SELECTING POTENTIALLY OPTIMAL ROUTES THROUGH OPTIMISTIC AND PESSIMISTIC NODE POTENTIALS

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Abstract
Existing assignment or route guidance models are not suitable to represent travellers` choices or to advise travellers in networks in which each link may be subject to failure with an unknown probability. A heuristic approach to select potentially optimal routes in such conditions is presented in this paper. Each link can be in failed or unfailed condition, but if failed the resulting delay can be time dependent. Conditions are discussed to select paths which are optimal in at least one scenario of link costs. Selection of potentially optimal links and route choice are decoupled. Both problems are solved by making use of node potentials under optimistic and pessimistic assumptions regarding link states. For pre-selection of a set of paths a simple heuristics is presented. Comparisons with an exhaustive method show that this approach may perform reasonably well in terms of potentially optimal links identified, and its run times are compatible with real world implementations. Finally, a route choice method based on traveller's attitude toward risk and regret is proposed.

1 Introduction
In the transport field, route guidance and flow assignment problems entail an implicit or explicit search for single or sets of shortest paths. The shortest path problem is a classical combinatorial optimization problem, for which efficient solutions exist both in the deterministic and the stochastic version (Ahuja et al., 1993, Pallottino and Scutellà, 1997). When the selection of the shortest path is affected by uncertainty (e.g. due to missing information or variability in users' perceptions) and hence by the risk of adopting wrong decisions, various decision principles are conceivable as discussed in Schmöcker (2010). These are minimising expected travel costs, optimism, pessimism and various forms of regret minimisation, including prospect theory.

Minimising expected costs remains one of the most common approaches. Such optimization can involve multiple path solutions, as in the case of the Stochastic User Equilibrium (Bell and Iida, 1997), and adaptive strategies, as it has been proven by Hall (1986) for time-dependent networks with stochastic arc costs and by Spiess and Florian (1989) for transit networks with random waiting time at stops. Studies on the route choice process show that models based only on expected total travel times may fail in capturing correctly traveller's decision making; especially the influence of travel time variability has to be considered (Noland and Polak, 2002). From a theoretical and implementation point of view the use of expected values is limited to cases in which the consideration of probabilities is meaningful and the probability functions are known. When this is not the case, for example because the traveller faces a given route choice only few times, or when suitable input data sets are not available because of for example low probability – high consequence problems, a different approach to cope with risks in decision making is needed.

The appropriateness of the different routing strategies to deal with risks can be studied within the theoretical framework of game theory where the traveller is aiming to minimise maximum risks (Bell et al., 2008, Hollander and Prashker, 2006). Bell (2000) models situation in which a "demon" strives to maximize the travel time between an OD pair by disrupting a single link over the network. Schmöcker et al. (2009) prove that the Spiess and Florian approach – applied by Bell (2009) to car navigation – is equivalent to a game in which a "local demon" can cause the failure of one of the options which might be attractive at each decision point. Restricting the possibility of failure to a single link – either at each point with alternative options or, *a fortiori*, over the entire network – may be not realistic. Szeto et al. (2007) and Schmöcker et al. (2009) discuss the possibility of multi demon games. Further doubts on the appropriateness of the Spiess and Florian algorithm for route choice are cast by the fact that the shortest path might not be part of the suggested hyperpath (Nguyen and Pallottino, 1988).

Route choice with a possibly unlimited number of scenarios of link parameters has been dealt with through robustness analysis. Min-max and min-max regret versions of the robust shortest path problem have been studied (Aissi et al., 2009). The former formulation, which provides the path with the best performance in the worst case, fits non recursive situations and situations in which a precautionary approach has to be adopted;

the latter looks for the solution which minimizes, over all the scenarios, the maximum difference between the path length and length of the optimal path in the corresponding scenario, and it is suitable when regret or rejoice for what could have happened is taken into account. The exact solution of the robust shortest path problem is at least NP-complete for networks with a finite number of scenarios, strongly NP-hard when link parameters vary within intervals; pseudo-polynomial algorithms have been provided for networks with a bounded number of scenarios (Yu and Yang, 1998). An approach considering exclusively min-max solutions might be too cautious for routing decisions in ordinary life because in most cases at least some vague information on incident probabilities is available. Therefore, Schmöcker (2010) proposes that mixing the min-max game with the expected value principle by limiting the attack probabilities is more likely to reflect reality in route choice decisions even of risk-averse travellers.

In this paper, similarly to the robust shortest path problem and expanding the results of Fonzone et al (2010) to the dynamic case, route choice is considered when each link may be subject to delay with an unknown probability. This is equivalent to consider a game against “nature” instead of a game against “demons” as in the aforementioned literature, in the sense that failures are random and not driven by the intent of maximizing travel times. The simplest discrete scenario case in which each link has just two possible travel times, corresponding to the free flow and delayed conditions is studied. FIFO is supposed to hold and delays are assumed time-dependent whereas the free flow travel time is assumed to be constant. Path selection and route choice are decoupled as in much literature on K–shortest path sets (see (Eppstein, 1998) for a general review or (Nielsen et al., 2005) for an application to hyperpaths). As in the hyperpath approach and differently from the robustness analysis one, a set of paths between a given OD pair is searched for, aiming to a solution endowed with four properties

- Routes are potentially optimal – i.e. each of them is optimal in at least one link state scenario –
- The undelayed conditions shortest path is included
- A small number of link state scenarios are explored
- Complete path enumeration is not required

The in the following proposed method is heuristic and relies on node potentials under optimistic and pessimistic assumptions of link states. The set identified through the heuristics can be used directly for assignment, considering all the potentially optimal links at each decision node equally attractive. Further, a specific path selection method is proposed based on traveller’s attitude toward risk and regret (Loomes and Sugden, 1982, Loomes and Sugden, 1987).

The paper is made up of two main sections. The first one deals with the problem of identifying links which are potentially optimal: initially assumptions regarding network properties are presented together with some definitions concerning potential optimality. Then a heuristic to identify a set of potentially optimal links is derived from definitions, and its performance on a small static network is compared with that of an exact method. The second section regards the use of the potentially optimal link set for route choice in assignment and navigation problems, bringing forward an approach considering the influence of risk aversion and fear of regret. Considerations on the scope of the results and proposals of further research conclude the paper.

2 Heuristics to identify potentially optimal links

2.1 Notation, network properties, and definitions

In the remainder of the paper the following notation is used

$\mathcal{G}(\boldsymbol{N}, \boldsymbol{A})$	Directed, acyclic graph with nodes $\boldsymbol{N}$ and arcs $\boldsymbol{A}$
I	Node $\in \boldsymbol{N}$
i	Link $\in \boldsymbol{A}$
O, D	Origin, destination
$H(i), T(i)$	Head node, tail node of link i
$FS(I)$	Forward star of node I
t	Generic time
$\pi_{[AB],[i]}$	Path [from A to B] [using link i]
$H(\pi), T(\pi)$	Destination, origin of path π
$c_i, d_i(t)$	Uncongested cost, potential additional cost on link i when entering the link at

	time t
$\omega \in \boldsymbol{\Omega}$	Cost scenario / Generic realization of link states (not congested/congested) in $\mathcal{G}$ within the set $\boldsymbol{\Omega}$ of all possible cost scenarios
$\chi_i^{\omega}(t)$	Actual cost on link $i = \begin{cases} c_i \text{ if } i \text{ is not congested} \\ c_i + d_i(t) \text{ if } i \text{ is congested} \end{cases}$ at link entry time t under cost scenario ω
Φ, Δ	Specific cost scenarios within set $\boldsymbol{\Omega}$ in which all links are not congested (optimistic scenario), congested (pessimistic scenario)
$\tau_I^{\omega}(\pi, t)$	Arrival/leaving time from I at t leaving from T(π) at time t and following π under cost scenario ω
$\gamma_{\pi}^{\omega}(t)$	Cost of path π at time t under cost scenario ω
$p_{AB,[i]}^{\omega}(t)$	Optimal path from A to B [using link i] at time t under cost scenario ω
$C_{A,[i]}^{\omega}(t)$	Cost of the optimal path from A to the destination (node potential of A) at time t under cost scenario ω [conditional on using link i]

For simplicity, for paths between nodes A and B, B is omitted in case it coincides with D. Elements between square brackets are also omitted when not relevant.
Costs are assumed coincident with travel times and waiting times at nodes are not considered, so that for a path π under a generic cost combination ω, the following relation holds

$$\tau_J^{\omega}(\pi, t) = \begin{cases} t \text{ if } J = T(\pi) \\ \tau_I^{\omega} + \chi_i^{\omega}\left(\tau_I^{\omega}(\pi, t)\right) \text{ otherwise, where } i=(I,J) \in \pi \end{cases}$$

FIFO networks are considered. This requires that

- Each link i is FIFO, i.e. the following holds for each delay function for all times t_1 and t_2

$$t_1 < t_2 \Rightarrow t_1 + d_i(t_1) < t_2 + d_i(t_2)$$

- The cost combinations ω are not time dependent, i.e. the state of a link does not change over time. However, if the state of a link is "delayed", the delay can be time dependent provided it complies with the previous condition.

The definitions below are considered in the following sections

DEFINITION 1: π_I is potentially optimal $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ ω and t such that $C_I^{\omega}(t) = \gamma_{\pi_I}^{\omega}(t)$, i.e. a cost combination ω and a time t exist such that under the cost combination ω, at the time t, π_I is the shortest path from I to D.

DEFINITION 2: i is potentially optimal $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ ω, t and $\pi_{T(i),i}$ such that $C_{T(i)}^{\omega}(t) = \gamma_{\pi_{T(i),i}}^{\omega}(t)$, i.e. there is at least a potentially optimal path from $T(i)$ to D including i.

DEFINITION 3: Let i and $j \in$ FS(I). i is potentially optimal with respect to j $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ ω, t and $\pi_{I,i}$ for which $\gamma_{\pi_{I,i}}^{\omega}(t) < C_{I,j}^{\omega}(t)$, i.e. a cost combination ω and a time t exist such that under the cost combination ω, at time t, there is a path from I to D using i which is shorter than all the paths from I to D using j.

DEFINITION 4: I is potentially optimal $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ potentially optimal i such that $I = H(i)$ or $I = O$, i.e. a node I is potentially optimal if it is the head of a potentially optimal link or is the origin.

2.2 Heuristics based on optimistic and pessimistic node potentials

From definitions 1 and 2 it follows immediately for nodes with several outgoing arcs that

LEMMA 1: i is potentially optimal $\Leftrightarrow \exists\ \omega$ and t for which i is optimal with respect to all $j \neq i \in FS(T(i))$

This means that the problem of identifying the set of the potentially optimal links from node I can be reduced to the study of the conditions of potential optimality between two links i and j leaving from the same node. Noting that the following holds

LEMMA 2: $C_I^{\Phi} < C_I^{\omega}(t) < C_I^{\Delta}(t)\ \forall I, \omega$ and t
from Definition 3 it can be deduced that

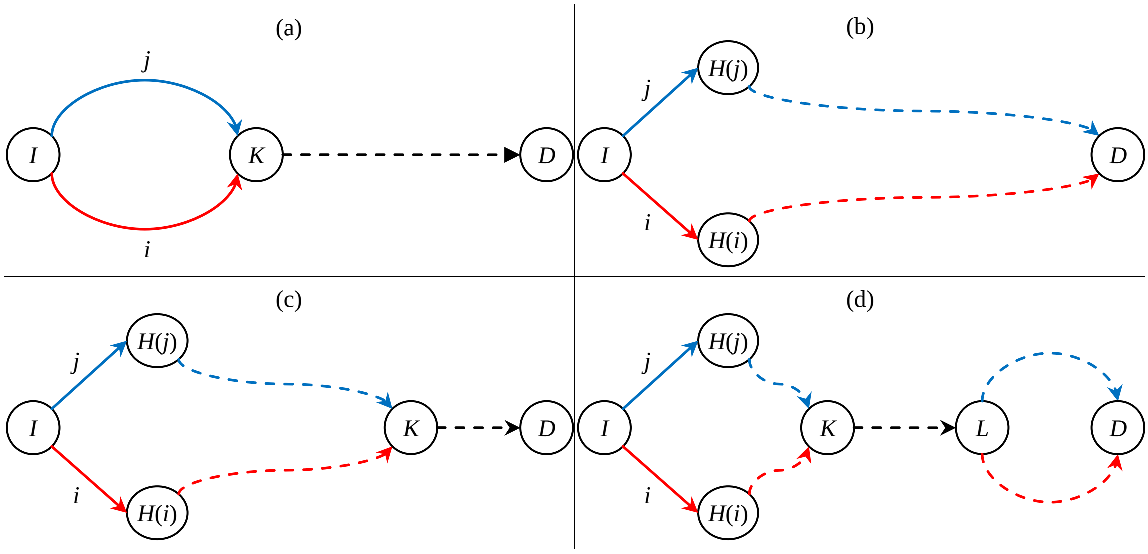
LEMMA 3 (Necessary condition for i to be potentially optimal with respect to j): i is potentially optimal with respect to $j \Rightarrow \exists\ t$ for which $c_i + C_{H(i)}^{\Phi} < c_j + d_j(t) + C_{H(j)}^{\Delta}\left(t + c_j + d_j(t)\right)$ [1]

Links for which [1] does not hold are never optimal and can hence be excluded from the set of potentially optimal links. Since $\{\Phi, \Delta\}$ constitute the minimum subset of the cost scenarios which has to be analysed to apply the necessary conditions, in the following it is analysed which subset of the actual potentially optimal links can be identified by using this scenario subset. Firstly note that [1] is not always valid as a sufficient condition for i to be potentially optimal with respect to j because of possible overlaps between the optimistic and pessimistic shortest paths departing from $H(i)$ and $H(j)$. The situation can be easily explained referring to a situation in which only two paths from I to D have to be compared. Let $i, j \in$ FS(I) and one path use link i whereas the other path includes link j. Three topological situations can be identified as shown in Figure 1:

- Case (a): $H(i) = H(j) = K$. The FIFO condition of the network involves that it is sufficient to study the optimality of the sub-path from I to K. It is evident that

  $$\exists\ t \text{ for which } c_i < c_j + d_j(t) \Rightarrow i \text{ is potentially optimal [2]}$$

  That is [1] is also a sufficient condition. Obviously at the times in which i is potentially optimal, it is optimal for every state of the links from K on. Note that Case (a) can only occur if the graph includes multiarcs.
- Case (b): $H(i) \neq H(j)$ and the two paths are completely disjoint. In this case it is possible that all links from i are not congested and all links from j are congested. For such a cost combination [1] is also a sufficient condition for potential optimality of i. i may not be optimal for every ω.
- Cases (c) and (d): $H(i) \neq H(j)$ and the two paths are partially overlapping. In such a situation the two conditions $C_{H(i)}(t+c_i) = C_{H(i)}^{\Phi}$ and $C_{H(j)}\left(t + c_j + d_j(t)\right) = C_{H(j)}^{\Delta}\left(t + c_j + d_j(t)\right)$ are incompatible because either all links from $H(i)$ to the destination, and in particular from K to D, are not congested or the entire path from $H(j)$ is congested. Therefore [1] is not valid as a sufficient condition. In the simple network in (c), the study of potential optimality of i from I to D can be reduced to the study of potential optimality of the subnetwork from I to K, which is of the kind described in Case (b). In general (Case (d)) to reach a conclusion on the optimality of i
  - Either all paths have to be enumerated to look for all situations to which [2] can be applied
  - Or the shortest path from I has to be found for all cost combinations. If n is the number of links connected both to I and D, the total number of cost scenarios to be considered is 2^n and hence the problem becomes rapidly not tractable as n increases. Note that to prove potentially optimality of a link , the number of cost combinations which need to be scanned can be much smaller than 2^n, but to prove that i is not potentially optimal, even though it complies with [1], all possible ω have to be analysed before the conclusion can be made.



Red = elements of path 1, Blue = elements of path 2, Black = common elements; Continuous = single links, dashed = set of links

Figure 1 – Topologies in a network with two paths

Heuristics to find potentially optimal links can be identified by utilising Lemma 4 which is a restatement of Definition 3:

LEMMA 4: (Sufficient condition for i to be potentially optimal with respect to j) $\exists$ ω and t for which $c_i+C^{\omega}_{H(i)}(t+c_i)<c_j+d_j(t)+C^{\omega}_{H(j)}\left(t+c_j+d_j(t)\right) \Rightarrow i$ is potentially optimal with respect to j

The direct implementation of this Lemma in cases in which node potentials coincide to those under Φ and Δ leads to

HEURISTIC 1 (Sufficient condition for i to be optimal with respect to j for any ω in which all the links of $p^{\Phi}_{H(i)}$ are not congested): $\exists$ **t** such that $c_i+C^{\Phi}_{H(i)}<c_j+d_j\left(c_j+d_j(t)\right)+C^{\Phi}_{H(j)}$ [3] $\Rightarrow$ i is potentially optimal with respect to j

HEURISTIC 2 (Sufficient condition for i to be optimal with respect to j for any ω in which all the links from H(i) and H(j) to the destination are congested): $\exists$ t such that $c_i+C^{\Delta}_{H(i)}(t+c_i)<c_j+d_j(t)+C^{\Delta}_{H(j)}\left(t+c_j+d_j(t)\right)$ [4] $\Rightarrow$ i is potentially optimal with respect to j

Note that [3] holds for the links of the shortest optimistic path ($p^{\Phi}_{H(i)}$is not time dependent because c_i's are not) and [4] for those of the shortest pessimistic one. In general, for an unknown ω only a necessary condition for link optimality can be established

HEURISTIC 3 (For unknown cost scenario): $\exists$ t such that $c_i+C^{\Delta}_{H(i)}(c_i+t)<c_j+d_j(t)+C^{\Phi}_{H(j)}$ [5] $\Rightarrow$ i is potentially optimal with respect to j

Since sufficient condition [5] is stricter than condition [3], all links found by Heuristic 3 are also found by Heuristic 2. Therefore, considering [3] and [4], a heuristic threshold can be defined using optimistic and pessimistic node potentials

$$HT_{i,j}(t) = \max\left\{c_j+d_j(t)+C^{\Phi}_{H(j)} - C^{\Phi}_{H(i)},\; c_j+d_j(t)+C^{\Delta}_{H(j)}\left(c_j+d_j(t)\right) - C^{\Delta}_{H(i)}(t)\right\}$$

This allows deriving from the sufficient condition of Lemma 1 the following Heuristics 4 to identify a subset of all potentially optimal links.

HEURISTIC 4: $\exists\, t$ such that $c_i(t) < HT_{i,j}(t)\ \forall\, j \neq i \in FS\big(T(i)\big) \Rightarrow i$ is potentially optimal

Note that Heuristic 4 implemented in static cases provides a sufficient condition larger than that identified in Fonzone et al (2010).

2.3 Performance of the heuristic

The performance of Heuristic 4 and an alternative exhaustive search algorithm to obtain the exact set of potential optimal links have been tested in searching for the potentially optimal links for a trip from the north-west to the south-east corner of a 3 by 5 grid network with all links oriented towards east or towards south (see Figure 2). The main objectives are to understand how many links and hence possibly optimal paths the proposed heuristic is missing, and to evaluate the reduction of elaboration time it brings about. To achieve this aim the algorithms in Table 1 and Table 2 have been coded in Matlab R2009b and run on an Intel(R) Core(TM)2 Duo processor at 3.00 GHz. Attempts with networks larger than 3 by 5 have been carried out, but the exact algorithm could not find a solution within the running time limit set equal to 12 hours.

Table 1 – Algorithm to find potentially optimal links using Heuristic 4

INITIALIZATION	Classify O as "**Potentially optimal**"
	Set all other nodes as "**Not examined**" and all links as "**Not classified**"
	Search for node potentials in Φ and Δ with a standard shortest paths algorithm such as Dijkstra
LOOP 1	For each potentially optimal node $I \neq D$ not examined
LOOP 2	For each link $i \in FS(I)$ not yet classified
LOOP3	For each $j \neq i \in FS(I)$
	If i complies with the necessary condition [Lemma 3]
	If $c_i > HT_{i,j}$ [Heuristic 4]
	Classify i as "**Undecidable**"
	Iterate LOOP 2
	Else iterate LOOP 3
	Else
	Classify i as "**Not potentially optimal**"
	Iterate LOOP 2
	If i is not classified
	Classify i as "**Potentially optimal**"
	Classify $H(i)$ as "**Potentially optimal**"
	Iterate LOOP 2
	Set I as "Examined"
	Iterate LOOP 1

Table 2 – Exact algorithm to look for potentially optimal links (exhaustive search)

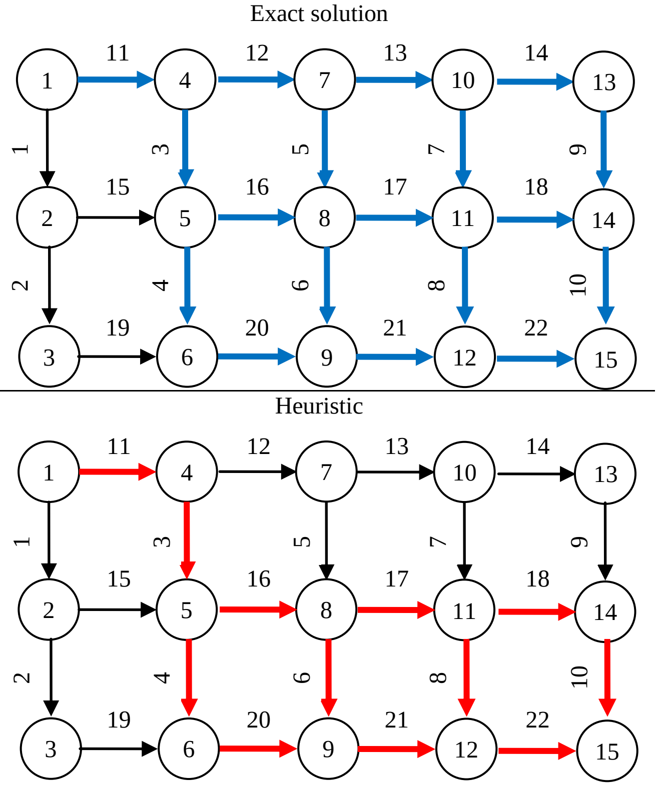
INITIALIZATION	Classify O as "**Potentially optimal**"
	Set all other nodes as "**Not examined**" and all links as "**Not classified**"
LOOP 1	For each potentially optimal node $I \neq D$ not yet examined
LOOP 2	For each link $i \in FS(I)$ not classified
LOOP3	While there are still cost scenarios ω_I not studied for the subnetwork having I as origin
	Choose a new ω_I starting with $\omega_I = \Phi_I$
	Determine the shortest path $p_I^{\omega_I}$

Classify all links and nodes of $p_I^{\omega I}$ as "**Potentially optimal**"
If there are $i \in FS(I)$ not classified iterate LOOP 3
Else iterate LOOP 2
If *i* is not classified, classify it as "**Not potentially optimal**"
Iterate LOOP 2
Iterate LOOP 1

50 trials have been carried out, each with different static link delays. Uncongested travel times and delays have been drawn randomly from uniform distributions with values between 0.1 and 100.0. As an example in Figure 2 link costs and results are shown concerning trial 28.

Link *i*	c_i	d_i	Link *i*	c_i	d_i
1	38.4	35.5	12	52.9	6.4
2	10.6	28.2	13	23.6	16.9
3	8.3	22.7	14	6.3	25.6
4	12.1	19.4	15	42.2	55.5
5	54.7	56.3	16	11.7	16.1
6	47.9	39.4	17	20	46.3
7	50.0	24.2	18	48.5	22.4
8	16.6	56.3	19	53.5	39.7
9	54.4	33.8	20	9.9	52.0
10	45.0	3.7	21	7.4	13.2
11	0.2	12.6	22	52.5	42.9

Indicator	Exact algorithm	Heuristic algorithm
Number of links potentially optimal	18	12
Running time (sec)	2,770.3	0.0090
Number of cost scenarios analysed	2,097,155	2



Thicker arrows identify links classified as potentially optimal

Figure 2 – Trial number 28

As it can be seen in Figure 3, the solution found heuristically matches the exact one in 5 cases (trials 6, 15, 30, 39, 47). In other 5 cases (trials 14, 16, 23, 31, 42) heuristic 4 finds only 6 potentially optimal links, that is in a network 3 by 5 just a single path; in 3 of such cases (trials 16, 31, 42) the exact algorithm classifies all links as potentially optimal.

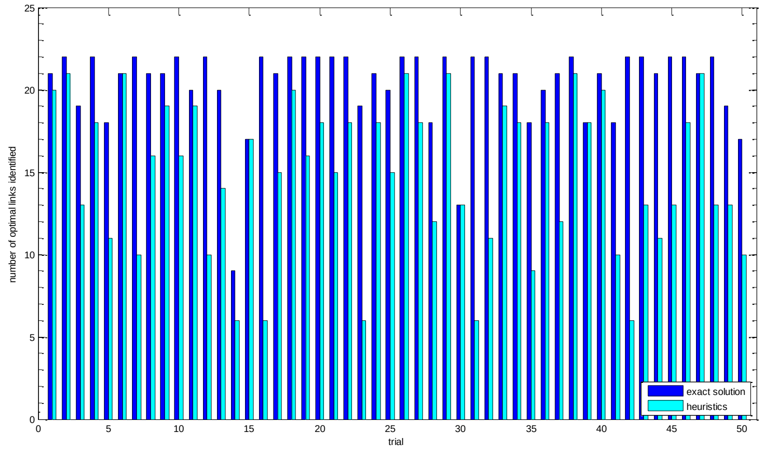


Figure 3 – Links classified as "potentially optimal"

The mean correctness of the approximate solutions, in terms of ratio between the number of the links that are identified as potentially by the heuristic and the number of those who are actually potentially optimal, is 0.72; the 25th percentile is 0.59. No special pattern can be recognized in the way the correctness varies with the number of actual potentially optimal links (Figure 4).

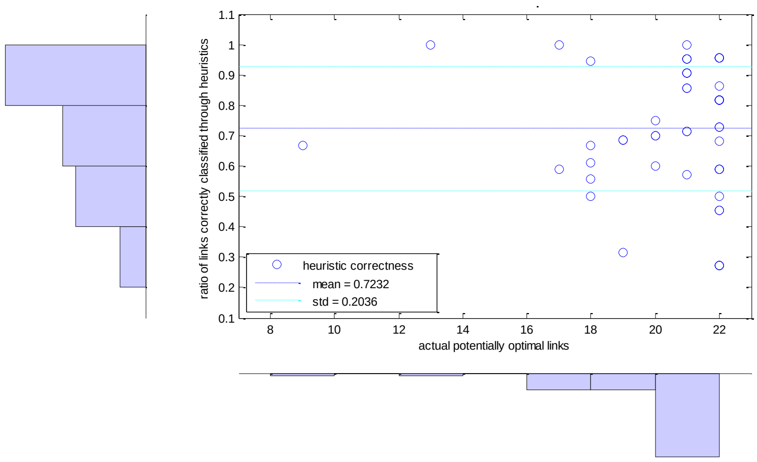


Figure 4 – Correctness of the heuristic solution vs. actual number of potentially optimal links

The heuristic-based algorithm remarkably outperforms the exact one in terms of run time, with means equal to 0.012 and 268.3 sec respectively and maximum to 0.029 sec and 46.26 min respectively. Nevertheless, in 6 cases (2, 12, 21, 26, 27, 38) the exact algorithm is faster (Figure 5).

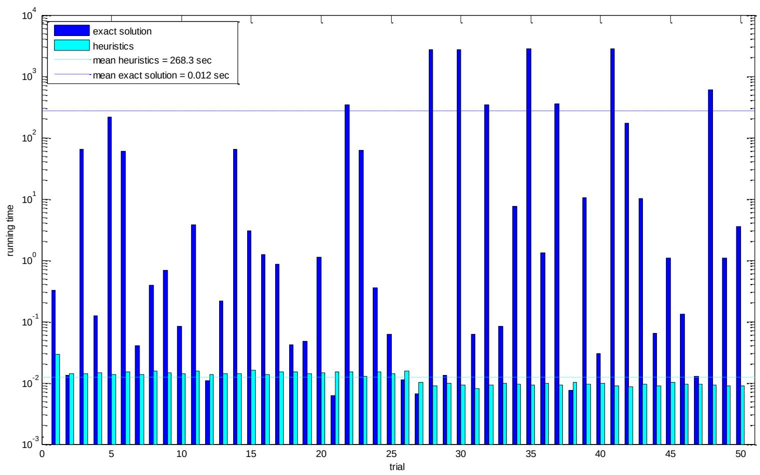


Figure 5 – Running times

Since the speed of execution depends on the characteristic of the processor, the number of shortest paths calculated while running the exact algorithm has been analyzed as an indicator of the complexity of the situations the algorithms faced (Figure 6**).**

Note that the number of cost scenarios analysed ranges from 6 to more than 2 millions scenarios, with a mean of 202,515 and a 25^{th} percentile of 76 (the number of possible cost scenarios in a network with 22 links is 4,194,024). The possibility of terminating the algorithm after just 6 iterations is provided by the implementation of a greedy choice of cost scenarios which exploits the peculiar topology of the networks considered. Cost scenarios are selected considering only the sub-network downstream with respect to the potentially optimal node under examination, and eliminating cost scenarios which are not favourable to the links not yet classified. As explained above, very early terminations are possible when in a few iterations potential optimal paths are found that include all links.

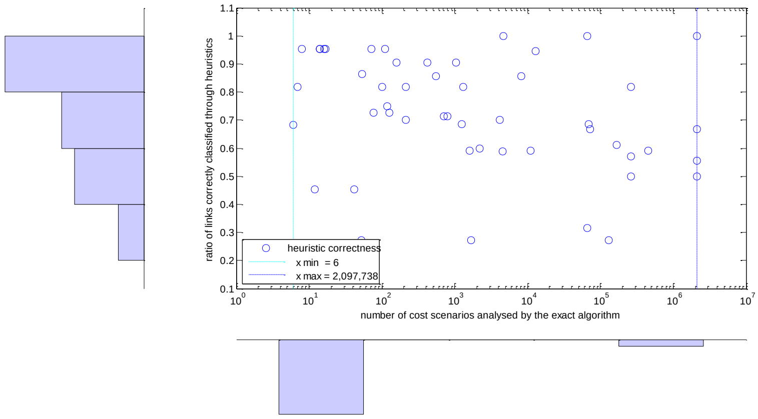


Figure 6 - Correctness of the heuristic solution vs. complexity of the network

3 Regret based route choice

A set of the potentially optimal links can be used as a base for route choice, both in the context of navigation assistance and of assignment. In the latter problem, the set identified through Heuristic 4 can be used as a whole, considering every link leaving from a given node equally attractive. This brings about a heuristic solution to the mixed route strategy problem with no limitation to number of links which can be spoiled.

For assignment in which one wants to take into consideration traveller's characteristics, and for route guidance when the complete route is decided before starting the trip, a heuristic route choice can be proposed depending on the traveller anticipations about cost scenarios and on his attitude towards regret. A decision criterion based on regret looks particularly suitable in a context characterized by high uncertainty such as that considered in this paper, in which travellers do not know the probabilities of cost scenarios and so cannot but base their choices on a "pure" guess about the state of network. In fact when probabilities are known, it can be argued that selecting the shortest path only on the basis of the expected values of travel times calculated over all the possible realizations of link states may lead to a wrong decision when a single extraction from the population of the cost scenarios is considered, but, according to the law of large numbers, it should eliminate regret (or, better, it should ensure that using the shortest path instead of other routes causes an "overall rejoicing" in travellers) when the path is used many times. On the contrary, when shortest paths are chosen on the basis of pure guesses, regret cannot be exluded by repetition of use. Consistently with the hypothesis for the selection of potentially optimal links according to Heuristic 4, only c_i, $d_i(t)$, $C^{\Phi}_{H(i)}(t)$ and $C^{\Delta}_{H(i)}(t)$ for each link i are assumed to be known.

Firstly consider the case in which the traveller has to choose between two links i and j. The "choiceless" utility of link i under a cost combination ω can be taken equal to the opposite of the minimum cost of going from *T(i)* to the destination using i

$$u_i^{\omega}(t)=-\left(\chi_i^{\omega}(t) + C^{\omega}_{H(i)}\left(t+\chi_i^{\omega}(t)\right)\right)$$

Supposing that when i is chosen instead of j the experienced regret, given cost scenario ω, can be measured by the linear function

$$R^{\omega}_{i,j}(t) = u^{\omega}_i(t) - u^{\omega}_j(t)$$

Then the modified utility of choosing link *i* with respect to link *j* is equal to

$$m^{\omega}_{i,j}(\beta, t) = u^{\omega}_i(t) + \beta \cdot \left(u^{\omega}_i(t) - u^{\omega}_j(t)\right)$$

It can be reasonably assumed that, instead of choosing by chance among the alternatives, the (possibly not optimal) choice is made on basis of the available, partial knowledge of the network. For each link, 4 cost scenarios can be evaluated (Table 3 reports those regarding *i*).

Table 3 – Computable cost scenarios for link *i*

Scenario	$\chi^{\omega}_i(t)$		$C^{\omega}_{H(i)}(t + \chi^{\omega}_i(t))$	
	Kind of anticipation	*Cost function*	*Kind of anticipation*	*Cost function*
$\omega_{i,1}$	Optimistic	c_i	Optimistic	$C^{\Phi}_{H(i)}(t+c_i)$
$\omega_{i,2}$	Pessimistic	$c_i + d_i(t)$	Optimistic	$C^{\Phi}_{H(i)}(t+c_i + d_i(t))$
$\omega_{i,3}$	Optimistic	c_i	Pessimistic	$C^{\Delta}_{H(i)}(t+c_i)$
$\omega_{i,4}$	Pessimistic	$c_i + d_i(t)$	Pessimistic	$C^{\Delta}_{H(i)}(t+c_i + d_i(t))$

As shown in Section 2.2, overlaps of paths can prevent the actual occurence of some of the 16 cost scenarios which can be obtained by combining each computable cost scenario for link *i* with every scenario for link *j*. The only combinations which can happen indepently from the network topology are $\omega_{i,1} \cup \omega_{j,1}$ and $\omega_{i,4} \cup \omega_{j,4}$. One might argue that, when deciding on the potential optimal path set, travellers want to be certain not to select links which might never be optimal. However, when travellers are subsequently choosing among actual potentially optimal links (from the previously determined set of links), without any information about the cost scenario probabilities they may be open to accept a non-optimal decision criterion. Therefore, for the calculation of a "heuristic expected modified utility" all 16 combined cost scenarios are taken into account. For a function in which the independent variable can assume 4 values, define the operator

$$\mathcal{A}(f(x_v)_{v=1\ldots4}) = \alpha^2 \cdot f(x_1) + (1-\alpha) \cdot \alpha \cdot f(x_2) + \alpha \cdot (1-\alpha) \cdot \alpha \cdot f(x_3) + (1-\alpha)^2 \cdot f(x_4)$$

If the traveller, because of his attitude towards risk, attaches a probability α to the optimistic anticipations and $(1-\alpha)$ to the pessimistic ones, the "choiceless" expected utility of choosing link *i* in the 4 four scenarios in Table 3 is

$$\bar{u}_i(\alpha, t) = \mathcal{A}\left(u_i^{\omega_{i,v}}\right)_{v=1\ldots4}$$

And the heuristic expected modified utility of link *i* with respect to link *j*

$$\bar{m}_{i,j}(\alpha, \beta, t) = \bar{u}_i(\alpha, t) + \beta \cdot \mathcal{A}\left(u_i^{\omega_{i,v}} - \bar{u}_j(\alpha, t)\right)_{v=1\ldots4}$$

According to regret theory, link *i* is preferred to *j* if $\bar{m}_{i,j}(\alpha, \beta, t) > \bar{m}_{j,i}(\alpha, \beta, t)$ [6]. This does not exclude non transitive pairwise choices. Therefore if a single link has to be selected and the traveller is faced with decisions between more than two links, avoiding this difficulty requires different heuristic modified expected utilities. Following Loomes and Sugden (1982), the utility of link *i* can be defined as

$$\bar{m}_{i,FS(I)}(\alpha, \beta, \gamma, t) = \sum_{j \neq i \in FS(I)} \frac{\gamma_j}{1-\gamma_i} \cdot \bar{m}_{i,j}(\alpha, \beta, t)$$

where γ is a vector of weights, adding up to 1, which indicate traveller's preferences between links. γ can be useful to choose paths when links are different because, for instance, they represent different modes or are characterized by different levels of safety. Link *i* is selected with the maximum $\bar{m}_{i,FS(I)}(\alpha, \beta, \gamma, t)$.

Note that Heuristic 4 and considering all 16 combined scenarios $\omega_{i,r} \cup \omega_{j,r}$ r,s = 1 ... 4 led to the definition of the operator $\mathcal{A}$, therefore changing $\mathcal{A}$ means changing the heuristics. α reflects the travellers fear on what could happen on his chosen link.The formulation proposed for the "choiceless" expected utility assumes that the risk associated with finding link *i* congested is the same as the risk associated with finding all links downstream of *H(i)* congested, which is a clearly a simplification. In further work one might weigh the risk according to the total uncertainty associated with link *i* and potential delay from *H(i)*. Alternative formulations, consistent with the hypothesis of traveller knowing only Φ and Δ, can be identified by associating the latter probability to the number of links in the optimistic and/or the pessimistic shortest paths. When the traveller is completely optimistic or risk prone, that is when α = 1, $\bar{u}_i(\alpha = 1,t)= u_i^{\omega_{i,1}}$ and the route is chosen considering only Φ. Vice versa, for completely pessimistic or risk averse travellers, α = 0 and $\bar{u}_i(\alpha = 1,t)= u_i^{\omega_{i,4}}$, which means that route choice is based only on Δ. α = 0.5 when the traveller is neither optimistic nor pessimistic, in the sense that he considers equally probable optimistic and pessimistic anticipations. β = 0 eliminates the influence of regret "on what could have happened on other routes" on route choice, but still the choice reflects the fact that different scenarios are possible on each link *i*.With $\beta \to \infty$ regret dominates route choice and the choiceless utility becomes neglectable. If preference for specific links is not an issue, γ_i is simply equal to $\frac{1}{n}$, where *n* is the number of links in *FS(I)*.
An alternative assignment could be obtained considering the subset of the identified potentially optimal links which are preferred to at least one different link departing from the same node according to [6]. The characteristics of such an assignment will be explored in the future.

4 Conclusions

In ordinary transport networks failures can occur on multiple links and representations in which they are modelled through malevolent agents may be over cautious. The probability distributions of travel times on links and waiting times at nodes are sometimes unknown, and do not have close and easy to implement expressions. Further, in many cases the correlations between failures are unknown.
Therefore, an approach which identifies sets of potentially optimal links and does not consider failure probabilities has been studied in the paper. It is believed that this as a step towards the development of a route choice model in which each link may be subject to a time-dependent delay. In particular for dynamic route guidance applications it is necessary to quickly suggest route sets and route selection criteria that ensure the traveller is not far of the best route but allows flexibility to choose between shorter and more reliable paths.
The exact solution of all potentially optimal path sets is computationally demanding even for small networks, therefore a heuristic is proposed, valid for FIFO networks and based on optimistic and pessimistic node potentials. An algorithm based on the heuristic has been tested and its performance has been compared to that of a greedy exact algorithm in a small and topographically simple network with static delays. The heuristic provides reasonably good results in terms of percentage of potentially optimal links identified, and it is generally much faster than the exact method. The exact algorithm involves run times which make it not suitable for practical purposes. From a theoretical point of view, the main shortcoming of the heuristic is related to the assumptions of the FIFO condition on the network, which requires that cost scenarios are not time-dependent. As to the implementation issues, the performance of the proposed algorithm has to be studied with large networks and dynamic link delays.
Sets of potentially optimal links can be implemented directly in assignment problems under the hypothesis of equal attractiveness of links. An alternative route selection method has been proposed, which takes into account traveller's optimism and attitude towards regret. Like the approach followed for the selection of the potentially optimal links, the choice criterion is based on optimistic and pessimistic node potentials. Both single and multiple paths can be obtained as outcome. The decision making approaches put forward need to be assessed in future works, to evaluate the plausibility of the routes proposed and their sensitivity to the parameters representing attitude towards risk and regret. Alternative ways of making use of optimistic and pessimistic node potentials (e.g. weighting differently the 16 scenarios mentioned in section 2, or attaching different reliability to the optimistic anticipation concerning χ_i^{ω} and to that regarding $C_{H(i)}^{\omega}$) will be explored as well.

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