}}{}^{n+1}$ and $\mathbf{OD}_{n+1}$ and record $\mathbf{C}_{n+1}$ and $\mathbf{F}_{n+1}$

Step 10: Use $\mathbf{OD}_{n+1}$ to determine $\mathbf{SO}_{n+1}$ and record $\mathbf{F}^{(SO_{n+1})}$, link costs $\mathbf{C}^{(SO_{n+1})}$ and $\text{TNTC}_{\text{SO}}{}^{n+1}$.

Stop if $\text{TNTC}_{\text{SUEED}}$ is sufficiently close to TNTC_{SO}

or set $n = n + 1$ and repeat from Step 5.

This is illustrated using the 9-node network given in figure 6.8. An elastic demand function as in (6.4) is used where $S_0 = 10$, D_0 is as given in figure 6.7 and $e = 0.7$; the stochastic assignment method used is logit SUEED where $\theta = 0.1$. ($c_a^{(0)}$ is free flow cost, Y_a is capacity, in labels alongside links in Fig 6.7).

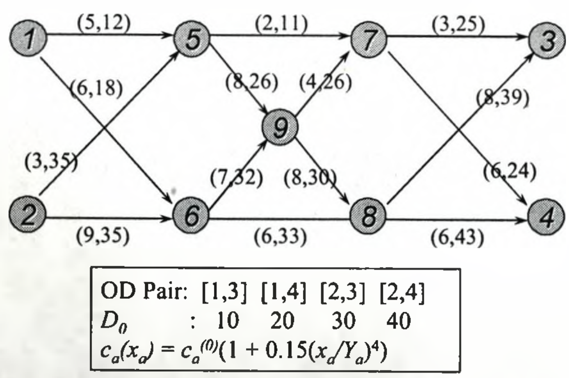


Figure 6.8: 9-node network

The iterative building of the resulting toll set is shown in Table 6.3; the heuristic has been stopped after 8 iterations as the Total Network Travel Cost at SUEED is within 1.1% of the Total Network Travel Cost at SO, which is deemed to be sufficiently close.

Iteration	0	1	2	3	4	5	6	7	8
t_1 (1-5)	-	-	-	-	-	-	**4**	4	4
t_2 (5-7)	-	**7.6**	7.6	7.6	7.6	7.6	7.6	7.6	7.6
t_3 (7-3)	-	-	-	-	-	-	-	-	-
t_4 (1-6)	-	-	-	-	-	-	-	-	-
t_5 (2-5)	-	-	-	-	-	-	-	-	-
t_6 (5-9)	-	-	-	-	-	-	-	-	-
t_7 (9-7)	-	-	-	-	2.5	2.5	2.5	2.5	2.5
t_8 (6-9)	-	-	-	-	-	**5**	5	5	5
t_9 (9-8)	-	-	-	**6.1**	6.1	6.1	6.1	**11**	11
t_{10} (7-4)	-		**5.3**	5.3	5.3	5.3	5.3	5.3	**7.5**
t_{11} (8-3)	-	-	-	-	-	-	-	-	-
t_{12} (2-6)	-	-	-	-	-	-	-	-	-
t_{13} (6-8)	-	-	-	-	-	-	-	-	-
t_{14} (8-4)	-	-	-	-	-	-	-	-	-
TNTC	2851	2650	2490	2256	2149	2008	1956	1858	1817
TNTC: SO	2626	2536	2406	2204	2107	1970	1926	1832	1797
	7.9%	4.3%	3.4%	2.3%	2.0%	1.9%	1.5%	1.4%	1.1%
REV	0	168	273	375	444	480	505	517	534

Table 6.3: Iterative Building of toll set

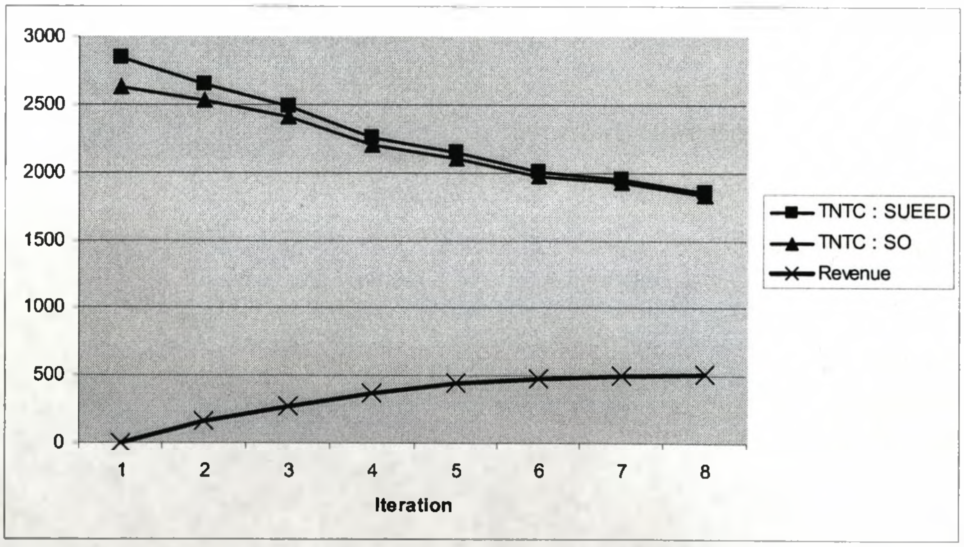


Figure 6.9: Reduction in TNTC as tolls are imposed

As the tolls are imposed the System Optimal TNTC for each particular value of demand will be smaller than the last, thus at each iteration the heuristic produces a TNTC value under SUEED which gradually becomes closer to an SO value, as can be seen in figure 6.9.

It can be seen that imposing the tolls given above will reduce the initial TNTC by 36.3%, but unlike in the fixed demand case where all the reduction in TNTC is due to more efficient routing through the network, in the elastic demand case much of this reduction is due to the additional suppression of demand. The change in the demand matrix with the imposition of tolls and the decrease in total demand are shown in table 6.4 below. It can be seen that the corresponding reduction in demand when the above tolls are used is 20.9%. As a general aim of introducing tolling is to reduce congestion the suppression of some traffic together with efficient re-routing would seem to be a sensible solution.

Iteration	OD (1-3)	OD (1-4)	OD (2-3)	OD (2-4)	Total demand: Q
0	12	21.5	35.5	41.4	110
1	11.6	21.2	34	40.5	107
2	12.2	19.5	35.5	36.9	104
3	11.6	18.4	33.9	34.8	99
4	11.1	18.1	32.6	34.2	96
5	10.4	17.3	31.2	33.2	92
6	9.4	16.3	31.7	33.4	91
7	9.1	15.8	30.8	32.3	88
8	9.2	15.4	31.3	31	87

Table 6.4: Decreasing Demand with tolling

This section has examined the possibility of achieving an optimal rerouting under stochastic assignment with elastic demand. The heuristic which was developed for the fixed demand case has been adapted to account for the target SO flow pattern changing as a result of the network demand varying with toll level imposed. The toll sets obtained will produce both a traffic suppression effect and an efficient re-routing effect. The tolls derived in this section will be compared with marginal social cost price tolls in section 6.4.

6.4 MSCP tolling under SUEED.

It would seem reasonable to suppose that it is possible to extend the formulation for SSO (Maher et al, 2005) to SSOED in an analogous manner to the extension of UEED to SOED given in section 6.2, and that consequently the SSOED solution may be obtained by using existing methods to solve for SUEED but by using marginal cost-flow functions in the place of the existing cost-flow functions. Thus the SSOED solution would be where economic benefit is maximised with elastic demand, as in Yang (1999) for the fixed demand case. The formulation of objective functions for the SSOED is contained in Chapter 7, but it is pertinent here to compare the results obtained from an MSCP tolling under SUEED with the toll sets calculated in section 6.3.

In the fixed demand case, the SSO (the flow pattern at which TPNTC is minimised and economic benefit maximised), does not optimise the system as efficiently as the 'true SO', i.e. the TPNTC is higher than the TNTC. It is also possible to achieve a flow pattern approaching the SO with a toll set which demands much less total revenue to be collected than when using MSCP tolls. The comparison in the elastic demand case however is not as clear as in the fixed demand case due to the obvious suppression of traffic with more costly toll sets. It would of course be possible to minimise the TNTC by setting such high network tolls that all trips were suppressed and all link flows were zero. This would also result in minimal revenue tolls as no tolls would be collected, but whilst pedestrianisation schemes are popular in some circumstances they would not be implemented by means of excessively high network tolls. The heuristic contained in section 6.3 was designed to optimise the network flows most effectively (by re-routing) whilst imposing a small cost on the user, however traffic suppression is very significant in the overall reduction of TNTC so that imposing higher MSCP tolls will produce a greater reduction in TNTC. This is illustrated in figures 6.10 and 6.11.

It can be seen that the MSCP toll solution in figure 6.10 results in a slightly smaller reduction in Total Network Cost, than the 'SO' solution in figure 6.11. The demand *Q* has been reduced to the same value in both cases (to a degree of accuracy). The SO tolls have a larger reduction in TNTC per unit of revenue collected, 0.068% as opposed to 0.052%, and fewer links are tolled than for MSCP tolls. The pattern of link tolls is not particularly similar, whilst link 2 (5-7) is highly tolled in both cases, the other tolled links in the 'SO' solution do not correspond with particularly highly tolled links with MSCP tolls, the MSCP tolls being far more evenly spread throughout the network.

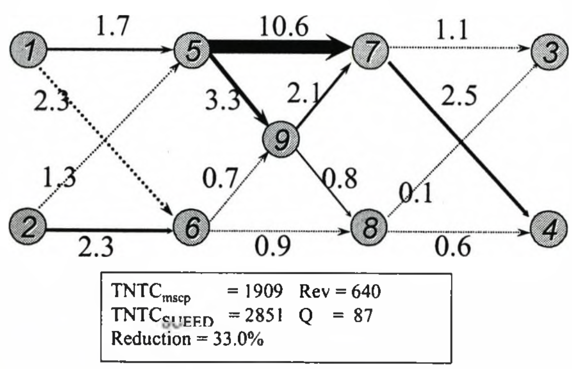


Figure 6.10: MSCP tolls under SUEED.

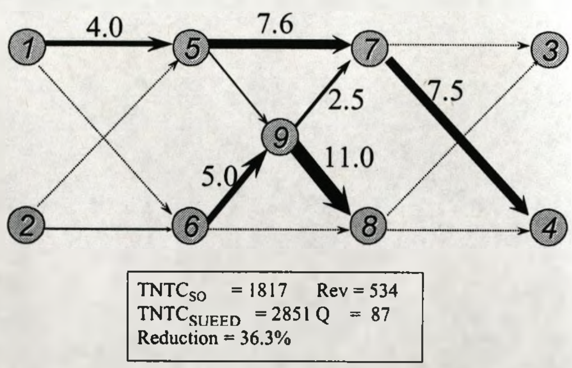


Figure 6.11: 'SO' tolls under SUEED (8-Iterations)

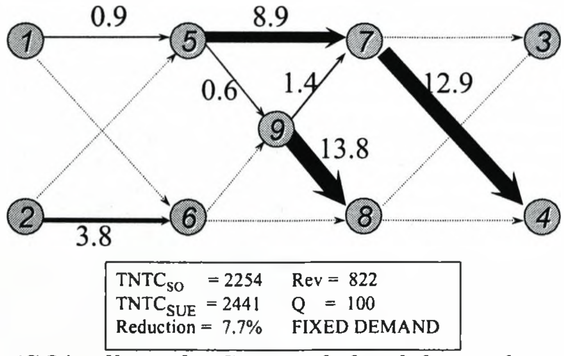


Figure 6.12: 'SO' tolls under SUE with fixed demand

As a significant portion of the TNTC reduction is due to suppressed traffic it is useful to compare these results with the toll set to achieve the 'SO' under fixed demand as shown in figure 6.12 above. It can be seen that the achievable reduction in TNTC by re-routing alone is much less than that achieved in either of the elastic demand cases by a combination of re-routing and traffic suppression. The revenue required to achieve this reduction was though significantly higher in the fixed demand case than in either of the elastic demand cases, which makes the case for tolling under elastic demand look rather more promising than under fixed demand from the perspective of the amount of potential network benefit which might be achieved by low-revenue tolling.

6.5 Conclusions

This chapter has considered network tolling in the case of stochastic user equilibrium with elastic demand. As in the case of stochastic assignment with fixed demand, the desired flow pattern to be achieved through tolling is not as obvious as in the deterministic case, and it is of interest to consider whether a 'true SO' flow pattern may be obtained through tolling. In the elastic demand case the deterministic SO flow pattern is not fixed and varies with the changing OD demand matrix as imposed tolls further suppress network demand from the SUEED solution. A heuristic derived to calculate tolls which would induce the SO in the deterministic case has been adapted to account for the changing desired SO flow pattern when demand is permitted to be elastic. It is seen that the reduction in TNTC achieved by tolling to obtain the 'SO' (to a sensible level of convergence) would appear to be slightly higher than that achieved by MSCP tolls in the elastic demand case, where the final value of demand is similar in each case. Consequently it would appear than the 'SO' tolling methodology is slightly preferable as it produces a greater reduction in network cost for a comparable amount of traffic suppression. Other scheme design factors may also need to be taken into account and finding a reasonable compromise between desired demand and desired revenue to be collected, as well as optimising the network as efficiently as possible with regard to re-routing is desirable.

It is suggested that achieving the SSOED through tolling under stochastic assignment with elastic demand would be analogous to achieving the SOED through tolling under deterministic assignment with elastic demand. In this case the revenue collected from any valid toll set would be expected to be constant and equal to that collected under MSCP tolling, as the economic benefit maximisation formulation would be expected to define an optimal OD matrix. As it was seen to be possible in this chapter, to achieve an 'SO' solution (for a particular value of demand) under SUEED, it will be likewise possible to achieve such an 'SSO' solution under SUEED. The formulation the SSOED and a discussion and comparison of relevant tolling methodologies is the subject of Chapter 7.

CHAPTER 7

Achieving SSO in the Stochastic Case with Elastic Demand

7.1 Introduction

This chapter examines tolling to achieve SSO flow patterns under stochastic assignment methods when elastic demand is permitted. Tolling to achieve the SO under deterministic assignment with elastic demand has been discussed in section 6.2 where an SOED formulation is given (Hearn and Yildirim, 2002). Under this formulation there is a lack of opportunity for additional optimisation to reduce the total revenue required to be collected from the users and this chapter will examine the extension of this result under stochastic assignment. The potential to derive meaningful reduced revenue toll sets to induce the deterministic SO under SUEED was also discussed in chapter 6 and this concept is extended to producing toll sets to induce stochastic system optimal (SSO) solutions in this chapter.

Section 7.2 presents a Stochastic System Optimal with Elastic Demand (SSOED) objective function where economic benefit will be maximised, this objective function is a natural extension of previous results. Section 7.3 examines tolling to induce the SSOED by implementing MSCP tolls under SUEED. Section 7.4 discusses the possibility of deriving toll sets with lower revenue (that MSCP tolls) that would produce 'an' SSO flow pattern and section 7.5 extends the heuristic used previously to obtain such toll sets. These are then compared with toll sets which induce deterministic System Optimal solutions under SUEED in section 7.6 and section 7.7 presents chapter conclusions.

7.2 Formulation of SSOED

It has been shown in previous work (Maher et al 2005) that under stochastic assignment modelling it is possible to define an objective function for a 'Stochastic Social Optimum' (SSO) where the total *perceived* network travel cost of the users is to be minimised. This definition can be seen to show a pleasing symmetry with the objective function for SO in the deterministic case;

Deterministic Objective functions:

$$Z_{SO}(\mathbf{h}) = \sum_{rs}\sum_{j} h_j^{rs} c_j^{rs}(\mathbf{h}) = \sum_{a} x_a t_a(x_a) \qquad 7.1$$

$$Z_{UE}(\mathbf{x}) = \sum_{a} \int_{0}^{x_a} t_a(\omega)d\omega \qquad 7.2$$

Stochastic Objective functions (logit formulation)

$$Z_{SSO}(\mathbf{h}) = \sum_{rs}\sum_{j} h_j^{rs} c_j^{rs}(\mathbf{h}) + \frac{1}{\theta}\sum_{rs}\sum_{j} h_j^{rs} \log h_j^{rs} \qquad 7.3$$

$$Z_{SUE}(\mathbf{h}) = \sum_{a} \int_{0}^{x_a} t_a(\omega)d\omega + \frac{1}{\theta}\sum_{rs}\sum_{j} h_j^{rs} \log h_j^{rs} \qquad 7.4$$

Stochastic Objective functions (general case)

$$Z_{SUE}(\mathbf{x}) = -\sum_{a} \int_{0}^{x_a} t_a(u)du + \sum_{a} x_a t_a(x_a) - \sum_{rs} q_{rs} S_{rs}[\mathbf{t}(\mathbf{x})] \qquad 7.5$$

$$Z_{SSO}(\mathbf{x}) = -\sum_{a} \int_{0}^{x_a} m_a(u)du + \sum_{a} x_a m_a(x_a) - \sum_{rs} q_{rs} S_{rs}[\mathbf{m}(\mathbf{x})] \qquad 7.6$$

where at the SSO solution it is therefore the case that:

$$y_a(\mathbf{m}(\mathbf{x})) = x_a \quad \forall\, a \qquad 7.7$$

where y_a are the auxiliary flows in a stochastic loading.

Deterministic Objective functions with elastic demand:

In the case of elastic demand, as presented in section 6.2, the objective functions in the deterministic case where the inverse demand function D_{rs}^{-1} has been included have been shown to be:

$$Z_{UEED} = \sum_a \int_0^{x_a} c_a(\omega)d\omega - \sum_{rs} \int_0^{q_{rs}} D_{rs}^{-1}(\omega)d\omega \qquad 7.8$$

$$Z_{SOED} = \sum_a x_a c_a(x_a) - \sum_{rs} \int_0^{q_{rs}} D_{rs}^{-1}(\omega)d\omega \qquad 7.9$$

In the case of stochastic assignment, it would seem tempting to subtract an extra term $\sum_{rs} \int_0^{q_{rs}} D_{rs}^{-1}(\omega)d\omega$ from the SUE objective function (7.5) to extend the formulation to include elastic demand, but as shown in Maher et al (1999), such a naïve approach would not result in an objective function which is convex at the SUEED solution.

Maher et al, instead proposed the single objective function (3.7) below;

Stochastic UE objective function with elastic demand: SUEED

$$Z_{SUEED} = \sum_a x_a c_a(x_a) - \sum_a \int_0^{x_a} c_a(x)dx + \sum_{rs} D_{rs}^{-1}(q_{rs}) D_{rs}(S_{rs}(\mathbf{c}(\mathbf{x})))$$

$$- \sum_{rs} S_{rs}(\mathbf{c}(\mathbf{x})) D_{rs}(S_{rs}(\mathbf{c}(\mathbf{x}))) + \sum_{rs} \int_0^{q_{rs}} D_{rs}^{-1}(q)dq - \sum_{rs} q_{rs} D_{rs}^{-1}(q_{rs}) \qquad 7.10$$

where the elastic demand q_{rs} is assumed to depend only on the expected minimum travel cost S_{rs}, (the satisfaction function) for OD pair r-s, such that;

$$q_{rs} = D_{rs}(S_{rs}) \quad \forall r,s \qquad 7.11$$

This was reduced (Rosa and Maher, 2002b) to a formulation which looks rather more like the naïve formulation, but which relies upon integrating over the Satisfaction function rather than the over the flow.

Alternative Stochastic UE objective function with elastic demand: SUEED

$$Z_{SUEED}(\mathbf{x}) = \sum_a x_a c_a(x_a) - \sum_a \int_0^{x_a} c_a(x)dx - \sum_{rs} \int_{S_{0rs}}^{S_{rs}(\mathbf{x})} D_{rs}\ (S_{rs}(\mathbf{c}(\mathbf{x})))dS_{rs} \qquad 7.12$$

There are several possible forms for the demand function (Maher et al, 1999), the Power Law form (7.13) will be used later in this chapter for illustrative purposes,

$$D_{rs}(S_{rs}) = D_0\left(\frac{S_0}{S_{rs}}\right)^e \qquad 7.13$$

where D_0 and S_0 are base values for the demand and satisfaction, and e is the fixed elasticity. The SUEED solution occurs where the Satisfaction value obtained from the inverse demand function associated with (7.13) above is equal to the Satisfaction value obtained from a stochastic loading. Using the balanced demand algorithm (BDA) (Maher et al, 1999) the demands are maintained in balance with the flows throughout. In the logit-based case (which is used for later numerical examples) the Satisfaction function may be written explicitly as (7.14) below (Williams, 1977), where i is summed over all paths between r and s:

$$S_{rs} = -\frac{1}{\theta}\ln(\sum_i \exp(-\theta(C_i))) \qquad 7.14$$

Thus it is possible to formulate an analogous objective function for the stochastic system optimum with elastic demand, i.e. where economic benefit is maximised. Combining the results of Maher et al (1999) and Maher et al (2005), a suitable objective function may be defined as (3.11) below;

Stochastic SO objective function with elastic demand: SSOED

$$Z_{SUEED} = \sum_a x_a m_a(x_a) - \sum_a \int_0^{x_a} m_a(x)dx + \sum_{rs} D_{rs}^{-1}(q_{rs}) D_{rs}(S_{rs}(\mathbf{m(x)}))$$

$$-\sum_{rs} S_{rs}(\mathbf{m(x)}) D_{rs}(S_{rs}(\mathbf{m(x)})) + \sum_{rs} \int_0^{q_{rs}} D_{rs}^{-1}(q)dq - \sum_{rs} q_{rs} D_{rs}^{-1}(q_{rs}) \qquad 7.15$$

where arguments regarding convexity are a direct analogy to those given in Maher et al (1999) and where at the SSO solution it is therefore the case that:

$$y_a(\mathbf{m(x)}) = x_a \quad \forall\ a \qquad 7.16$$

where y_a are the auxiliary flows in a stochastic loading.

Thus the stochastic social optimum solution with elastic demand (SSOED) may be determined by replacing the unit-cost functions in an SUEED assignment, with marginal social cost functions.

The more compact objective function (Rosa and Maher, 2002b), may be used in the same way.

Alternative Stochastic SO objective function with elastic demand: SSOED

$$Z_{SUEED}(\mathbf{x}) = \sum_a x_a m_a(x_a) - \sum_a \int_0^{x_a} m_a(x)dx - \sum_{rs} \int_{S_{0rs}}^{S_{rs}(\mathbf{x})} D_{rs}\ (S_{rs}(\mathbf{m(x)}))dS_{rs} \qquad 7.17$$

Minimisation of the objective functions (7.10) and (7.15) is illustrated using the 2-link network as shown in figure 7.1, using logit-based stochastic assignment ($\theta = 0.1$), with a single demand function of the form (3.9) where $D_0 = 2779$, $S_0 = 5.26$ and $e = 0.7$.

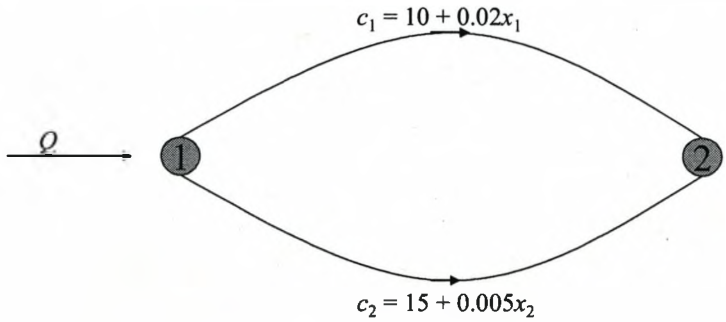


Figure 7.1: 2-link network with elastic demand

The objective functions for SUEED and SSOED are illustrated in figures 7.2 and 7.3, below.

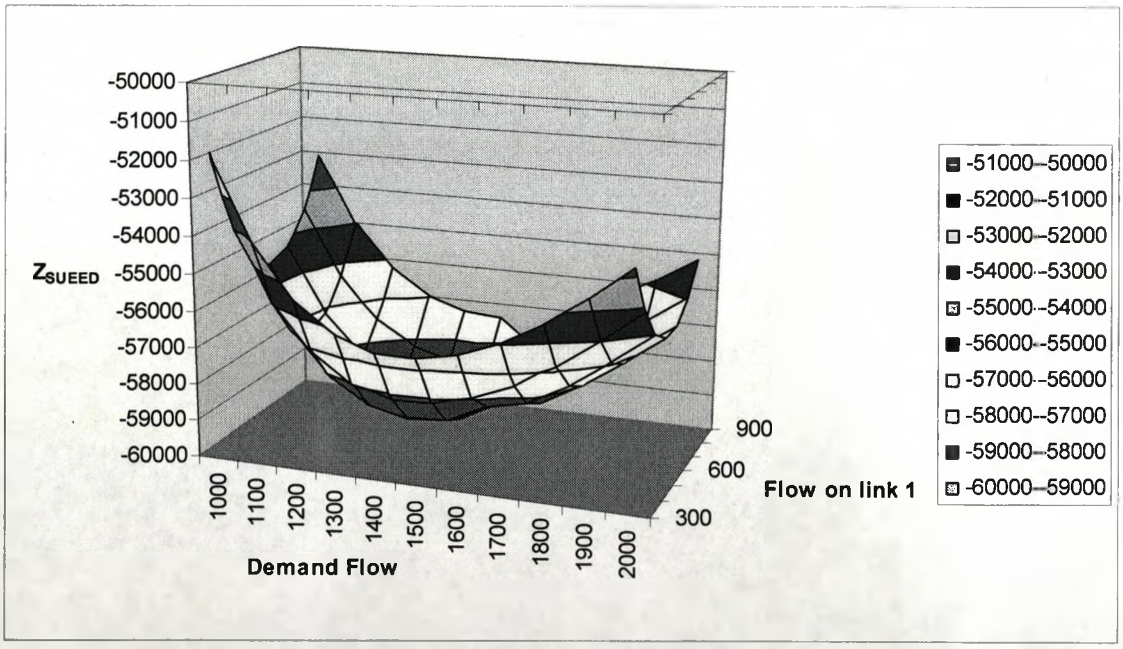


Figure 7.2: Objective Function for Stochastic User Equilibrium with Elastic Demand.

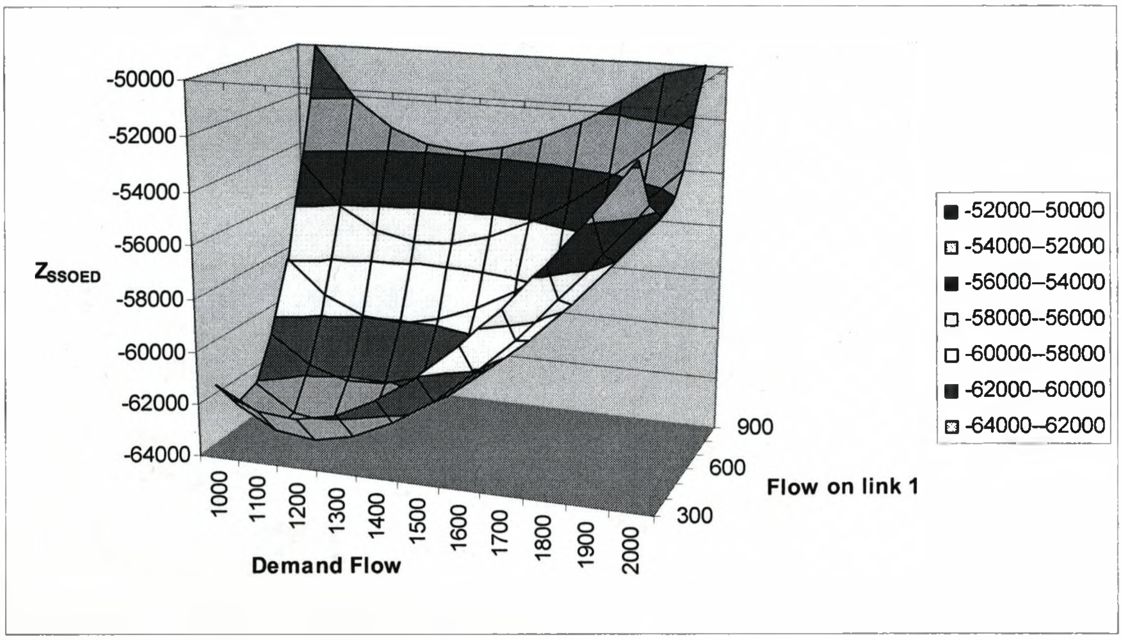


Figure 7.3: Objective Function for Stochastic Social Optimum with Elastic Demand

The solutions for the 2-link network are given in table 7.1 below, where x_a is the flow on link a, Q is the equilibrium demand flow and Z is the minimum value of the objective function:

	x_1	x_2	Q	Z
SUEED	607.4	822.8	1430.2	-59384
SSOED	441.3	743.6	1184.9	-63549.5

Table 7.1: SUEED and SSOED solutions for 2-link network

It can be seen that the socially optimal solution has the effect both of suppressing overall network flow (by 20.7%; from 1430 for SUEED to 1185 for SSOED, and of reducing the relative flow on link 1 w.r.t link 2 (from 42.5% of the total flow on link 1 for SUEED to 37.2% for SSOED).

It is possible to verify the correctness of these objective functions by performing standard stochastic assignment to determine the SUE and SSO flows and costs with the values of demand obtained. It can then be checked that the correct demand value is recovered when these costs are used in the demand function. For the above results the correctness was verified in this way.

7.3 Achieving the SSO through tolling MSCP tolls: Welfare Maximisation

The socially optimal flow pattern and network demand may be derived theoretically by using marginal social cost price functions in the place of the unit cost flow functions in an SUEED assignment. This flow and demand pattern will however not occur in reality unless drivers are tolled and forced to pay for the cost of their externalities on each link, i.e. when marginal social cost price (MSCP) tolls are applied.

As for the deterministic case given in section 6.2, when examining the case of stochastic assignment with elastic demand, the presence of the demand function means that the SSOED solution achieved by applying MSCP tolls to an SUEED assignment will not be achieved by a set of 'reduced revenue' tolls. Any reduction in the total revenue of the toll-set, will result in more demand flow being permitted onto the network, and thus the objective function (7.15) not being minimised.

For the 2-link network given in figure 7.1 (using the Balanced Demand Algorithm (BDA) rather than the pseudo-link methodology), the SSOED flow/demand pattern and MSCP tolls are given in table 7.2 below.

Link	x_i	m_i	c_i	*mscp* t_i
1	441.3	27.7	18.8	8.83
2	743.6	22.4	18.7	3.72

Table 4: mscp tolls to achieve SSOED under SUEED assignment

Total Demand Q = 1184.9, Total Revenue = 6662.9 and TNTC = 22201.8

As in the deterministic case, whilst there is a single unique value of total revenue which must be imposed upon users to achieve the SSOED, in a general network there may be scope for some additional optimisation with regard to which links along a path are to be tolled. Thus the total path toll must remain fixed at the MSCP value, but there may be some scope for some zero-toll links while other links are made more expensive to maintain the total revenue requirement. Therefore some additional optimisation may be possible with respect to which links are to be tolled but any reduction in total toll revenue will necessarily not produce the SSOED. It is however possible that it would not be politically desirable to impose a set of tolls for which the total cost imposed upon users is too high and consequently it is of interest to determine lower revenue tolls that produce some desirable rerouting effect whilst relaxing the requirement for full economic benefit maximisation. The nature of such tolls is discussed in section 7.4 and a method for their calculation is presented in section 7.5.

7.4 Minimal Revenue SSO tolls.

As for the deterministic case with elastic demand that was discussed in section 6.2 it would seem feasible that a minimal revenue toll set could be used to achieve an SSO flow pattern under SUEED assignment, but that the SSO flow pattern achieved would not be that which was associated with maximising net user benefit. Thus it would be possible to impose tolls that would result in an SSO solution for a particular value of demand, rather than the SSOED.

In the case of the previously used 2-link example (figure 7.1) in the fixed demand case a minimal revenue toll set could be easily derived from the marginal social cost price (mscp) toll set by setting the lower tolled path to zero and placing a toll equal to the difference in mscp tolls on the other path (as given in section 5.5). In the elastic demand case any reduction in network cost will result in the release of suppressed traffic onto the network. If mscp tolls were imposed they would result in the SSOED flow pattern being produced under an SUEED assignment and the traffic being suppressed to a particular demand value. If minimal revenue tolls were to be calculated from the mscp tolls the reduction in costs on the network would lead to an increase in demand and the 'target' SSO flow pattern being altered. However if the 'minimal revenue' toll set (t_1, t_2) = (5.11, 0) [where $t_1 = mscpt_1 - mscpt_2$] were to be used as an initial toll set, it is possible to recalculate the network demand flows and iterate (in a similar manner to that illustrated in figure 6.7) to achieve a converged solution as given in table 7.3 below.

Link	x_i	m_i	c_i	***Min Rev*** t_i
1	484.0	29.4	19.7	5.377
2	860.5	23.6	19.3	0

Table 7.3: Min Rev tolls to achieve an SSO solution under SUEED assignment

Total Network Demand Q = 1344.5, Total Revenue = 2602.5 and TNTC = 26142.5

For comparative purposes, the SUEED solution for the 2-link network is given in table 7.4 below;

Link	x_i	c_i
1	607.4	22.1
2	822.8	19.1

Table 7.4: SUEED solution for 2-link network

Total Network Demand Q = 1430.2, Total Revenue = 0 and TNTC = 29181.6

Thus the SSOED solution provides a reduction in TNTC of 23.9%, and the SSO(Min-Rev) 10.4% when compared to the TNTC for SUEED. Much of this reduction for the SSOED however, may be accounted for by the larger suppressed demand associated with larger revenue toll sets, the SSO(Min Rev) in fact shows a larger reduction in TNTC per 1000 units of revenue collected; 4.0%, as opposed to 3.6% for SSOED.

This section has demonstrated for the simple 2-link network that it is possible to use the minimal revenue toll set that would have been determined in the fixed demand case as an initial toll set to find a converged 'minimal revenue' toll set in the elastic demand case. In general networks however it is not so simple to calculate minimal revenue toll sets and it would not be desirable to have to do so multiple times as part of an iterative procedure. Section 7.5 presents a heuristic to calculate reduced revenue toll sets that induce SSO flow patterns under SUEED.

7.5 Heuristic to achieve SSO toll sets under SUEED

Whilst a 2-link network has been used here for illustrative purposes, it is clearly desirable to consider larger networks where the determination of a minimal revenue toll set is not as obvious as in the 2-link case. It is however easy to adapt the heuristic presented in previous chapters, to achieve an SSO solution by reduced revenue toll sets in the elastic demand case, in the same way that it was adapted to calculate SO toll sets under elastic demand in chapter 6. It is simply required that instead of recalculating a deterministic SO flow pattern as a target flow pattern for particular demand matrices, that an SSO flow pattern is used. The adapted heuristic is presented below.

Step 1: Link toll vector set to zero: $\mathbf{T}_0 = \mathbf{0}$

Step 2: Set $n = 0$

Step 3: Perform SUEED with current link tolls $\mathbf{T}_n$ and obtain OD matrix, $\mathbf{OD}_n$, link costs $\mathbf{C}_n$ and link flows $\mathbf{F}_n$

Step 4: Find the deterministic SSO solution $\mathbf{SSO}_n$ using $\mathbf{OD}_n$ and determine link flows $\mathbf{F}^{(SSO_n)}$, link costs $\mathbf{C}^{(SSO_n)}$, and the total network travel cost $TNTC_{SSO}{}^n$.

Step 5: Calculate: $P_j = \left(F_j^{(n)} - F_j^{(SSO_n)}\right)\left(\left|C_j^{(n)} - C_j^{(SSO_n)}\right|\right)$

Step 6: Determine link j where P_j is greatest.

Step 7: Perform iteration to calculate t_j s.t $F_j^{(n)} = F_j^{(SSO_n)}$ to required degree of accuracy.

7a: Set $t_{j_0} = \left|C_{j_0} - C_j^{(SSO_n)}\right|$ where C_{j_0} is the current cost on link j (as per step 4)

7b: Set m=1

7c: Perform SUEED assignment, calculate $\left|C_{j_m} - C_j^{(SSO_n)}\right|$

7d: Set $t_{j_m} = t_{j_{m-1}} + \left| C_{j_m} - C_j^{(SSO_n)} \right|$

7e: Calculate P_{j_m} : Stop if sufficiently close to zero and let $t_j = t_{j_m}$, or set

$m = m + 1$ and repeat from step 7c.

Step 8: $\mathbf{T}_{n+1} = \mathbf{T}_n + \mathbf{t}$; where $t_i = t_j$ when $i = j$ and $t_i = 0$ otherwise

Step 9: Calculate $TNTC_{SUEED}{}^{n+1}$ and $\mathbf{OD}_{n+1}$ and record $\mathbf{C}_{n+1}$ and $\mathbf{F}_{n+1}$

Step 10: Use $\mathbf{OD}_{n+1}$ to determine $\mathbf{SSO}_{n+1}$ and record $\mathbf{F}^{(SSO_{n+1})}$, link costs $\mathbf{C}^{(SSO_{n+1})}$ and $TNTC_{SSO}{}^{n+1}$.

Stop if $TNTC_{SUEED}$ is sufficiently close to $TNTC_{SSO}$

or set $n = n + 1$ and repeat from Step 5.

Thus it is possible to achieve reduced revenue toll sets to produce SSO or SO solutions for particular values of demand using a similar methodology under SUEED assignment methods. A comparison of tolling to achieve either SO or SSO flow patterns while allowing demand to be elastic is given in section 7.6. Minimal revenue tolls are compared using the 2-link network and the above heuristic is utilised to obtain SSO results for the previously used 9-node network which are then compared to the SO results given in section 6.3.

7.6 Comparison of SSO tolls with SO tolls

This section compares toll sets derived to induce either SO or SSO flow patterns under SUEED. Results are compared for the 2-link and 9-node networks which have been previously defined.

For the 2-link network (figure 7.1), the minimal revenue tolls required to achieve an SSO solution under SUEED are compared with the minimal revenue tolls that would be required to achieve a deterministic SO solution. The results are given in tables 7.5 and 7.6.

Link	x_i	c_i	***Min Rev*** t_i
1	352.1	17.0	11.98
2	908.6	19.3	0

Total Network Demand Q = 1260.755, Total Revenue = 4218.2 and TNTC = 23759.1

Table 7.5: Min Rev tolls to achieve an SO solution under SUEED assignment

Link	x_i	c_i	***Min Rev*** t_i
1	484.0	19.7	5.377
2	860.5	19.3	0

Total Network Demand Q = 1344.5, Total Revenue = 2602.5 and TNTC = 26142.5

Table 7.6: Min Rev tolls to achieve an SSO solution under SUEED assignment

Whilst the minimum revenue required to force the network into a deterministic SO flow pattern is 62% larger than the minimum revenue required to achieve an SSO flow pattern, the reduction in TNTC relative to SUEED (per 1000 unit revenue reduction) is 4.4% which is a greater reduction than the 4.0% achieved by the SSO min rev tolls.

Thus in the 2-link network the SO produces the greatest reduction in TNTC per unit revenue imposed, but this is at the expense of a much greater required revenue than for the SSO. The TNTC per unit demand is also lower for the SO (with min rev tolls) 18.8 as opposed to 19.4 for the SSO (with min rev tolls).

Whilst in theory the SO may seem to be more desirable as a target flow solution (in terms of network optimisation), the larger costs that it would impose upon the user may make the SSO with reduced revenue tolling seem the most desirable in practice. The 2-link example is however a special case, so the comparison is also made for the more general 9-node network (as defined in figure 6.8).

Using the heuristic presented in section 7.5, and applying it to Bergendorff's 9-node network, the iterative building of the toll-set is given in table 7.7.

Iteration	0	1	2	3	4	5	6	7	8
t_1 (1-5)	-	-	-	-	-	-	-	**0.1**	0.1
t_2 (5-7)	-	**11**	11	11	11	**10.6**	10.6	10.6	10.6
t_3 (7-3)	-	-	-	-	-	-	-	-	-
t_4 (1-6)	-	-	-	-	-	-	-	-	-
t_5 (2-5)	-	-	-	-	-	-	-	-	-
t_6 (5-9)	-	-	-	**1.8**	1.8	1.8	**1.9**	1.9	1.9
t_7 (9-7)	-	-	**3**	**3.2**	3.2	3.2	3.2	3.2	3.2
t_8 (6-9)	-	-	-	-	-	-	-	-	-
t_9 (9-8)	-	-	-	-	-	-	-	-	-
t_{10} (7-4)	-	-	-	-	**0.5**	0.5	0.5	0.5	0.6
t_{11} (8-3)	-	-	-	-	-	-	-	-	-
t_{12} (2-6)	-	-	-	-	-	-	-	-	-
t_{13} (6-8)	-	-	-	-	-	-	-	-	-
t_{14} (8-4)	-	-	-	-	-	-	-	-	-
TNTC	2851	2581	2447	2362	2347	2354	2350	2348	2345
TNTC: SSO	2719	2515	2419	2349	2337	2345	2341	2340	2337
	4.6%	2.6%	1.1%	0.6%	0.4%	0.4%	0.4%	0.3%	0.3%
REV	0	216	310	363	373	368	370	371	373

Table 7.7: Iterative building of toll-set to seek an SSO.

It can be seen that in 8-iterations of the heuristic, the solution is much closer to the desired 'SSO' (0.3%), than was the case in chapter 6 (table 6.3) for a desired 'SO' (1.1%). The revenue extracted from the users is much lower ('SSO': 373 vs 'SO': 534

vs MSCP: 640). The suppression in network demand associated with the above toll sets is given in table 7.8. It can be seen that the suppression in traffic with 'SSO' tolls is smaller than that for 'SO' tolls, and hence the revenue extracted from the users for the 'SO' would be expected to be relatively larger. It can be seen that after 8-iterations the 'SSO' tolls would reduce the TNTC by 21.6% ('SO': 36.3%, MSCP: 33.0%), and the corresponding reduction in demand is 9.23% ('SO': 20.9%, :MSCP: 20.9%). The reduction in TNTC per unit of revenue collected is 0.058% ('SO': 0.068%, MSCP: 0.052%). Thus the 'SSO' tolls do not suppress demand to the extent of the MSCP tolls and the 'SO' tolls, but whilst the 'SO' tolls result in the greatest percentage reduction in TNTC per unit of extracted revenue, the 'SSO' tolls are somewhere between the 'SO' and the MSCP.

Iteration	OD (1-3)	OD (1-4)	OD (2-3)	OD (2-4)	Total demand: Q
0	12.0	21.5	35.5	41.4	110.5
1	11.4	21.0	33.1	39.9	105.3
2	10.9	20.6	31.9	39.2	102.6
3	10.8	20.3	31.2	38.5	100.8
4	10.7	20.3	31.1	38.5	100.6
5	10.8	20.1	31.2	38.1	100.3
6	10.8	20.2	31.3	38.2	100.5
7	10.8	20.2	31.3	38.2	100.4
8	10.8	20.1	31.3	38.2	100.3

Table 7.8: Decreasing Demand with tolling: 'SSO'.

The toll sets derived from 8-iterations of the heuristic, for both 'SSO' and 'SO' tolls are given in figures 7.4 and 7.5

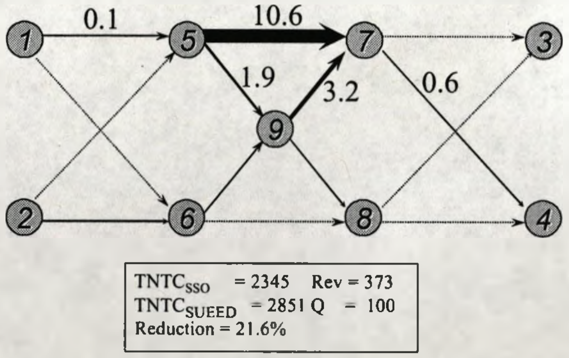


Figure 7.4: 'SSO' tolls under SUEED (8-Iterations).

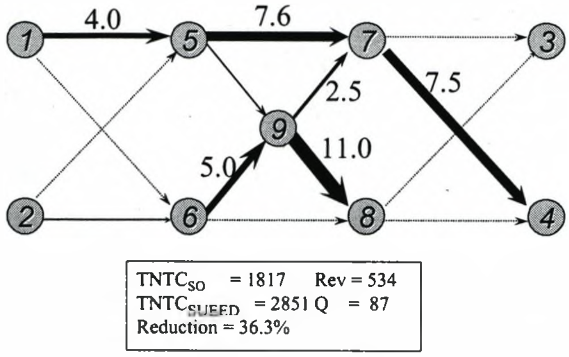


Figure 7.5: 'SO' tolls under SUEED (8-Iterations)

It can be see that the results for both the 2-link and 9-node networks show similar trends in that the 'SO' tolls produce the greatest reduction in Total Network Travel cost per unit of revenue charged to the users, but in absolute values the amount of revenue extracted is least in the case of 'SSO' tolls which do not suppress traffic to the same extent.

7.7 Conclusions

This chapter has considered network tolling in the case of stochastic user equilibrium with elastic demand, and has presented an objective function for the Stochastic Social Optimum with Elastic Demand (SSOED). As in the case of stochastic assignment with fixed demand, the desired flow pattern to be achieved through tolling is not as obvious as in the deterministic case. It is possible to derive toll-sets which will produce full economic benefit maximisation at the SSOED solution (MSCP tolls) or reduced revenue toll sets which can produce solutions approaching either the SSO or SO solutions for particular values of demand. Thus it may be seen that a range of possible effects may be achieved by the choice of tolls to be used. Whilst economic benefit maximisation may be desirable in certain contexts, the high revenue required to be imposed upon users is generally thought to make such toll sets unrealistic from a political viewpoint. It is of relevance therefore to consider the implications of applying reduced or minimal revenue tolls which produce flow patterns which are sub-optimal in terms of total economic benefit, but which are optimal for a particular level of demand. In this case it may be considered whether it is more desirable to seek an SSO flow pattern (where total perceived network travel cost is minimised for a particular demand value) or an SO flow pattern (where 'true' total network cost (TNTC) is minimised).

Further work is required to apply the heuristics to realistic networks so that comparisons between the different possible desirable flow patterns, and the revenue required to achieve them may be made. This is discussed in chapter 8.

CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

8.1 Introduction

This thesis has examined minimal-revenue and low-revenue tolling methodologies to induce or approach System Optimising flow patterns under stochastic user equilibrium assignment. In the stochastic case the desired flow pattern to be achieved by tolling is not as clear as in the deterministic case; either the deterministic SO where actual TNTC is minimised or the stochastic SSO where perceived TNTC is minimised (and in the case of MSCP-tolls economic benefit is maximised) are possible desired flow patterns. Tolling methodologies have been extended from the fixed demand case to the elastic demand case, where viable SO or SSO solutions are desired for particular demand values under SUEED.

This final chapter summarises and concludes this thesis. Conclusions from Chapters 4-7 are included in section 8.2, Implications for the real-world implementation of link or path-based tolling schemes are considered in section 8.3 and areas for future proposed or potential research are outlined in section 8.4.

8.2 Conclusions

Jones (2003) comments that there is a commonly held public misconception that a significant drop in traffic levels, such as of the order of 50%, is required to result in the better network conditions that are experienced during school holidays. In fact this effect is achieved from traffic reduction of the order of around 15% and significant network benefit may result from lower traffic reduction. Whilst small toy-networks have been utilised in this thesis to illustrate the tolling methodologies proposed, it is nonetheless of interest to compare the network cost reductions achieved in each case.

In the fixed demand case, for Bergendorff's 9-node toy network, the SO represents a possible 8.3% reduction in TNTC from the SUE solution. A 7.1% reduction in TNTC would be achieved from the use of the preferred low-rev toll set where the heuristic was halted after 4-iterations. In the idealised Edinburgh network, the potential reduction in TNTC (if the SO flow pattern was induced) was 10.5% and the preferred low-rev toll set achieved in 2-iterations of the heuristic produced a reduction in TNTC due to rerouting of 6.1%. The reduction in TNTC is clearly not synonymous with the reduction in traffic (being more comparable with a reduction in congestion). The London scheme monitoring report compared a roughly 30% reduction in congestion with a 15% reduction in circulating traffic (TfL, 2004a). Thus assuming reduction in congestion is roughly double the reduction in traffic (for reductions of similar orders of magnitude), a 6-7% reduction in congestion might be comparable to that expected from roughly a 3% reduction in traffic, and this reduction in congestion is in fact achieved with no reduction in traffic as it is due entirely to re-routing. Clearly the methodology presented should be applied to real networks to assess to potential reduction in TNTC from such tolls due entirely to re-routing. The heuristic presented is essentially a hillclimbing algorithm and there is no supposition that in the general case toll sets created would approach the desired SO to any desired degree of accuracy. Testing on larger networks would be required to assess the potential reduction in TNTC that could be achieved from such tolls in the more general case. As mentioned in Chapter 1, FORTRAN re-coding did not comprise part of this project, and this would have to be completed before testing could be undertaken on realistic networks. The principle however of assessing and charging the 'worst' links for tolling would seem to be most reasonable, as the extra revenue required to reach the SO is unlikely to be justifiable and achieving most of the SO benefits would be acceptable. In the case of Minimal-revenue tolls to achieve the SSO only a 4.5% reduction in TNTC was achieved for Bergendorff's 9-node network, as compared to 7.1% after 4-iterations of the heuristic to seek the SO. It would seem therefore that in the fixed demand case it is reasonable to utilise Low-Rev SO tolls due to their significantly better result for network re-routing.

In the elastic demand case both re-routing and traffic suppression occur whether an SO or an SSO flow pattern is desired under SUEED. After 8-iterations of the heuristic applied to Bergendorff's 9-node network, the SSO tolls resulted in a reduction in TNTC of 21.6% where the corresponding reduction in demand was 9.23%, and for SO tolls the reduction in TNTC was 36.3% where the corresponding reduction in demand was 20.9%. Clearly in the elastic demand case the desired demand level may be desired to be set by the planner in order to limit congestion levels but allow a certain level of access. SO tolls obtained from this methodology give the most favourable reduction in TNTC per unit of revenue collected, but in practice this might not be the most desired objective and alternative objectives could be used utilising the same methodology.

The basic assumptions for this research were based on wardropian principles of equilibrium, namely that drivers will act selfishly to minimise their own travel costs rather than altruistically to minimise costs of the overall system. There is some evidence however that users may be less selfish and more altruistic than generally assumed under Wardrop's principles. If an individual for instance believes themselves to be partly responsible for a problem, they might view themselves as having a responsibility to help to solve that problem (Schade, 2003) without necessarily being subject to a financial penalty. Thus if a particular choice has greater utility for an individual but a different choice has greater benefit to society as a whole the individual will not always choose the first option (Jaensirisak, 2003). This tendency towards some degree of societal cooperation was demonstrated in an agent-based simulation (Avineri, 2006), where actual route choice behaviour was observed to be somewhere between the expected wholly selfish UE and the totally cooperative altruistic SO (in the experiment perfect network knowledge was given so deterministic assignment was expected). If users are not as selfish in practice as has been assumed then the potential benefit in achieving an SO flow pattern might be less than expected. However the cost to achieve it would similarly be less than expected, so the benefit wrt cost imposed might be similar. Further work would be required to assess whether Avineri's results might be observed in a real world situation.

8.3 Considerations for the Implementation of a link-based tolling scheme

Since the publication of the Smeed report in 1964 (now over 40 years ago), it has been hypothesised that the technology required to successfully implement Road User Charging is always around 10-15 years in the future (Goodwin , 2005, in LTT 409). Goodwin however claims that as in the London case, where political will exists appropriate technology can be found. This section discusses some of the considerations inherent in the actual implementation of a link or path-based tolling scheme, such as proposed in this thesis.

8.3.1 Practical Implementation Considerations

Operational road user charging schemes have used a cordon system, which has the benefit of being transparent and easy to implement, and acts to discourage drivers from entering the controlled area, but once the driver is within the cordon, there is no additional incentive to choose a route which would be beneficial to the system as a whole. Trials have however been carried out in which tolling schemes have been tested with road pricing measures such as; distance travelled, time spent travelling and congestion caused (Cambridge study (May and Milne, 2000)). It may then be possible, with the current advances in Intelligent Transport Systems (ITS) to implement a path- or link-based tolling system, such as would be required to implement the type of tolling outlined in this thesis. Various technology is in existence which could make link- or path-based schemes possible. For link-based schemes, any existing electronic charging scheme could potentially be implemented, if a roadside beacon were to be placed on each link to be charged, and some sort of IVU capable of electronic payment were installed in each vehicle. This may require a larger than desired investment in infrastructure, and may result in an unacceptably high adverse visual impact in the area. Siting of numerous on-street beacons may also impede pedestrian movements. The effects could potentially be reduced by implementing a toll set which maximised the number of uncharged links, whilst still optimising with respect to other required conditions. Such a scheme would however require each vehicle to have an IVU, such as the scheme operating in Singapore, which might be more politically difficult to implement in the UK. If separate link charges were to be made, it would not be feasible to operate a paper permit type scheme, such as existed in Singapore prior to ERP, or to operate a pre-payment scheme as has recently been introduced in London. For path-based tolling schemes however, the vehicle would have to be tracked through the network, as individual links would not necessarily have the same charge. Thus the payment card (or similar in the IVU), would not be debited directly on passing a link beacon. This raises some privacy issues, as discussed below, but also requires more complex technology. It is possible to track vehicles through a network by using simply number plate recognition cameras (or manual data collectors) at suitable route points, and to track the plates. This method has been generally used to estimate OD matrix data in peak periods etc (Clegg, 2003). There are however various problems with this technique, which would make it an unreliable basis for a charging scheme, such as the difficulties in camera setup to ensure that all vehicles are reliably tracked, and the need for a suitable enforcement system. Such considerations would indicate that the use of a GPS in-car navigation system combined with a vehicle information and communication system (VICS), and a payment system would be preferable. Whilst such technology is not in general use for charging motorists in the UK there is a currently ongoing 'pay-as-you-drive' scheme run by Norwich Union (Norwich Union, 2006) where motorists are charged for their insurance based on a fixed monthly fee plus a usage charge, the usage being monitored by use of in-car GPS units. The trial started with 6,500 GPS units installed in the cars of volunteers and this was extended in March 2006 with an additional 10,000 units and was oversubscribed indicating the willingness of some drivers to permit this type of monitoring. Japan currently has a VICS service which covers virtually the whole country (Iguchi, 2002), and whilst not currently used for ERP, 7 million car navigation systems are in use, of which 2 million have VICS capability to receive real time information. A navigation system would have tracking capabilities and so, in combination with VICS and a payment system, could be used to implement a path based charging scheme. While GPS technology has not yet been implemented as a payment method for road user charging the existence of GPS units in cars for other purposes, such as paying for insurance or receiving navigation information, suggests that their utilisation for road user charging could be feasible.

8.3.2 User Uncertainty

A common criticism of any variable or multiple charge tolling scheme, is that they are not transparent and therefore result in user uncertainty. Whilst it is undoubtedly true that a fixed charge single cordon type scheme is the easiest to understand, uncertainty can be reduced by the use of advanced ITS solutions. A GPS and real-time VICS system could give advance cost information for a particular route, for a static scheme, or cost estimates for a dynamic or time variable scheme. The final charge for a trip could also be clearly displayed at the trip end, so that charge knowledge would be built up over time. Thus whilst uncertainty concerns might be a basis for initial public resistance, they might not in fact be insurmountable in practice.

Whilst the tolling methodology derived in this thesis is based on static rather than dynamic assignment, it would be expected that were it to be implemented in practice, it should be time variant with respect to differing OD matrices for particular time intervals. In practice it is currently difficult to obtain accurate OD data, and so it is not generally available for short time intervals, but if route data was constantly being transmitted to a central VICS then this could be used to constantly update OD matrices, and potentially to produce toll path vectors in real time. This information could then be available to the driver as an integrated part of the information system.

8.3.3 Privacy Issues

The combination of advanced ITS and ERP systems has the potential to provide large amounts of detailed real time location and identification information (Ogden, 2001). Privacy issues are a serious concern in Western society, where privacy is viewed as a basic human right; for example there is strong opposition in the UK to the introduction of personal identity cards. If a route-based tolling scheme were to be implemented, the user would have to be reassured that detailed tracking information would not be linked to them personally, or combined with other electronic data (such as financial details), and used for example for marketing purposes. Whilst it is important to be able to identify vehicles in the case of payment violation, the user would expect that their details were not stored for other purposes. In the case of the London scheme, the enforcement scheme is designed to store details only of violators after a particular day has passed, so that cumulative historical data that could identify users personally is not stored. Whilst a route-based scheme could potentially record more detailed information, if it were only matched to vehicle registration details in the event of a violation, then non-violators' privacy would be maintained. It has further been claimed that as the population becomes accustomed to being tracked through voluntarily used devices such as mobile phones, aversion to such surveillance will lessen.

8.4 Future Work:

This work has been based on the assumption of a static stochastic equilibrium model (with either fixed or elastic demand) with a single user class. An obvious extension to the completed work is to include multiple user classes and this extension is summarised in section 8.4.1. Static assignment methods have been used throughout this thesis; it is however reasonable to wish to model the variable and peaked features of traffic throughout a day and possibly weekly and seasonal variations. It is possible to apply static assignment models for time slices throughout the day to approximate to the dynamic nature of traffic, but various genuinely dynamic models exist and it would be of interest to examine optimal tolling using dynamic assignment models. This is discussed in section 8.4.2.

8.4.1 Multiple User Classes.

It is of interest to consider the case for tolling to achieve optimal traffic flows through a network when potentially a different set of link tolls may be applied to different segments of the driving population. It may well be of interest to have different charging rates for different classes of vehicles, such as standard cars and HGVs and additionally it may be politically desirable to exempt certain classes of vehicles from paying any toll, such as PSVs Taxis etc. Such differences in vehicle classification may be modelled as Multiple User Classes (MUC), where the different groups of drivers are assigned different generalised cost functions to account for the difference in travel time and route constraint between groups. In stochastic modelling this may be achieved by assigning different values to the dispersion parameter (Maher and Hughes, 1997a), and objective functions for MUCSUE have been defined as extensions of the SUE objective function. (Maher and Hughes 1996, Maher 1998). In an MUC SUE assignment model no driver could reduce their perceived travel cost by changing route in each class, whilst all classes are assigned in the network allowing for interaction.

Proposed future work will extend the formulation for SSO to include MUC, such that total perceived network travel costs for all classes are minimised and will present an objective function for MUC SSO. To achieve such a flow pattern under an MUCSUE assignment, link tolls will be applied, which may differ for each user class. Marginal social cost price (MSCP) tolling will be examined for differing user classes, and then the possibility of reduced (or minimal) revenue tolling strategies to produce the same effect will be illustrated. It may be politically desirable to allow that one or more MUCs should be exempt from tolls, and the possibility of producing toll sets that apply to less than the total number of user groups, but still result in the MUCSSO flow pattern being achieved will be considered. The methods used to derive toll sets will be equally applicable to any stochastic assignment method.

It is clear that imposing tolls on a network, will directly affect demand as well as being able to influence route choice. Elastic Demand (ED) as previously discussed may be readily included in stochastic equilibrium models (Maher and Hughes, 1997b), and in the SSO case, MSCP tolls may be derived by using marginal cost functions in an SUEED algorithm. It is of interest to allow for both MUC's and ED, and the proposed future work will utilise the methodology presented in Chapters 6 and 7 of this thesis to present an objective function for MUCSSOED as an extension to that for MUCSUEED (Maher and Zhang, 2000), and will further discuss the options for reduced revenue tolls to create MUCSSO solutions under MUCSUEED for particular values of demand. The possibility of allowing for zero tolled user classes will be considered.

8.4.2 Dynamic Assignment

It would be of interest to examine tolling to achieve optimal flow patterns in the case of Dynamic assignment. It is possible to model traffic dynamically either macroscopically (as for the static assignment models utilised in this thesis), or microscopically by modelling each vehicle independently. Several microsimulation packages are currently available (PARAMICS, VISSIM), and it would be of interest to develop toll sets which might replicate system optimal flow patterns in an aggregate sense for specific time slices. Paramics routes traffic essentially by an all-or-nothing assignment, but the existence of a feedback feature enables the user to require the model to recalculate least cost paths every time increment (minimum of every one minute) (S-Parmics, 2005). Thus over an aggregate time interval multi-routing as in a static model could be achieved. As for any simulation based model there are issues regarding repeatability and the necessity to complete multiple runs which would make the derivation of 'optimal' toll sets difficult. It should however be possible to at least examine the effect of total network travel cost over a specific time slice in relation to tolls imposed.

Analytical macroscopic assignment models continue to be a difficult research problem (e.g. Heydecker and Addison, 2006), and no generally accepted dominant approach exists. It would be desirable to define a dynamic user equilibrium (DUE) and a dynamic system optimal (DSO) and extend tolling methodologies accordingly in both the deterministic and stochastic cases. Maher and Rosa (2006) have been developing a probit-based stochastic dynamic traffic assignment model utilising a cell transmission model, but numerical results remain at a preliminary stage. Chow (2006) has presented a formulation for a dynamic system optimal, and has also presented work relating to developing as general dynamic road pricing framework (Chow, 2005), but this is limited at present to a whole-link model (i.e. a single link). Much analytical work in dynamic assignment remains focussed on whole link models while network assignment models tend to be simulation-based (e.g. Florian et al, 2006). Dynamic modelling is further not limited to within day traffic variations, but also day-to-day variations and Yang and Szeto (2006) have examined pricing towards a System Optimal using day-to-day dynamic models. Tolling in the dynamic case has many varied possibilities which remain open research questions.

REFERENCES

AFFORD project (1999). Oslo, www.vatt.fi/afford/oslo.htm

Avineri E. (2006). Modelling social interactions in multi-agent traffic simulations. *38th Annual Conference of the University Transport Study Group, Trinity College, Dublin*, 4-6 January 2006.

Bai L, Hearn D.W. and Lawphongpanich S. (2004). Decomposition Techniques for the Minimum Toll Revenue Problem. *Networks*, 44(2), 142-150.

Beckmann M, McGuire C.B and Winston C.B, (1956). *Studies in the Economics of Transportation*. Cowles Commission Monograph, Yale University Press, New Haven.

Bekhor S. and Toledo T. (2005). Investigating path-based solution algorithms to the stochastic user equilibrium problem. *Transportation Research* 39B, 279-295.

Ben-Akiva M, Bergman M.J, Daly A.J. and Ramaswamy R. (1984). Modelling inter urban route choice behaviour, in Volmuller J, Hamerslag, R. (Eds.), *Proceedings of the 8th IFAC Symposium on Transportation Systems*, 299-330.

Bergendorff P, Hearn D.W. and Ramana M.V, (1997). Congestion Toll Pricing of Traffic Networks, Network Optimization. Pardalos P, Hearn D.W, Hager W.W, (Eds.), *Lecture Notes in Economics and Mathematical Systems*, Springer-Verlag, 450, 51-71.

Cascetta E, Nuzzolo A, Russo F. and Vitetta A. (1996). A modified logit route choice model overcoming path overlapping problems: specification and some calibration results for interurban networks. In: Lesort, J.B. (Ed.), *Proceedings of the International Symposium on Transportation and Traffic Theory*, Lyon, 697-711.

Cascetta E, Russo F. and Vitetta A. (1997). Stochastic user equilibrium assignment with explicit path enumeration: comparison of models and algorithms. In: Papageorggiou M, Pouliezos A. (Eds.), *Proceedings of the 8th IFAC Symposium on Transportation Systems*, 1078-1084.

Cascetta E. (1990). Metodi Quantitativi per la Pianificazione dei Sistemi di Trasporto. Cedam. P71.

Chen M. and Alfa A. S. (1991). Algorithm for solving Fisk's stochastic traffic assignment model. *Transportation Research* 25B(6), pp 405-412.

Chow A. (2005). Toward a general framework for dynamic road pricing. *4th IMA International conference on Mathematics in Transport, 7-9 September 2005, University College London.*

Chow A. (2006). Analysis of dynamic system optimum and externalities with departure time choice. *Proc. 1st International Symposium on Dynamic Traffic Assignment, Leeds,* June 2006.

The City of Edinburgh Council (1999). Interim Local Transport Strategy.

The City of Edinburgh Council (2001). Integrated Transport Initiative for Edinburgh and South-East Scotland, Application to the Scottish Executive for Approval in Principle, Appraisal Summary Table and Technical Report.

City of Edinburgh Council (2004). The integrated Transport Initiative for Edinburgh and South East Scotland, proposed congestion charging scheme: statement of case. Tie Ltd. and the City of Edinburgh Council.

Clark C.E. (1961). The greatest of a finite set of random variables. *Operations Research* 9, 145-162.

Clegg R.G. (2003). A Freely Available Data Set For Modelling Day-To-Day Route Choice. *Proceedings of the 35th Annual UTSG conference (Loughborough University).*

Commission for Integrated Transport (2002). *Paying for Road Use*, www.cfit.gov.uk

Connors R D, Sumalee A and Watling D P. Equitable Network Design. *Journal of the Eastern Asia Society for Transportation Studies* 6, pp.1382-1397 2005

DETR (1998). *A New Deal for Transport: Better for everyone*, TSO, London.

Department of Transport Standing Advisory Committee on Trunk Road Appraisal (SACTRA) (1994). *Trunk Roads and the Generation of Traffic*, HMSO, London.

DfT (2003). Managing Our Roads. July 2003.

Dial R.B. (1971). A probabalistic multipath traffic assignment modal which obviates path enumeration, *Transportation Research* 5, 83-111.

Dial R.B. (1999). Minimal-revenue congestion pricing part I: A fast algorithm for the single origin case. *Transportation Research* 33B(3), 189-202.

Dial R.B. (2000). Minimal-revenue congestion pricing part II: An efficient algorithm for the general case. *Transportation Research* 34B(8), 645-665.

Florian M, Mahut M. and Tremblay N. (2006). A simulation-based dynamic traffic assignment model. *Proc. 1st International Symposium on Dynamic Traffic Assignment, Leeds,* June 2006.

Frey B.S. (2003). Why are efficient transport policy instruments so seldom used?, in *Acceptability of Transport Pricing Strategies*, Schade, J. and Schlag, B. Eds.Elsevier. 27-62.

Gaunt M. (2006). Public acceptability of road user charging: the case of Edinburgh and the 2005 referendum. *38th Annual Conference of the University Transport Study Group, Trinity College, Dublin*, 4-6 January 2006.

Glaister S, Burnham J, Stevens H. and Travers T. (1998). *Transport Policy in Britain*, Macmillan.

Harsman B. (2003). Success and Failure: Experiences from Cities, in *Acceptability of Transport Pricing Strategies*, Schade, J. and Schlag, B. Eds.Elsevier. 27-62.

Hearn D.W. and Ramana M.V. (1998). Solving Congestion Toll Pricing Models. *Equilibrium and Advanced Transportation Modeling*, Marcotte P, Nguyen (eds.), Kluwer Academic Publishers, 109-124.

Hearn D.W. and Yildirim M.B. (1999). A Toll Pricing Framework for Traffic Assignment Problems with Elastic Demand. *Transportation and Network Analysis - Current Trends (papers in honor of Michael Florian)*, Gendreau M. and Marcotte P. (Eds.), Kluwer Academic Press.

Hearn D.W. and Yildirim M.B. (2002). A Toll Pricing Framework for Traffic Assignment Problems with Elastic Demand. *Current Trends in Transportation and Network Analysis: Papers in honor of Michael Florian, M. Gendreau and P. Marcotte (Eds.),* Kluwer Academic Publishers, 135-145.

Hearn D.W. and Lawphongpanich S. (2003). Solving second best toll pricing problems, *Proc. The Theory and Practice of Congestion Charging: An International Symposium, Imperial College London.*

Heydecker B.G. and Addison J.D. (2006). Analysis of dynamic traffic assignment. *Proc. 1st International Symposium on Dynamic Traffic Assignment, Leeds,* June 2006.

Iguchi M. (2002). A perspective on ITS deployment. *JSAE Review* 23, 173-176.

Ison S. (1998). A concept in the right place at the wrong time: congestion metering in the city of Cambridge. *Transport Policy* 5(3), 139-146.

ITC (2003). Transport Pricing Better for Travellers. The Independent Transport Commission, June 2003.

Jaensirisak S, May A.D. and Wardman M. (2003). Acceptability of road user charging: the influence of selfish and social perspectives, in *Acceptability of Transport Pricing Strategies*, Schade, J. and Schlag, B. Eds.Elsevier. 203-215.

Jones P. (2003). Acceptability of Road User Charging: Meeting the Challenge, in *Acceptability of Transport Pricing Strategies*, Schade, J. and Schlag, B. Eds.Elsevier. 27-62.

Larsson T. and Patriksson M. (1998). Side Constrained Traffic Equilibrium Models- Traffic Management Through Link Tolls, in: *Equilibrium and Advanced Transportation Modeling*, (P. Marcotte and S. Nguyen Eds.), pp.125-151, Kluwer Academic Publishers.

Leurent F. (1994). Elastic Demand, logit-based equilibrium traffic assignment with efficient dual solution algorithm. *Unpublished preprint*, INRETS, Paris France.

Local Transport Today, (2002). Issue 351,Edinburgh referendum on charging could set decision-making trend. p1.

Local Transport Today, (2002). Issue 351, First Road-User Charging Scheme. p1.

Local Transport Today, (2005). Issue 409, Road-User Charging: don't wait for the big bang. p17.

Local Transport Today, (2005). Issue 421, Road pricing packages and infrastructure to aid GDP will share TIF resources. p1.

Local Transport Today, (2005). Issue 432, From B to W...the plans of the seven TIF pilot areas. p5.

Local Transport Today, (2006). Issue 443, Further rise in C-charge considered. p5.

M6 toll (2006). http://www.m6toll.co.uk/

Ma K-R. and Chatterjee K. (2006). Insights on travel behaviour dynamics from the London congestion charging panel survey. *38th Annual Conference of the University Transport Study Group, Trinity College, Dublin*, 4-6 January 2006.

Maher M.J, Hughes P.C. (1995). A Probit-based Stochastic User Equilibrium Assignment Model, *Transportation Research* 31B, 341-355.

Maher M.J, Hughes P.C. (1996). Estimation of the potential benefit from an ATT system using a multiple user class stochastic user equilibrium assignment model. In: *Applications of Advanced Technologies in Transportation Engineering* (eds: Stephanedes Y.J. and Filippi F.), American Society of Civil Engineers, 700 -704.

Maher M.J. (1997), Algorithms for logit-based Stochastic User Equilibrium Assignment Model, *Transportation Research*32B, 539-545.

Maher M.J, Hughes P.C. (1997a). A probit-based stochastic user equilibrium assignment model. *Transportation Research* 31B(4), 341-355.

Maher M.J, Hughes P.C. (1997b). An Algorithm for SUEED Stochastic User Equilibrium with elastic demand. *Presented at the 8th IFAC Symposium on Transportation Systems, Chania, Crete.*

Maher M.J. (1998). Algorithms for logit-based stochastic user equilibrium assignment. *Transportation Research,* 32B(8), 539-550.

Maher M.J. and Hughes P.C. (1998). New algorithms for the solution of the stochastic user equilibrium problem with elastic demand. In: *Proceedings of the 8th World Conference on Transport Research.*

Maher M.J., Hughes P.C. and Kim K.-S. (1999). New Algorithms for the Solution of the Stochastic User Assignment Problem with Elastic Demand. *Proceedings of the 14thInternational Symposium on Transportation and Traffic Theory,* (A. Ceder Ed.).

Maher M.J, Zhang X. (2000). Formulation and algorithms for the problem of stochastic user equilibrium assignment with elastic demand. In: *Proceedings of the 8th Euro Working Group meeting on Transportation, Rome*, September 2000.

Maher M.J, Stewart K. and Rosa A. (2005). Stochastic Social Optimum Traffic Assignment. *Transportation Research,* 39B(8), 753-767.

Maher M.J. and Rosa A. (2006). The Development of a probit-based stochastic dynamic traffic assignment model. *Proc. 1st International Symposium on Dynamic Traffic Assignment, Leeds,* June 2006.

May A.D, Milne D.S. (2000). Effects of alternative road pricing systems on network performance. *Transportation Research* 34A(6), 407-436.

May A.D, Liu, R. Shepherd, S.P. and Sumalee, A. (2002). The impact of cordon design on the performance of road pricing schemes. *Transport Policy* 9, 209-220.

Mayeres I. (2003). Reforming Transport Pricing: An economic perspective on equity efficiency and acceptability, in *Acceptability of Transport Pricing Strategies*, Schade, J. and Schlag, B. Eds.Elsevier. 27-62.

Meng K.T, Bus A.B and AM A.I. (1999). Tolling Strategies and their Impacts on the Evaluation of Road Transport Projects, MSc Dissertation, Institute for Transport Studies, University of Leeds.

MVA consultancy (1997). *Impact of Highway Capacity Reductions*. Report on Modelling. MVA.

MVA consultancy (1999). City of Edinburgh Road User Charging Study, Transport Modelling, Presentation to Report Progress, 16 Dec.

MVA consultancy (2000). City of Edinburgh Road User Charging Study, Transport Modelling, Final Report, project number C31154.

Newbery D. and Santos G. (2003). Cordon tolls in eight English towns: theory simulations and impacts, *Proc. The Theory and Practice of Congestion Charging: An International Symposium, Imperial College London.*

Norwich Union (2006). 'Pay-as-you-drive' Insurance. www.norwichunion.com/pay-as-you-drive/

Ogden K.W. (2001). Privacy issues in electronic toll collection. *Transportation Research* 9C, 123-134.

Oppenheim N. (1995). *Urban travel demand modelling: from individual choices to general equilibrium*. Wiley.

Ortuzar J. and Wilumsen L. (1994). *Modelling Transport*, Wiley.

Penchina C. (2002). Flexibility of tolls for optimal flows in networks with fixed and elastic demands. *Proc. 2002 Annual Meeting, Transportation Research Board, Washington D.C.*

Penchina C. (2004). *Minimal-revenue congestion pricing: some more good-news and bad news. Transportation Research* 38B, 559-570.

Pigou A.C. (1920) *The Economics of Welfare.* MacMillan, New York.

PLANNING (2005). Selling Congestion charging. 4th March 2005, p16.

PROGRESS project (2000). www.progress-project.org

PROGRESS (2004). Pricing Road Use for Greater Responsibility Efficiency and Sustainability in Cities, Main Project Report, July 2004.

Richards M, Gilliam C, Larkinson J. (1996). The London Congestion Charging Research Programme, 1. The programme in overview, Traffic Engineering and Control, February pp66-71.

Road Charging Options for London: A Technical Assessment, http://www.open.gov.uk/glondon/transport/rocol.htm

Rosa A. (2001). Path-based Traffic Assignment with Probit Analytical Methods. *Proc. 33rd annual Universities Transport Study Group conference (Oxford University).*

Rosa A, Maher M.J. (2002a). Algorithms for solving the probit path-based stochastic user equilibrium traffic assignment problem with one or more user classes. In: *Transportation and Traffic Theory in the 21st Century. Proceedings of the 15th International Symposium on Transportation and Traffic Theory* (ed. M.A.P. Taylor), Pergamon Press. 371-392.

Rosa A. and Maher M.J. (2002b). Stochastic user equilibrium traffic assignment with multiple user classes and elastic demand. *Proceedings of the 9th Meeting of the Euro Working Group on Transportation.* Bari, Italy. 392-397.

S-Paramics (2005). Reference Manual, SIAS Ltd.

SATURN version 8.4, (1993). User Manual, The Institute of Transport Studies, The University of Leeds.

Schade J. (2003). European research results on transport pricing acceptability, in *Acceptability of Transport Pricing Strategies*, Schade, J. and Schlag, B. Eds.Elsevier. 109-136.

Seik F.T. (2000). An advanced demand management instrument in urban transport Electronic road pricing in Singapore, *Cities*, Volume 17, Issue 1, February 2000, pp33-45.

Sheffi Y. and Powell W. (1982). An algorithm for the equilibrium assignment problem with random link times. *Networks* 12, 191-207.

Sheffi Y. (1985). *Urban Transportation networks: Equilibrium Analysis with Mathematical Programming Methods*, Prentice-Hal.

Shepherd S.P, May A.D. and Milne D.S. (2000). The Design of Optimal Road Pricing Cordons. *9th World Conference in Transportation Research, Seoul.*

Sikow-Magny C. (2003). Efficient Pricing in Transport-overview of European Commissions's Transport Research Programme, in *Acceptability of Transport Pricing Strategies*, Schade, J. and Schlag, B. Eds.Elsevier. 13-26.

Smeed R.J (1964). *Road Pricing: the economic and technical possibilities.* HMSO, London.

Smith M.J. (1979). The marginal cost taxation of a transportation network. *Transportation Research* 13B, 237-242.

Smith T.E, Eriksson E. A, Lindberg P.O. (1994). Existence of Optimal Tolls under Conditions of Stochastic User-Equilibria. in *Road Pricing: Empirical Assessment and Policy*, Johansson B. and Mattsson L.G. (eds.), Kluwer Academic Publishers, Dordrecht, The Netherlands, 65-87.

Stewart K. and Maher M.J. (2005). Minimal revenue network tolling: system optimisation under stochastic assignment with elastic demand. *4th IMA International conference on Mathematics in Transport, 7-9 September 2005, University College London.*

Stewart K. (2006). Minimal Revenue Network Tolling: stochastic social optimisation under stochastic assignment with elastic demand. *38th Annual Conference of the University Transport Study Group, Trinity College, Dublin*, 4-6 January 2006.

Stewart K. and Maher M.J. (2006). Minimal revenue network tolling: system optimisation under stochastic assignment, in Hearn D.W, Lawphongpanich S, Smith M, (Eds.), *Mathematical and Computational Models for Congestion Charging, Applied Optimization*, Vol 101, New York, Springer.

Stewart K. (2007). Tolling Traffic Links under Stochastic Assignment: Modelling the relationship between the number and price level of tolled links and optimal traffic flows. *Transportation Research A (in press).*

Sumalee A. (2004a). Optimal road user charging cordon design: A heuristic optimisation approach. Computer-Aided Civil and Infrastructure Engineering 19, 377-392.

Sumalee A. (2004b). An innovative approach to option generation for road user charging scheme design: Constrained and multi-criteria design, *Proc. 10th World Conference on transport research, Istanbul.*

Sumalee A. May, A. and Shepherd, S. (2005). Comparison of judgemental and optimal road pricing cordons. *Transport Policy, forthcoming.*

Sumalee A. Connors R. and Watling D. (2006). Optimal toll design problem with improved behavioural equilibrium model: the case of probit model, in: Hearn D.W, Lawphongpanich S, Smith M, (Eds.), *Mathematical and Computational Models for Congestion Charging, Applied Optimization*, Vol 101, New York, Springer.

Taha H.A., (1976). *Operations Research: an Introduction.* Collier Macmillan.

Thomas R. (1991). Traffic Assignment Techniques, Avebury Technical

Tie Ltd. (2000). Transport Initiatives Edinburgh: http://iti.tiedinburgh.co.uk/

Transport for London, TfL (2004a). *Congestion Charging: Update on scheme impacts and operations*, February 2004.

Transport for London, TfL (2004b). *Impacts monitoring: second annual report*, April 2004.

Tretvik, T. (2003). Urban Road Pricing in Norway:Public Acceptability and Tracel Behaviour, in *Acceptability of Transport Pricing Strategies*, Schade, J. and Schlag, B. Eds.Elsevier. 77-92.

Trivector (2005). Evaluation of the congestion charge trial in Stockholm. Trivector Traffic AB, 16th February 2005.

US Bureau of Public Roads (1964). *Traffic Assignment Manual*, U.S. Department of Commerce, Washington D.C.

Verhoef E. (2002). Second-best congestion pricing in general static transportation networks with elastic demands. *Regional Science and Urban Economics* 32, 281-310.

Verhoef E. (2002b). Second-best congestion pricing in general static transportation networks. Heuristic Algorithms for finding second-best optimal toll levels and toll points. *Transportation Research B* 36, 707-729.

Vickrey W. (1955). Some implications of marginal cost pricing for public utilities. *American Economic Review* 45, p605-620.

Walters A. (1954). Track costs and motor taxation. *Journal of Industrial Economics* 2, p135-146.

Wardrop J.G. (1952). Some theoretical aspects on road traffic research. *Proc. Inst. Civil Engineers* 11 (1), 325-378.

Wong S.C, Ho H.W, Yang H. and Loo B. (2003). The first-best and cordon-based second-best congestion pricing in a continuum traffic equilibrium system. *Proc. The Theory and Practice of Congestion Charging: An International Symposium, Imperial College London.*

Williams H.C.W.L. (1977). On the formulation of travel demand models and economic evaluation measures of user benefit. *Environment and Planning A*, 9(3), 285-344.

Yang F. and Szeto W.Y. (2006). Day-to-day congestion pricing policies towards system optimal. *Proc. 1st International Symposium on Dynamic Traffic Assignment, Leeds,* June 2006.

Yang H. (1997). Sensitivity analysis for the elastic demand network equilibrium problem with applications. Transportation Research 31B(1), 55-70.

Yang H. (1999). System Optimum, Stochastic User Equilibrium, and Optimal Link Tolls. *Transportation Science* 33(4), 354-360.

Yildirim M.B. and Hearn D.W. (2005). A first best toll pricing framework for variable demand traffic assignment problems. *Transportation Research* 39B, 659-678.

Zhang X. and Yang H. (2004). The optimal cordon-based network congestion pricing problem. *Transportation Research* 38B, 517-537.

Personal Communication

Simpson M. (2006). City of Edinburgh Council: Cycling officer. Meeting on 28th April 2006.

APPENDIX A.1

	Path-flow	Path-Cost	tp(2-6) -tp1	tp(3-6)-tp2	tp(4-6)-tp3	tp(5-6)-tp4	tp6-tp5	order(asc)			
1	2.23721162	15.3635253						tp4	0	tp1	12.28273
2	1.38574603	23.3494241	-3.195982411					tp6	1.741054	tp2	9.086749
3	0.0062255	30.603592	43.603246	46.79922841				tp2	9.086749	tp3	55.88598
4	3.45516869	23.2998466	**-12.28273127**	**-9.086748862**	**-55.88597727**			tp1	12.28273	tp4	0
5	0.03237252	30.5540145	27.16626714	30.36224955	-16.43697886	**39.44899841**		tp5	39.449	tp5	39.449
6	2.88327564	23.3682348	-10.54167691	-7.345694501	-54.14492291	**1.741054361**	-37.707944	tp3	55.88598	tp6	1.741054
			tp(8-12)-tp7	**tp(9-12)-tp8**	**tp(10-12)-tp9**	**tp(11-12)-tp10**	**tp12-tp11**				
7	3.50287712	18.0325692						tp12	0	tp7	15.42842
8	1.4181727	26.018468	1.056255734					tp11	12.13793	tp8	16.48468
9	0.91744207	29.2112459	2.218828258	1.162572524				tp7	15.42842	tp9	17.64725
10	1.51566495	25.9688904	0.440982868	-0.615272866	-1.77784539			tp10	15.8694	tp10	15.8694
11	1.59955251	29.1616684	-3.290491708	-4.346747442	-5.509319966	-3.731474576		tp8	16.48468	tp11	12.13793
12	11.0462907	21.9758888	-15.42841976	-16.4846755	-17.64724802	-15.86940263	**-12.137928**	tp9	17.64725	tp12	0
			tp(14-18)-tp13	**tp(15-18)-tp14**	**tp(16-18)-tp15**	**tp(17-18)-tp16**	**tp18-tp17**				
13	7.68724119	13.7034098						tp18	0	tp13	11.02066
14	10.4128277	21.6893086	-11.02066389					tp14	0.91218	tp14	0.91218
15	0.05407945	28.9434765	34.32856244	45.34922633				tp16	5.047461	tp15	45.34923
16	4.20259673	25.7152064	-5.973202935	5.047460954	-40.30176537			tp13	11.02066	tp16	5.047461
17	0.06844804	32.9693743	27.94645949	38.96712338	-6.382102949	33.91966242		tp17	38.96712	tp17	38.96712
18	7.57480691	25.7835946	-11.93284373	-0.912179844	-46.26140617	-5.959640798	-39.879303	tp15	45.34923	tp18	0
			tp(20-24)-tp19	**tp(21-24)-tp20**	**tp(22-24)-tp21**	**tp(23-24)-tp22**	**tp24-tp23**				
19	7.79391048	16.3724537						tp24	0	tp19	16.27123
20	5.98872569	24.3583525	-5.351258075					tp21	7.011025	tp20	10.91997
21	6.43335075	27.5511304	-9.26020469	-3.908946615				tp20	10.91997	tp21	7.011025
22	0.70895933	28.3842502	11.96120181	17.31245988	21.2214065			tp19	16.27123	tp22	28.23243
23	1.28594775	31.5770282	2.813892709	8.165150784	12.0740974	-9.1473091		tp23	19.08512	tp23	19.08512
24	17.789106	24.3912486	-16.27123011	-10.91997204	-7.011025424	-28.23243192	**-19.085123**	tp22	28.23243	tp24	0

SO path tolls for 9-node network: $\theta = 0.1$

APPENDIX A.2

Edinburgh toy-network: Link Data

Origin	Destination	Free-flow cost (c_0)	Capacity X
201	202	4.7	1200
202	201	4.7	1200
202	203	2.3	1800
203	202	2.3	1800
203	204	2.3	3500
204	203	2.3	3500
204	205	3.5	3500
205	204	3.5	3500
205	206	2.7	3500
206	205	2.7	3500
206	207	2.7	3500
207	206	2.7	3500
207	208	2.5	1500
208	207	2.5	1500
208	209	3	1500
209	208	3	1500
209	201	6.1	1200
201	209	6.1	1200
201	212	3.7	900
212	201	3.7	900
202	212	6.9	1000
212	202	6.9	1000
203	212	8.6	1000
212	203	8.6	1000
204	213	8.6	1200
213	204	8.6	1200
205	213	5.9	900
213	205	5.9	900
206	211	6.2	900
211	206	6.2	900
207	210	1	2500
210	207	1	2500
210	211	6	800
211	210	6	800
209	211	3.8	900
211	209	3.8	900
211	212	2.7	1000
212	211	2.7	1000
212	213	2.2	800
213	212	2.2	800
213	211	1.8	1000
211	213	1.8	1000

$c_a(x_a) = c_a(0)[\ 1 + k(x_a/X_a)^p]$ $\quad k = 1 \quad p = 4$

Edinburgh toy-network: OD Data

O \ D	201	202	203	204	205	206	207	208	210	211	212	213
201		150	75	150	150	150	400	225	200	400	300	100
202	190		180	275	190	180	350	95	250	375	375	90
203	190	125		125	10	10	550	350	250	325	400	250
204	300	155	300		100	100	375	190	190	190	310	100
205	125	125	250	95		95	325	125	130	190	200	95
206	50	125	250	95	100		320	190	250	225	230	95
207	290	125	500	125	130	100		100	325	350	400	350
208	300	200	250	100	100	125	150		300	400	400	250
210	25	25	100	50	50	50	100	100		50	50	50
211	60	60	250	75	75	75	300	350	250		150	125
212	100	100	250	100	100	100	200	200	200	50		50
213	100	100	250	100	100	100	100	100	150	50	50	

[NB. Node 209 is neither and origin nor destination]

APPENDIX A.3

0.01

mscp	path	minrev p	minrev l
5.662554	15.9107	12.98549	12.83795
9.398137	21.24762	18.32241	0
0.850005	21.10008	18.17487	0.147541
2.224155	12.14386	9.218653	0
1.857636	11.99632	9.071112	5.992468
9.626424	2.925205	0	5.336919
5.108633	22.82275	19.21776	0
3.961065	28.15967	24.55468	9.071112
5.522764	21.77986	18.17487	0
7.762055	19.05591	15.45092	6.379807
0.288333	12.6761	9.071112	0
5.26472	3.60499	0	0
0.412717	12.10578	6.140008	0
0.968117	17.4427	11.47693	0
	17.29516	11.32939	
	15.18442	9.218653	
	15.03688	9.071112	
	5.96577	0	
	19.01783	12.37227	
	24.35475	17.70919	
	17.97494	11.32939	
	22.09647	15.45092	
	15.71667	9.071112	
	6.645554	0	
Rev			
1577.105			1030.958

0.1

mscp	path	minrev p	minrev l
2.226369	17.63734	10.93942	2.914132
14.06763	12.19928	5.50136	5.438058
1.343346	9.612055	2.914132	2.587228
4.362869	10.0321	3.334173	0
2.057901	7.444868	0.746945	2.376139
5.56164	6.697923	0	0
3.067929	20.37456	12.20967	0
1.257952	14.9365	6.771612	0.746945
1.658949	11.07902	2.914132	0
4.080562	12.76931	4.604425	3.85748
0.165097	8.911833	0.746945	0
4.732394	8.164888	0	0
2.169957	17.46887	10.40143	0
1.632062	12.03082	4.963367	0
	9.443587	2.376139	
	10.40162	3.334173	
	7.814393	0.746945	
	7.067448	0	
	20.20609	11.67168	
	14.76803	6.233619	
	10.91055	2.376139	
	13.13884	4.604425	
	9.281358	0.746945	
	8.534413	0	
Rev			
1185.502			401.8251

0.2

mscp	path	minrev p	minrev l
1.666363	18.30589	11.50489	0.963943
14.91775	11.4466	4.645613	6.859282
1.721775	7.764935	0.963943	3.68167
5.127652	10.48266	3.68167	0
2.181513	6.800992	0	2.164609
5.1238	8.6538	1.852809	0
2.934667	19.76668	11.15352	0
0.698567	12.9074	4.294238	0
0.86084	9.577102	0.963943	0
3.182567	11.94345	3.330295	3.330295
0.113932	8.613159	0	0
4.442136	10.46597	1.852809	0
3.412216	18.82104	12.70556	1.852809
1.9261	11.96175	5.846279	0
	8.280084	2.164609	
	9.797146	3.68167	
	6.115476	0	
	7.968285	1.852809	
	20.28183	12.35419	
	13.42255	5.494904	
	10.09225	2.164609	
	11.25794	3.330295	
	7.927643	0	
	9.780452	1.852809	
Rev			
1200.141			459.2978

0.5

mscp	path	minrev p	minrev l
1.3268188	19.49522	13.00888	0.235601
15.792933	11.81511	5.328765	7.680112
2.3754653	6.721942	0.235601	5.093163
5.7526763	11.5795	5.093163	0
2.3526407	6.486341	0	2.93348
4.9930635	10.863	4.376656	0
3.1197575	19.62394	10.97554	0
0.3316047	11.94383	3.29543	0
0.3383703	8.884001	0.235601	0
2.5041891	11.70823	3.059828	3.059828
0.0636893	8.648399	0	0
4.0806191	13.02505	4.376656	0
5.0466306	20.52104	15.70676	4.376656
2.225748	12.84093	8.026644	0
	7.747764	2.93348	
	9.907447	5.093163	
	4.814283	0	
	9.190939	4.376656	
	20.64976	13.67342	
	12.96965	5.993309	
	9.909823	2.93348	
	10.03617	3.059828	
	6.976342	0	
	11.353	4.376656	
Rev			
1293.9343			632.6207

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mscp	path	minrev p	minrev l
1.224449	20.30331	13.88472	0.058431622
16.25584	12.40921	5.99063	7.894094193
2.823013	6.477014	0.058432	5.932198247
5.97701	12.35078	5.932198	0
2.451794	6.418582	0	3.372912308
5.024988	12.00682	5.588239	0
3.336763	19.73859	11.00464	0
0.213995	11.8445	3.110543	0
0.183762	8.792385	0.058432	0
2.258298	11.78607	3.052112	3.052111714
0.043815	8.733954	0	0
3.889874	14.32219	5.588239	0
5.985996	21.53065	17.1992	5.588238632
2.359187	13.63656	9.305111	0
	7.704359	3.372912	
	10.26364	5.932198	
	4.331446	0	
	9.919685	5.588239	
	20.96594	14.31912	
	13.07184	6.425024	
	10.01973	3.372912	
	9.69893	3.052112	
	6.646818	0	
	12.23506	5.588239	
Rev			
1368.475			733.7943325

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mscp	path	minrev p	minrev l
1.153548	21.29793	14.91845	0
16.77149	13.28893	6.909453	8.009
3.372894	6.379474	0	6.909453
6.144866	13.29318	6.913704	0.004251
2.563494	6.383725	0.004251	3.86105
5.114328	13.22753	6.848055	0
3.648157	19.99077	11.16324	0
0.127261	11.98177	3.154244	0
0.083558	8.827528	0	0
2.065739	11.98602	3.158495	3.154244
0.02804	8.831779	0.004251	0
3.689511	15.67558	6.848055	0
7.054623	22.70787	18.7795	6.843804
2.476094	14.69887	10.7705	0
	7.78942	3.86105	
	10.83782	6.909453	
	3.92837	0	
	10.77217	6.843804	
	21.40072	15.02429	
	13.39172	7.015294	
	10.23747	3.86105	
	9.530667	3.154244	
	6.376423	0	
	13.22023	6.843804	
Rev			
1464.173			850.9196

SSO path tolls for 9-node network: θ varies

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