Potentially optimal paths and route choice in networks with arc delays

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Abstract

The shortest path problem has been widely studied in graph theory and its solutions have been largely used in the transport field, both in assignment and vehicle navigation applications. Solutions have been proposed under different assumptions concerning the time-dependency and the randomness of arc costs. In particular it has been shown that in networks affected by not completely predictable events travellers can minimize their expected travel times by adopting adaptive strategies, i.e. sets of paths, instead of single routes. The present paper focuses on the problem of route guidance in a network in which the travel time on each link can assume only two values, corresponding to the free flow situation and to the delayed one. Path set selection and route choice are decoupled. The problem of finding the set of the potentially optimal paths is solved implicitly by identifying the set of the potentially optimal links. Sufficient and necessary conditions for a link to be optimal do not coincide and therefore an algorithm is presented which can identify only a subset of all potentially optimal links. An approach considering risk aversion is proposed for route choice.

1. Introduction

Finding a single or set of minimum paths in networks has applications in a large number of disciplines. In transport, for example, finding the shortest path (see Pallottino & Scutellà 1997 for a critical introduction) is the basis for normative as well as predictive problems such as assignment or route guidance. Of particular interest are problems when delays might occur that are random and/or time–dependent. Sources of delays are numerous including congestion, accidents or network disruptions caused by natural disasters.

When trips over a transport network are affected by such random or unpredictable en–route events, travellers can minimize their expected travel times by adopting adaptive strategies instead of single paths. This is proven by Hall (1986) in the case of stochastic time–dependent networks with random link travel times, and by Spiess and Florian (1989) for transit networks in which randomness is given by the arrival order of services at stops.

Schmöcker et al (2009) show that the path set created by the Spiess and Florian approach and utilised in Bell (2009) for car navigation can also be derived by a "local demon game" in which a traveller fears that at each decision point exactly one "failure" among the options that might be attractive occurs. Though in some scenarios, such as waiting for a transit service, where one can be certain that within a certain time interval one of the services arrives, this might be a reasonable assumption, especially for route guidance this might not be the case. Link delays are often correlated so that the likelihood of delay on one route increases rather than decreases the likelihood of delays on other also possible attractive routes. Szeto et al (2007) and Schmöcker et al (2009) discuss the possibility of multi demon games where the traveller is even more pessimistic and fears that several "failures" or "delays" occur. A second problem of the Spiess and Florian approach when applied to route guidance in scheduled transit networks or for car navigation is that the shortest path, in case the potential delay is not occurring, is not necessarily included. Therefore a traveller who is at least to some degree optimistic might be mislead to longer routes.

In the following we hence firstly decouple the problem of path set selection and route choice. This is similar to a large set of literature concerning the creation of K–shortest path sets under various assumptions of potential delays (see Eppstein 1998 for a general review; Nielsen et al 2005 for an application to hyperpaths). In contrast our objective is to find the most complete possible set of paths that are potentially optimal given that each link might be subject to a delay. We simplify the problem by considering that each link has only the two potential states of being delayed or not. In the game theoretic approach this would mean modelling as many demons as there are links in the network with a power to cause the link specific potential delay. Whether or not the demons strike is however uncertain. As the problem turns out to be non–trivial we firstly develop necessary and sufficient conditions for a link to be included in the set of optimal paths. Since the sufficient and necessary conditions do not coincide we can only identify a subset of paths which are proven that they must be in the set of optimal paths. In particular it shown that the shortest and the most reliable path will be among this subset. In the final part of this paper we then outline how this set of paths can be used for route guidance applications.

The paper is organized as follows: Section 2 studies some properties of potentially optimal links. Section 3 presents an algorithm which finds the most complete set of links which are demonstrated potentially optimal by using such properties. Section 4 proposes a criterion to select a specific route from the set of the potentially of optimal links, based on risk aversion. Conclusions summarize the paper and delineate further research.

2. Properties of the Potentially Optimal Links

Define the following notation

$G(\boldsymbol{N}, \boldsymbol{A})$	Directed, acyclic graph with nodes $\boldsymbol{N}$ and arcs $\boldsymbol{A}$
I	Node $\in \boldsymbol{N}$
R, S	Origin, destination
i	Arc $\in \boldsymbol{A}$
$H(i)$, $T(i)$	Head node, tail node of link i
$FS(A)$	Forward star of node A
c_i, d_i	Uncongested cost, potential additional cost on link i
χ_i	Actual cost on link $i = \begin{cases} c_i & \text{if } i \text{ is not congested} \\ c_i + d_i & \text{if } i \text{ is congested} \end{cases}$
$\boldsymbol{X}$	Vector of actual links costs denoting the current network condition
π_{AB}	Path from A to B
$\pi_{AB,i}$	Path from A to B using link i
$p_{AB,i}$	Optimal path from A to B using link i
p^{Γ}_{AB}	Optimal path from A to B when all links are not congested
p^{H}_{AB}	Optimal path from A to B when all links are congested
$\tau_{AB,\pi}$	Cost of path π from A to B
t_{AB}	Cost of the optimal path from A to B
$t_{AB,i}$	Cost of the optimal path from A to B using link i
t^{Γ}_{AB}	Cost of the optimal path from A to B when all links are not congested
t^{H}_{AB}	Cost of the optimal path from A to B when all links are congested

For simplicity, for paths between nodes A and B, B is omitted in case it coincides with S.

DEFINITION 1: π_A is potentially optimal $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ at least one $\boldsymbol{X}$ for which π_A is the minimum cost path from A to S.

DEFINITION 2: i is potentially optimal $\overset{\text{def}}{\Leftrightarrow}$ $p_{H(i),i}$ is potentially optimal.

LEMMA 1: If A and B are connected, $t_{AB} \geq t^{\Gamma}_{A} - t^{\Gamma}_{B}$.

Proof: 1) If $\chi_k = c_k \ \forall \ k \in p^{\Gamma}_{AB}$, given that each subpath of an optimal path is the optimal path between the two extreme nodes, $t_{AB} = t^{\Gamma}_{A} - t^{\Gamma}_{B}$. 2) Assume that $\exists| \ k \in p^{\Gamma}_{AB} \ni' \chi_k = c_k + d_k$ [1]. $t_{AB,k} = t_{AT(k)} + \chi_k + t_{H(k)B} = (t^{\Gamma}_{A} - t^{\Gamma}_{T(k)}) + c_k + d_k + (t^{\Gamma}_{H(k)} - t^{\Gamma}_{B}) \geq (t^{\Gamma}_{A} - t^{\Gamma}_{T(k)}) + c_k + (t^{\Gamma}_{H(k)} - t^{\Gamma}_{B}) \geq (t^{\Gamma}_{A} - t^{\Gamma}_{T(k)}) + (t^{\Gamma}_{T(k)} - t^{\Gamma}_{H(k)}) + (t^{\Gamma}_{H(k)} - t^{\Gamma}_{B}) = t^{\Gamma}_{A} - t^{\Gamma}_{B}$. If [1] holds for more than one $k \in p^{\Gamma}_{AB}$, the thesis can be easily verified by iterating the same mechanism of proof used for the case of a single k. QED

LEMMA 2: Let $B \in p^H{}_{AB}$, $\boldsymbol{NC} \subseteq \boldsymbol{A}$, and $\boldsymbol{NC_{AB}} = \{k \in \boldsymbol{NC} \mid \exists$ a path from A to B including $k\}$.

$\chi_k = \begin{cases} c_k \text{ if } k \in NC \\ c_k + d_k \text{ otherwise} \end{cases} \Rightarrow t_{AB} \geq t^H{}_A - t^H{}_B - \sum_{j \in NC_{AB}} d_j$.

Proof: Let $\tau^{con}{}_{AB,\pi}$ be the cost of π_{AB} when all links are congested. $\tau^{con}{}_{AB,\pi} = \sum_{j \in \zeta_{AB}} (c_j + d_j)$. Given that each subpath of an optimal path is the optimal path between the two extreme nodes, $\tau^{con}{}_{AB,\pi} = \sum_{j \in \zeta_{AB}} (c_j + d_j) \geq t^H{}_A - t^H{}_B$. $\chi_k = \begin{cases} c_k \text{ if } k \in NC \\ c_k + d_k \text{ otherwise} \end{cases} \Rightarrow \tau_{AB,\pi} = \sum_{j \in \zeta_{AB} \cap NC} c_j + \sum_{j \in \zeta_{AB} \setminus j \in NC} (c_j + d_j) = \sum_{j \in \zeta_{AB} \cap NC_{AB}} c_j + \sum_{j \in \zeta_{AB} \setminus j \in NC_{AB}} (c_j + d_j) = \tau^{con}{}_{AB,\pi} - \sum_{j \in \zeta_{AB} \cap NC_{AB}} d_j \geq t^H{}_A - t^H{}_B - \sum_{j \in \zeta_{AB} \cap NC_{AB}} d_j \geq t^H{}_A - t^H{}_B - \sum_{j \in NC_{AB}} d_j \;\; \forall \; \pi_{AB} \Rightarrow t_{AB} = \min\{\tau_{AB,\pi}\} \geq t^H{}_A - t^H{}_B - \sum_{j \in NC_{AB}} d_j$. QED

COROLLARY 1: Let $B \in p^H{}_{AB}$, $\boldsymbol{NC} \subseteq \boldsymbol{A}$, and $\boldsymbol{NC_{AB}} = \{k \in \boldsymbol{NC} \mid \exists$ a path from A to B including $k\}$. $\chi_k = \begin{cases} c_k \text{ if } k \in NC \\ c_k + d_k \text{ otherwise} \end{cases} \Rightarrow t_{AB} \geq \max\{t^\Gamma{}_A - t^\Gamma{}_B,\ t^H{}_A - t^H{}_B - \sum_{j \in NC_{AB}} d_j\}$.

Proof: The thesis follows immediately from lemmas 1 and 2. QED

THEOREM 1: Let $j \in FS(T(i))$, $I = p^H{}_{H(j)} \cap p^\Gamma{}_{H(i)}$, and $t_{H(j)I}{}^{inf} = \max\{t^\Gamma{}_{H(j)} - t^\Gamma{}_I,\ t^H{}_{H(j)} - t^H{}_I -$ Σ_j("k ∈ " "p" _"H(j)" "∩" "\" "p" _"I" "∩") "d" _"k" $\}$. If $\chi_k = \begin{cases} c_k \text{ if } k = i \text{ or } k \in p^\Gamma_{H(i)} \\ c_k + d_k \text{ otherwise} \end{cases}$, $c_i \leq c_j + d_j + t_{H(j)I}{}^{inf} - (t^\Gamma{}_{H(i)} - t^\Gamma{}_I) \Rightarrow t_{T(i),i} \leq t_{T(i),j}$.

Proof: $t_{T(i),k} = t_I + t_{T(k)I,i} = t_I + t_{H(k)I} + \chi_k \;\forall\; k \in \mathrm{FS}(T_i)$. $\chi_k = \begin{cases} c_k \text{ if } k = i \text{ or } k \in p^\Gamma_{H(i)} \\ c_k + d_k \text{ otherwise} \end{cases} \Rightarrow t_{T(i),i} = c_i + t_{H(i)} = c_i + t^\Gamma{}_{H(i)}$, $t_{T(i),j} = c_j + d_j + t_{H(j)I} + t^\Gamma{}_I$. From lemma 2, $t_{T(i),j} = c_j + d_j + t_{H(j)I} + t^\Gamma{}_I \geq c_j + d_j + t_{H(j)I}{}^{inf} + t^\Gamma{}_I$. Therefore $c_i \leq c_j + d_j + t_{H(j)I}{}^{inf} - (t^\Gamma{}_{H(i)} - t^\Gamma{}_I) \Rightarrow t_{T(i),i} = c_i + t^\Gamma{}_{H(i)} \leq c_j + d_j + t_{H(j)I}{}^{inf} - (t^\Gamma{}_{H(i)} - t^\Gamma{}_I) + t^\Gamma{}_{H(i)} = c_j + d_j + t_{H(j)I}{}^{inf} + t^\Gamma{}_I \leq t_{T(i),j}$. QED

THEOREM 2: Let $j \in FS(T(i))$. $t_{T(i),i} \leq t_{T(i),j} \Rightarrow c_i \leq c_j + d_j + t^H_{H(j)} - t^\Gamma_{H(i)}$.

Proof: $t_{T(i),j} = \chi_j + t_{H(j)} \leq c_j + d_j + t_{H(j)} \leq c_j + d_j + t^H_{H(j)}$. But $t_{T(i),i} = \chi_i + t_{H(i)} \geq c_i + t_{H(i)} \geq c_i + t^\Gamma_{H(i)}$. Therefore $t_{T(i),i} \leq t_{T(i),j} \Rightarrow c_i + t^\Gamma_{H(i)} \leq t_{T(i),i} \leq c_j + d_j + t^H_{H(j)} \Leftrightarrow c_i \leq c_j + d_j + t^H_{H(j)} - t^\Gamma_{H(i)}$.QED

LEMMA 3 (sufficient condition for potential optimality): Let $t_{H(j)}^{inf} = \max\{t^\Gamma_{H(j)}, t^H_{H(j)} - \sum_{a_k \in p^\Gamma_{H(i)}} d_k\}$. $c_i \leq c_j + d_j + t_{H(j)I}^{inf} - t^\Gamma_{H(i)}\ \forall j \in FS(T(i)) \Rightarrow i$ is potentially optimal.

Proof: Given $j \in FS(T(i))$, the conclusion of theorem 1 holds for every $I = p^H_{H(j)} \cap p^\Gamma_{H(i)}$, in particular for S. Consider $\boldsymbol{X} \ni' X_k = \begin{cases} c_k & \text{if } k = 1 \text{ or } k \in p^\Gamma_{H(i)} \\ c_k + d_k & \text{otherwise} \end{cases}$. By applying theorem 1 and noting that when $I = S$, $t^\Gamma_I = t^H_I = 0$, it turns out that $c_i \leq c_j + d_j + t_{H(j)}^{inf} - t^\Gamma_{H(i)} = c_j + d_j + t^\Gamma_{H(j)S}{}^{inf} - (t^\Gamma_{H(i)} - t^\Gamma_S) \Rightarrow t_{T(i),i} \leq t_{T(i),j}\ \forall j \in FS(T(i)) \Rightarrow \exists$ at least one $\boldsymbol{X}$ for which $t_{T(i),i} \leq t_{T(i),j}\ \forall j \in FS(T(i)) \stackrel{\text{def}}{\Longleftrightarrow} i$ is potentially optimal. QED

COROLLARY 2 (inclusion of p^Γ_A among potentially optimal paths): The hypothesis of lemma 3 holds for $i \in p^\Gamma_{T(i)}$.

Proof: For a generic $\boldsymbol{X}$, $t_{T(i),j} = \chi_i + t_{H(i),j}$. $i \in p^\Gamma_{T(i)} \Leftrightarrow t_{T(i),i} \leq t_{T(i),j}\ \forall j \in FS(T(i))$ when $\chi_k = c_k\ \forall k \Leftrightarrow c_i + t^\Gamma_{H(i)} \leq c_j + t^\Gamma_{H(j)} \leq c_j + d_j + t^\Gamma_{H(j)} \Leftrightarrow c_i \leq c_j + d_j + t^\Gamma_{H(j)} - t^\Gamma_{H(i)} \leq c_j + d_j + \max\{t^\Gamma_{H(j)}, t^H_{H(j)} - \sum_{a_k \in p^\Gamma_{H(i)}} d_k\} - t^\Gamma_{H(i)}\ \forall j \in FS(T(\mathrm{i}))$. QED

COROLLARY 3 (non inclusion of p^H_A among potentially optimal paths): The hypothesis of lemma 3 might not hold for $i \in p^H_{T(i)}$.

Proof: If we find a feasible case in which $i \in p^H_{T(i)}$ and the conclusion of lemma 3 doesn't hold for i, the corollary is proved. $i \in p^H_{T(i)} \Rightarrow c_i + d_i + t^H_{H(i)} \leq c_j + d_j + t^H_{H(j)} \Leftrightarrow c_i \leq c_j + d_j + t^H_{H(j)} - d_i - t^H_{H(i)}\ \forall j \in FS(T(i))$. The conclusion of lemma 3 doesn't hold for $i \Leftrightarrow \exists m \in FS(T(i)) \ni' c_i > c_m + d_m + \max\{t^\Gamma_{H(m)}, t^H_{H(m)} - \sum_{a_k \in p^\Gamma_{H(i)}} d_k\} - t^\Gamma_{H(i)}$. Therefore for each of such

m's, it is necessary that $c_m + d_m + \max\{t^\Gamma_{H(m)}, t^H_{H(m)} - \sum_{a_k \in p^\Gamma_{H(i)}} d_k\} - t^\Gamma_{H(i)} < c_i \le c_m + d_m + t^H_{H(m)} - d_i - t^H_{H(i)}$, which is possible if $\max\{t^\Gamma_{H(m)}, t^H_{H(m)} - \sum_{a_k \in p^\Gamma_{H(i)}} d_k\} - t^\Gamma_{H(i)} < t^H_{H(m)} - d_i - t^H_{H(i)}$ [1] holds. [1] is feasible, in fact 1) If $t^\Gamma_{H(m)} \ge t^H_{H(m)} - \sum_{a_k \in p^\Gamma_{H(i)}} d_k$, [1] $\Leftrightarrow t^\Gamma_{H(m)} - t^\Gamma_{H(i)} < t^H_{H(m)} - d_i - t^H_{H(i)} \Leftrightarrow d_i < (t^H_{H(m)} - t^\Gamma_{H(m)}) - (t^H_{H(i)} - t^\Gamma_{H(i)})$, which is feasible when $(t^H_{H(m)} - t^\Gamma_{H(m)}) > (t^H_{H(i)} - t^\Gamma_{H(i)})$ (otherwise d_i should be negative). 2) $t^\Gamma_{H(m)} < t^H_{H(m)} - \sum_{a_k \in p^\Gamma_{H(i)}} d_k$, [1] $\Leftrightarrow t^H_{H(m)} - \sum_{a_k \in p^\Gamma_{H(i)}} d_k - t^\Gamma_{H(i)} < t^H_{H(m)} - d_i - t^H_{H(i)} \Leftrightarrow d_i < \sum_{a_k \in p^\Gamma_{H(i)}} d_k - (t^H_{H(i)} - t^\Gamma_{H(i)})$, which is feasible when $\sum_{a_k \in p^\Gamma_{H(i)}} d_k > (t^H_{Hi} - t^\Gamma_{Hi})$. QED

LEMMA 4 (necessary condition for potential optimality): i is potentially optimal $\Rightarrow c_i \le c_j + d_j + t^H_{H(j)} - t^\Gamma_{H(i)}\ \forall j \in FS(T(i))$.

Proof: i is potentially optimal $\overset{\text{def}}{\Leftrightarrow}$ $\exists$ at least one $\boldsymbol{X}$ for which $t_{T(i),i} \le t_{T(i),j}\ \forall j \in FS(T(i))$. For such a $\boldsymbol{X}$, from theorem 2 it follows that $t_{T(i),i} \le t_{T(i),j} \Rightarrow c_i \le c_j + d_j + t^H_{H(j)} - t^\Gamma_{H(i)}\ \forall j \in FS(T(i))$. QED

3. An Algorithm to Find the Proven Potentially Optimal Links

The algorithm described below finds the most complete set of links potentially optimal when one travels from R to S, which can be deduced by the properties stated in the previous paragraph. Such a set of proven potentially optimal links (referred to as $\boldsymbol{A_{PPOL}}$ in the following) is made up of links which lemma 3 allows classifying as potentially optimal for sure, plus the links of p^H_R which violate the condition in lemma 3. Other links for which the necessary condition of potential optimality (lemma 4) is valid but the sufficient not, are not included in the set provided by the algorithm (unless they are part of p^H_R).

$t^{\Gamma}_{I}, t^{H}_{I} \leftarrow +\infty, I \in \boldsymbol{N} \backslash \{S\};\ t^{\Gamma}_{S}, t^{H}_{S} \leftarrow 0$

sd_I[1] $\leftarrow -\infty, I \in \boldsymbol{N}$

a^{Γ}_{I}[2] $\leftarrow \emptyset, I \in \boldsymbol{N} \backslash \{S\}$

$\boldsymbol{A_t}$[3] $\leftarrow \boldsymbol{A}, \boldsymbol{A_{PPOL}} \leftarrow \emptyset$

$\boldsymbol{N_t}$[4] $\leftarrow \{R\}$

While $\boldsymbol{A_t} \neq \emptyset$

$\quad$ Find $i \in \boldsymbol{A_t}$ with minimum $t^{\Gamma}_{H(i)} + c_i$

$\quad\quad \boldsymbol{A_t} \leftarrow \boldsymbol{A_t} \backslash \{i\}$

$\quad$ If $t^{\Gamma}_{T(i)} > t^{\Gamma}_{H(i)} + c_i$

$\quad\quad t^{\Gamma}_{T(i)} \leftarrow t^{\Gamma}_{H(i)} + c_i$

$\quad\quad a^{\Gamma}_{T(i)} \leftarrow i$

$\quad\quad \boldsymbol{A_{PPOL}} \leftarrow \boldsymbol{A_{PPOL}} + \{i\}$

$\boldsymbol{A_t} \leftarrow \boldsymbol{A}$

While $\boldsymbol{A_t} \neq \emptyset$

$\quad$ Find $i \in \boldsymbol{A_t}$ with minimum $t^{H}_{H(i)} + c_i + d_i$

$\quad \boldsymbol{A_t} \leftarrow \boldsymbol{A_t} \backslash \{i\}$

$\quad$ If $t^{H}_{T(i)} > t^{H}_{H(i)} + c_i + d_i$

$\quad\quad t^{H}_{T(i)} \leftarrow t^{H}_{H(i)} + c_i + d_i$

$\quad\quad$ If $i \notin \mathrm{A_{PPOL}}$

$\quad\quad\quad \boldsymbol{A_{PPOL}} \leftarrow \boldsymbol{A_{PPOL}} + \{i\}$

While $\boldsymbol{N_t} \neq \{S\}$

$\quad$ Select $T(i) \in \boldsymbol{N_t}$

$\quad \boldsymbol{N_t} \leftarrow \boldsymbol{N_t} \backslash \{T(i)\}$

$\quad\quad$ For each $k \in FS(T(i))$

$\quad\quad\quad$ If $k \notin \boldsymbol{A_{PPOL}}$

$\quad\quad\quad\quad pol_k$[5] $= 1$

$\quad\quad\quad\quad$ For each $j \in FS(T(i)) \backslash \{k\}$

$\quad\quad\quad\quad\quad$ If $c_k \leq c_j + d_j + t^{H}_{H(j)} - t^{\Gamma}_{H(k)}$

$\quad\quad\quad\quad\quad\quad$ If $sd_{H(k)} = -\infty$

$\quad\quad\quad\quad\quad\quad\quad NN$[6] $\leftarrow H(j)$

$\quad\quad\quad\quad\quad\quad\quad sd_{H(k)} = 0$

$\quad\quad\quad\quad\quad\quad\quad$ While $NN \neq S$

$\quad\quad\quad\quad\quad\quad\quad\quad nl \leftarrow a^{\Gamma}_{NN}$

$\quad\quad\quad\quad\quad\quad\quad\quad sd_k \leftarrow sd_k + d_{nl}$

$\quad\quad\quad\quad\quad\quad\quad\quad NN \leftarrow H_{nl}$

$\quad\quad\quad\quad\quad\quad t^{inf} \leftarrow \max\{t^{\Gamma}_{H(j)}, t^{H}_{H(j)} - sd_k\}$

$\quad\quad\quad\quad\quad\quad$ If $c_k > c_j + d_j + t^{inf} - t^{\Gamma}_{H(k)}$

$\quad\quad\quad\quad\quad\quad\quad pol_k = 0$

$\quad\quad\quad\quad\quad$ Else

$\quad\quad\quad\quad\quad\quad pol_k = 0$

$\quad\quad$ If $pol_k = 1$

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[1] $sd_i = \sum_{a_k \in p^{\Gamma}_{H(i)}} d_k$

[2] a^{Γ}_{I} = link which follows node I along p^{Γ}_{I}

[3] $\boldsymbol{A_t}$ = set of links to be examined

[4] $\boldsymbol{N_t}$ = set of nodes which are potentially optimal and whose forward star has not yet been examined

[5] $pol_k = 1$ if k is potentially optimal, 0 otherwise

[6] NN = Next node on $p^{\Gamma}_{H(i)}$

42 $\boldsymbol{A_{PPOL}} \leftarrow \boldsymbol{A_{PPOL}} + \{k\}$
43 $\boldsymbol{N_t} \leftarrow \boldsymbol{N_t} + \{H(k)\}$

LINES 1–5: Initialization.

LINES 6–13: Search for t^{Γ}_{R} and add all $i \in p^{\Gamma}_{R}$ to $\boldsymbol{A_{PPOL}}$, even though corollary 2 guarantees that lines 21–43 perform the latter operation if line 25 is eliminated. The adopted solution allows avoiding operations in lines 26–43 if the link has been already classified as potentially optimal by lines 6–13.

LINES 14–20: Search for t^{H}_{R} and add to $\boldsymbol{A_{PPOL}}$ all $i \in p^{H}_{R}$ which do not already belong to it. This is necessary to ensure that all $i \in p^{H}_{R}$ are classified as potentially optimal, because corollary warns that lines 21–43 might not include some $i \in p^{H}_{R}$ in $\boldsymbol{A_{PPOL}}$.

LINES 21–43: Add to $\boldsymbol{A_{PPOL}}$ all links for which the sufficient condition stated in lemma 3 holds, and which have not yet been classified as potentially optimal.

Lines 22-43: Potential optimality of the links belonging to the forward star of $T(i)$ is evaluated.

LINES 22 and 43: The algorithm evaluates the potential optimality of an arc only if at least one potential optimal path exists between the origin R and its tail node.

LINE 28 AND 39-40: Stops checking whether a link is potentially optimal if it violates the necessary condition expressed in lemma 4.

LINES 29–35: Calculate $\sum_{a_k \in p^{\Gamma}_{H(i)}} d_k$ if it is still unknown.

LINES 37–38: If condition in lemma 3 is violated for at least one $j \in FS(T(i))$, link i cannot be classified as potentially optimal for sure and so it is not included in $\boldsymbol{A_{PPOL}}$.

The example in figure 1 allows understanding how the algorithm works. Links 2, 3, 4, 5, 6, 8 and 9 (thicker in the figure) are classified as surely potentially optimal. The red path from 1 to 5 is the optimal one when all links are not congested, whereas the green one is optimal if all links are congested. Eliminating links 3 and 6 would nott affect t^{Γ} and t^{H} of any node (while link 9 is crucial for node 3) but they are included in the set of the proven potentially optimal links because they are backup links: link 3 is optimal when links 2, 4, 5, 6 are congested, whereas link 6 has to be used when 5 is congested. Link 7 is not included among the potentially optimal because it violates the necessary condition, in fact even when links 8 and 9 are congested the path 3→4→5 is faster (the concerning travel time is 16 in the worst case) than 3→5 (whose travel time is at least 18). The algorithm failed in including link 1 – which is optimal if links 2, 3, 4 and 9 are congested – in A_{PPOL} because it does not comply with the sufficient condition, in particular the inequality in the hypothesis of lemma 3 does not hold for link 4.

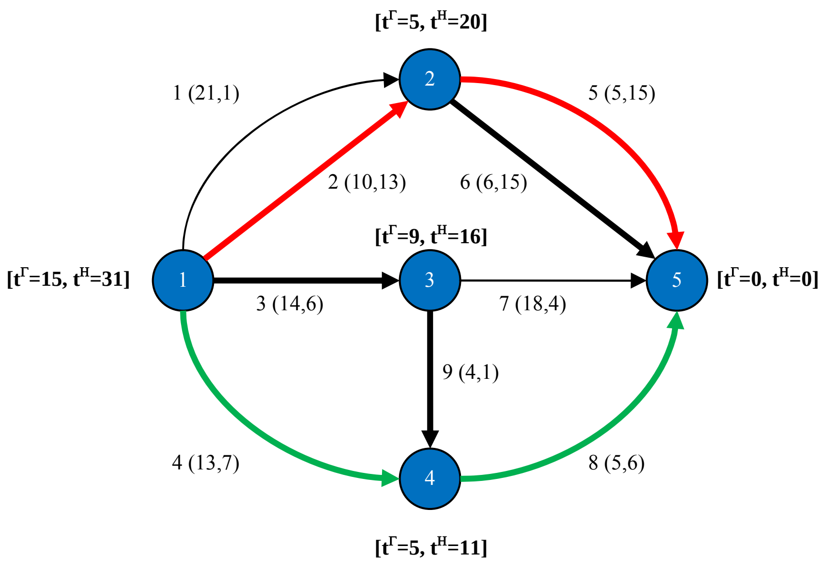

FIGURE 1: Example network

4. Route Choice for Navigation Assistance

Navigation assistance usually provides users with a single or a small set of routes whereas the set of proven potentially optimal links can implicitly contain a large number of alternative paths. The route selection problem can be dealt be with by considering risk aversion.

At each node $T(i)$, the optimistic or risk–prone traveller expects all the downstream arcs not to be congested and so he will choose $p^{\Gamma}{}_{T(i)}$. The following theorem proves that this is equivalent to select link i for which $c_i = p^{\Gamma}{}_{T(i)} - p^{\Gamma}{}_{H(i)}$ or, equivalently, $i \in FS(T(i)) \ni' c_i + p^{\Gamma}{}_{H(i)} = \min\{c_k + p^{\Gamma}{}_{H(k)} \,|\, k \in FS(T(i))\}$.

THEOREM 3: $i \in p^{\Gamma}{}_{T(i)} \Leftrightarrow c_i = t^{\Gamma}{}_{T(i)} - t^{\Gamma}{}_{H(i)}$

Proof: 1) The implication “$\Rightarrow$” is a trivial consequence of the definitions of p^{Γ} and t^{Γ}. 2) As to “$\Leftarrow$”, suppose it doesn’t hold, i.e. $c_i = t^{\Gamma}{}_{T(i)} - t^{\Gamma}{}_{H(i)}$ [1] but $\exists\, j \in FS(T(i)), j \neq i \ni' j \in p^{\Gamma}{}_{T(i)}$. $j \in p^{\Gamma}{}_{T(i)} \Leftrightarrow c_j + t^{\Gamma}{}_{H(j)} < c_k + t^{\Gamma}{}_{H(k)}\ \forall\ k \in FS(T(i)), k{\neq}j$. But from the first part of the proof, $j \in$

$p^{\Gamma}_{T(i)} \Rightarrow c_j = t^{\Gamma}_{T(i)} - t^{\Gamma}_{H(j)}$. Therefore $i \in FS(T(i)) \Rightarrow c_j + t^{\Gamma}_{H(j)} = t^{\Gamma}_{T(i)} - t^{\Gamma}_{H(j)} + t^{\Gamma}_{H(j)} = t^{\Gamma}_{T(i)} < c_i + t^{\Gamma}_{H(i)}$ which contradicts [1]. As the contradiction comes out as a consequence of assuming that the implication "$\Leftarrow$" is not true, it follows that the implication "$\Leftarrow$" must hold. QED

Analogously the pessimistic or risk–averse traveller, being afraid of finding congestion on all the potentially optimal links to the destination, will opt for $p^{\Gamma}_{T(i)}$, i.e. he will pick link i for which $c_i + d_i = p^H_{T(i)} - p^H_{H(i)}$ or, equivalently, $i \in FS(T(i)) \ni' c_i + d_i + p^H_{H(i)} = \min\{c_k + d_k + p^H_{H(k)} \mid k \in FS(T(i))\}$. Intermediate levels of aversion to risk could be represented by introducing a parameter $\alpha \in [0,1]$ such that the traveller chooses $i \in FS(T(i))$ which minimize the cost function

$$\alpha * (c_k + d_k + p^H_{H(k)}) + (1 - \alpha) * (c_k + p^{\Gamma}_{H(k)}),\ k \in FS(T(i)) \quad (4.1)$$

A highly risk–averse driver with $\alpha \rightarrow 1$ will hence choose the safest route p^H_{Ti} whereas an optimistic driver with $\alpha \rightarrow 0$ will choose the shortest path p^{Γ}_{Ti}. Drivers with medium level of risk–aversion might choose paths from the set of optimal paths that are neither the shortest nor the safest but satisfy both criteria to some degree.

Note that $\alpha * (c_k + d_k + p^H_{H(k)}) + (1 - \alpha) * (c_k + p^{\Gamma}_{H(k)}) = (c_k + p^{\Gamma}_{H(k)}) + \alpha * (p^H_{H(k)} + d_k - p^{\Gamma}_{H(k)})$. $(p^H_{H(k)} + d_k - p^H_{H(k)})$ is the difference between the cost of the path using link k at $T(i)$ when all downstream links from $T(i)$ are congested and when they are all not congested. In other words, it is the maximum regret the traveller can experience selecting link k. This means that adopting minimization function (4.1) as a route choice criterion is equivalent to extract paths taking into account both (optimistic) travel time and (maximum) regret (Loomes & Sugden, 1982), and considering the latter at most as important as the former.

Finally, it is worth noting that α could be related also to the network condition, in the sense that if the traveller has experienced congestion on most of the links he has travelled so far (and the conditions of different links are not independent) then it can be assumed that his risk aversion or his pessimism tends to increase.

5. Conclusions

Properties have been studied of potentially optimal links in a network in which each arc is characterized by two possible travel times, corresponding to the non congested and the congested condition. A potentially optimal link is a link which is included in the optimal path from its tail node to the destination for at least one combination of costs. An algorithm has been presented to find the set of links which such properties allow classifying definitely as potentially optimal when travelling between an OD pair. Finally a criterion to select a single route within the set of proven potentially optimal links has been proposed, considering risk aversion.

Sufficient and necessary conditions of potential optimality do not coincide. Further research should aim to find a condition of sufficiency larger than that identified in the paper. A feasible approach could be combining conditions for optimality when only two links are examined (theorem 1) in a way different from what lemma 3 does.

A static network has been studied in the paper. The properties of potentially optimal links have to be extended to the dynamic case, in which c_i can be interpreted as the free flow travel time and assumed constant while d_i changes over time.

When potentially optimal paths are investigated in the context of navigation assistance, the speed of the algorithm to identify $\boldsymbol{A_{PPOL}}$ becomes crucial. The complexity of the proposed algorithm needs to be analysed, in the search for faster solutions for both the static and the dynamic case.

The proposed route choice cost function has to be implemented in real world network examples to test the acceptability of paths suggested. Alternative combinations of travel time and regret can be considered.

Finally applications of the concept of potentially optimal links in assignment problems should be explored.

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