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On the Rodrigues Formula Solution of the Hypergeometric-Type Differential Equation

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Abstract

In this paper, we present a new systematic approach to the solution of the hypergeometric-like differential equation and its associated equation. The method produces, *tout court*, the general solution of these equations in the form of a combination of a standard Rodrigues formula and a 'generalized' Rodrigues formula, of a type due originally to Gonçalves [5] and recently considered, again, by Area *et al* [1]. In addition, a novel analysis of a class of integrals determining the generalized Rodrigues formulae is given, which complements an original analysis of Area *et al* [1]. Finally, the relation between the hypergeometric-type differential equation and the hypergeometric equation is elucidated further, following the work of Koepf and Masjed-Jamel [7].

Mathematics Subject Classification: 33C25; 33C45

Keywords: hypergeometric-type differential equation; Rodrigues formula; generalized Rodrigues formula; integration technique

1. Introduction

Consider the 'hypergeometric-type' second-order linear ordinary differential equation (with *non-negative* integer n)

$$p(z)y_n''(z) + q(z)y_n'(z) + \lambda_n y_n(z) = 0 \tag{1}$$

where $p(z)$ is a quadratic function, $q(z)$ a linear function and λ_n is independent of z. As usual, the dashes denote differentiation with respect to the function argument (in this case the variable z). Our interest here, is in extracting, by a new systematic process, a Rodrigues formula solution to equation (1). The method is inspired by an old paper by Gonçalves [5], although in the current work we are able to eliminate particular assumptions made by Gonçalves, for example, that the λ_n of equation (1) takes the particular form that we *derive* below.

The method that we adopt has three parts.

(a) First, (1) is rewritten in *self-adjoint* form, in the usual way [8], then transformed into the *formal adjoint* of equation (1).

(b) Next, we *assume* the solution of the formal adjoint of (1), (6) below, is given as the nth-derivative of another second-order linear ordinary differential equation of (effectively) the same class as (1), but, this time of a particular *inhomogeneous* form.

(c) Finally, we assume that this second differential equation is *exact* and may be integrated directly.

The outcome of the above three-part recipe is a system of four equations in four unknowns that emerge from the method, which, when solved and combined with the rest of the analysis, leads to the general solution of (1) which has, as a special case, a Rodrigues formula format. In fact, as an advance on previous work, we are able to derive the general solution to equation (1) with the 'second solution' arising automatically from the general method. This 'second solution' is given in terms of a *generalized* Rodrigues formula considered recently by Area *et al* [1], though derived, already, by Gonçalves [5]. Further, we are able to apply the method to the so-called *associated equation* [6] of equation (1), equation (24) below, and obtain the general solution to (24) as a combination, also, of a Rodrigues formula and a *generalized* Rodrigues formula.

When considering the *generalized* Rodrigues formulae, it proves necessary to determine a particular class of integrals. This problem was studied in detail by Area *et al* [1], but there is a 'gap' in their discussion which we fill-in here, by re-analysing the problem of evaluating the said integrals in an entirely different manner. The connection with the hypergeometric equation is also discussed.

2. The Solution Method

First, (1) is rewritten in *self-adjoint* form as [8]

$$\frac{d}{dz}[p(z)\omega(z)y_n'(z)] + \omega(z)\lambda_n y_n(z) = 0 \tag{2}$$

where $\omega(z)$ the solution of the Pearson differential equation

$$\frac{d}{dz}[p(z)\omega(z)] = q(z)\omega(z) \tag{3}$$

is given by

$$\omega(z) = \frac{e^{\int \frac{q(z)}{p(z)}dz}}{p(z)} \tag{4}$$

Following Gonçalves [5], we transform equation (2) by the change of dependent variable

$$w_n(z) = \omega(z)y_n(z) \tag{5}$$

so that equation (2) becomes

$$p(z)w_n''(z) + [2p'(z) - q(z)]w_n'(z) + [p''(z) - q'(z) + \lambda_n]w_n(z) = 0 \tag{6}$$

with equation (6) being the *formal adjoint* of equation (1). Equation (6) is the starting point, also, of the analysis of Area *et al* [1]. However, the *details* of the analysis presented here differs from that of Gonçalves [5] and Area *et al* [1]. Next, we associate equation (6) with an *inhomogeneous* second-order linear ordinary differential equation of the same class. We write this equation, with new dependent variable $v_n(z)$, as

$$p(z)v_n''(z) + r_n(z)v_n'(\mathrm{z}) + \mu_n v_n(z) = P_{n-1}(z) \tag{7}$$

with $r_n(z)$ (a linear function) and μ_n (independent of z) to be determined and $P_{n-1}(z)$ an *arbitrary polynomial* of degree $n-1$. We now assume that the nth derivative of (7) reduces to (6) exactly. This gives rise to the identities

$$\frac{d^n}{dz^n}[p(z)v_n''(z) + r_n(z)v_n'(z) + \mu_n v_n(z)]$$
$$\equiv p(z)w_n''(z) + [2p'(z) - q(z)]w_n'(z) + [p''(z) - q'(z) + \lambda_n]w_n(z) \tag{8}$$

and

$$w_n(z) \equiv v_n^{(n)}(z) \tag{9}$$

Finally, the last part of the method assumes that equation (7) is *exact* in that we may write (following [5] again)

$$p(z)v_n''(z) + r_n(z)v_n'(z) + \mu_n v_n(z) \equiv \frac{d}{dz}[p(z)v_n'(z) + \ell_n(z)v_n(z)] = P_{n-1}(z) \tag{10}$$

with $\ell_n(z)$ to be determined.

The identities (8) and (10) enable the identification of the four unknowns: λ_n, μ_n, $\ell_n(z)$ and $r_n(z)$. Using Leibnitz rule for the nth-derivative, we find that the identity (8) yields

$$r_n(z) = (2-n)p'(z) - q(z) \tag{11a}$$

and

$$\mu_n + n r'_n(z) + \frac{n(n-1)}{2} p''(z) = \lambda_n + p''(z) - q'(z) \tag{11b}$$

while enforcing the identity (10) leads to

$$\ell_n(z) = r_n(z) - p'(z) \tag{12a}$$

and

$$\ell'_n(z) = \mu_n \tag{12b}$$

So, from equations (11) and (12), we have

$$r_n(z) = (2-n)p'(z) - q(z) \tag{13a}$$

and

$$\ell_n(z) = (1-n)p'(z) - q(z) \tag{13b}$$

and

$$\mu_n = (1-n)p''(z) - q'(z) \tag{13c}$$

and

$$\lambda_n = -\frac{n}{2}[(n-1)p''(z) + 2q'(z)] \tag{13d}$$

3. The Solution(s)

We can now start to 'roll-back' the above process by solving (9) for $v_n(z)$ and then substitute in (9) for $w_n(z)$ and then substitute in (5) for $y_n(z)$. First, we rewrite (10) using (13b) to get

$$\frac{d}{dz}[p(z)v'_n(z) + [(1-n)p'(z) - q(z)]v_n(z)] = P_{n-1}(z) \tag{14}$$

Integrating equation (14), repeatedly, we get

$$v_n(z) = p^n(z)\omega(z)\left(\int \frac{1}{p^{n+1}(z)\omega(z)}[P_n(z) + D_n]dz + C_n\right) \tag{15}$$

with C_n and D_n (for *arbitrary* n) integration constants and $P_n(z)$ an *arbitrary* *polynomial* of degree n. Next, we substitute (15) back through (9) and (5) to get the *general solution* of equation (1) as (see [5] again)

$$y_n(z) = \frac{1}{\omega(z)} \frac{d^n}{dz^n}\left[p^n(z)\omega(z)\left(\int \frac{1}{p^{n+1}(z)\omega(z)}[P_n(z) + D_n]dz + C_n \right)\right] \quad (16)$$

If we set $P_n(z) = D_n = 0$ in (16), then we get a particular solution to (1) as

$$y_n(z) = \frac{C_n}{\omega(z)} \frac{d^n}{dz^n}[p^n(z)\omega(z)] \quad (17)$$

and (17) is indeed the Rodrigues formula solution to (1), *subject to the condition* (13d). The actual forms that the C_n are presented, for example, in reference [2]. Interestingly, if we set $P_n(z) = C_n = 0$ in (16), then we get a *second* particular solution to (1) as

$$y_n(z) = \frac{D_n}{\omega(z)} \frac{d^n}{dz^n}\left[p^n(z)\omega(z)\int \frac{1}{p^{n+1}(z)\omega(z)} dz\right] \quad (18)$$

and (18) is the *extended* Rodrigues formula solution to (1), discussed recently by Area *et al* [1]. Solution (18) is *subject to the condition* (13d) also.

4. The Associated Equation

We consider, now, the relationship [6] between the basic equation (1) and its so-called *associated equation*, with an eye to producing a Rodrigues formula solution for the associated equation. If we differentiate (1) m times, $\mathrm{m} \le \mathrm{n}$, we find that, with $y_n^{(m)}(z) \equiv \frac{d^m}{dz^m}[y_n(z)]$, we get

$$p(z)[y_n^{(m)}(z)]'' + q_m(z)[y_n^{(m)}(z)]' + \lambda_n^m y_n^{(m)}(z) = 0 \quad (19)$$

where

$$q_m(z) = mp'(z) + q(z), \qquad \lambda_n^m = \frac{m(m-1)}{2} p''(z) + mq'(z) + \lambda_n \quad (20)$$

Apparently, equation (19), with (20) in mind, is of the *same class* as equation (1), so that equation (19) comes within the compass of the Rodrigues' formula solution methodology and we can write down a Rodrigues formula solution to equation (19). Indeed, since $y_n^{(m)}(z)$ is a polynomial of degree $n-m$, (20) has a

Rodrigues' formula solution of the form (16), but with allowance for the difference in degree of the polynomials and the specific form of the coefficients $q_m(z)$ and λ_n^m. Making these allowances, we find that the Rodrigues formula for $y_n^{(m)}(z)$ is

$$y_n^{(m)}(z) = \frac{1}{\omega_m(z)} \frac{d^{n-m}}{dz^{n-m}} \left[p^{n-m}(z)\omega_m(z) \left(\int \frac{1}{p^{n-m+1}(z)\omega_m(z)} [P_{n-m}(z) + D_n^m] dz + C_n^m \right) \right] \tag{21}$$

with C_n^m and D_n^m independent of z and where $\omega_m(z)$ is a solution of

$$\frac{d}{dz}[p(z)\omega_m(z)] = q_m(z)\omega_m(z) \tag{22}$$

so that

$$\omega_m(z) = p^m(z)\omega(z) \tag{23}$$

The *associated equation* is obtained from (19) by transforming (19) into the same format as (1), that is, we wish to write (19) as

$$p(z)y''_{n,m}(z) + q(z)y'_{n,m} + \lambda_{\text{n,m}} y_{n,m}(z) = 0 \tag{24}$$

with $\lambda_{\text{n,m}}$ to be determined. In fact, it is well-known that [6]

$$y_{n,m}(z) = (-1)^m p^{\frac{m}{2}}(z) y_n^{(m)}(z) \tag{25}$$

and

$$\lambda_{n,m}(z) = \lambda_n - \lambda_m - \frac{\frac{m}{2}\left(p''p + p'[q + (\frac{m}{2} - 1)p'] \right)}{p} \tag{26}$$

Relations (24) and (26) yield the usual expressions for the associated equation. Finally, from (21), (23) and (25), we get the Rodrigues formula solution to (24), the associated equation, as

$$y_{n,m}(z) = \frac{(-1)^m p^{-\frac{m}{2}}(z)}{\omega(z)} \frac{d^{n-m}}{dz^{n-m}} \left[p^n(z)\omega(z) \left(\int \frac{1}{p^{n+1}(z)\omega(z)} [P_{n-m}(z) + D_n^m] dz + C_n^m \right) \right] \tag{27}$$

Setting $P_{n-m}(z) = D_n^m = 0$ in (27), we get a more standard Rodrigues formula solution to (24) as

$$y_{n,m}(z) = \frac{(-1)^m C_n^m p^{-\frac{m}{2}}(z)}{\omega(z)} \frac{d^{n-m}}{dz^{n-m}}[p^n(z)\omega(z)] \tag{28}$$

(See Table 1 for some examples, to within a *conventional* normalizing constant.)
While, setting $P_{n-m}(z) = C_n^m = 0$ in (27), we get a *second* type of Rodrigues formula solution to (24) as

$$y_{n,m}(z) = \frac{(-1)^m D_n^m p^{-\frac{m}{2}}(z)}{\omega(z)} \frac{d^{n-m}}{dz^{n-m}}\left[p^n(z)\omega(z)\int \frac{1}{p^{n+1}(z)\omega(z)}dz\right] \tag{29}$$

Associated Equation	**Equation Form**	**Rodrigues Formula**
Legendre	$(1-z^2)P''_{n,m} - 2z P'_{n,m} + \left[n(n-1) - \frac{m}{1-z^2}\right]P_{n,m} = 0$	$P_{n,m}(z) = \frac{(-1)^m C_n^m}{(1-z^2)^{\frac{m}{2}+1}} \frac{d^{n-m}}{dz^{n-m}}[(1-z^2)^{n+1}]$
Hermite	$H''_{n,m} - 2z H'_{n,m} + 2(n-m)H_{n,m} = 0$	$y_{n,m}(z) = (-1)^m C_n^m e^{z^2} \frac{d^{n-m}}{dz^{n-m}}[e^{-z^2}]$
Generalized Laguerre	$zL''_{n,m} + (a+1-z)L'_{n,m} + \frac{1}{2}[(2n-m) - \frac{m(2a+m)}{2z}]\, L_{n,m} = 0$	$L_{n,m}(z) = \frac{(-1)^m C_n^m e^z}{z^{a+\frac{m}{2}}} \frac{d^{n-m}}{dz^{n-m}}[z^{n+a} e^{-z}]$
Bessel	$z^2 J''^{(a,b)}_{n,m} + [(a+2)z + b]J'^{(a,b)}_{n,m} - [n(a+n+1) + \frac{mb}{z}]J^{(a,b)}_{n,m} = 0$	$J^{(a,b)}_{n,m}(z) = \frac{(-1)^m C_n^m}{z^{m+a} e^{-\frac{b}{z}}} \frac{d^{n-m}}{dz^{n-m}}[z^{2n+a} e^{-\frac{b}{z}}]$
Jacobi	$(1-z^2)P''^{(a,b)}_{n,n} + \left[b-a-(a+b+2)z\right]P'^{(a,b)}_{n,n} + \left[\begin{matrix} n(n+a+b+1) \\ -m(m+a+b) - \frac{m[(b-a)-(m+a+b)z]z}{1-z^2} \end{matrix}\right]P^{(a,b)}_{n,n} = 0$	$y_{n,m}(z) = \frac{(-1)^m C_n^m}{(z-1)^{(m+2a)/2}(z+1)^{(m+2b)/2}} \times \frac{d^{n-m}}{dz^{n-m}}[(z-1)^{(n+a)}(z+1)^{n+b}]$

Table 1. Important Associated ODE and their Rodrigues Formulae (with a, b and c constant)

5. The Integrals

The integrals

$$I_{n+1}(z) = \int \frac{1}{p^{n+1}(z)\omega(z)} dz \tag{30}$$

have been studied in detail by Gonçalves [5] and Area *et al* [1]. However, the analysis presented in this section should be seen as a complement to the work of Area *et al* [1]. Elementary differentiation shows that the $I_n(z)$ satisfy the differential equation

$$I_n''(z) + \left(\frac{(n-1)p'(z) + q(z)}{p(z)} \right) I_n'(z) = 0 \tag{31}$$

or

$$p(z)I_n''(z) + [(n-1)p'(z) + q(z)]I_n'(z) + I_n(z) = I_n(z) \tag{32}$$

which suggest looking for *raising and lowering operators* (with the $K's$ *independent* of z)

$$\left[a_n(z)\frac{d}{dz} + K_n^+ \right] I_n = I_{n+1} \tag{33a}$$

and

$$\left[b_n(z)\frac{d}{dz} - K_n^- \right] I_n = I_{n-1} \tag{33b}$$

where we have assumed that (32) can be written as

$$\left[a_n(z)\frac{d}{dz} + K_{n-1}^+ \right]\left[b_n(z)\frac{d}{dz} - K_n^- \right] I_n = I_n \tag{34}$$

with

$$a(z)b(z) = p(z) \tag{35a}$$

and

$$b_n'(z) + K_{n-1}^+ b_n(z) = (n-1)p'(z) + q(z) \tag{35b}$$

and

$$K_{n-1}^+ K_n^- = -1 \tag{35c}$$

Taking

$$a_n(z) = c_n z + d_n \tag{36}$$

(33a) becomes

$$\left[(c_n z + d_n)\frac{d}{dz} + K_n^+ \right] I_n = I_{n+1} \tag{37}$$

which is where Area *et al* [3] begin their analysis of I_n; so, c_n, d_n and K_n^+ may be taken as *known*. Substituting back, we get $b_n(z)$ and K_n^- from (35b) and (35c).

5. Discussion and Conclusions

In this final section we will discuss the relation of our methodology to related work, with particular emphasis given to a comparison with the 'root paper' of Gonçalves [5], which inspired the current approach, and, also, to the work of Area *et al* [1] to which the results of the current enquiry are also closely related. Also, for completeness, we will elucidate the connection between the hypergeometric-like equations discussed above, and the hypergeometric equation itself.

First, with regard to Gonçalves paper [5], we note that, while the current method has been developed after a close study of this work, Gonçalves approach was that of following a *particular* path towards his goal and he did *not* set-out the general procedure presented here. Indeed, using the methodology set-out in section 1, we have *derived* the expressions (13) for the differential equations' parameters, assuming only the *general forms* (7) and (10), whereas Gonçalves approach assumed these results, explicitly or implicitly.

Further, Gonçalves [5] does *not* discuss the associated equation, (24), although he does discuss the derivatives of (16), but does not consider the general case discussed in section 3. We note in passing, though, that Jafarizadeh and Fakhri [6] discuss the associated equation and quote an alternative form of the Rodrigues formula (28); also, special cases of (28) are presented in Chenaghlou and Fakhri [3,4]. Finally, Gonçalves treatment of the class of integrals represented by relations (30) is very specific and a somewhat different problem that that treated here and by Areal *et al* [1] is considered [5]. (The methodology, being so specific, is different too.)

Next, turning to the recent work of Area *et al* [1], we note that as well as developing their approach to equation (1) in a different manner to either Gonçalves [5] or the current paper, they have to utilize variation of parameters to produce the generalized Rodrigues formula, whereas in the current approach, as in Gonçalves [5] *original* work, the full general solution appears *at once* from the basic methodology. This is so, also, for the associated equation, equation (24), as well as the original equation (1). As to the integrals, $I_{n+1}(z)$, appearing in the generalized Rodrigues formulae, the analysis of section 4 develops their solution methodology *ab initio*, with the results 'dove-tailing' into the work of Area *et al* [1], thus closing a 'gap' in their discussion and completing the analysis of the

$I_{n+1}(z)$. All details of the calculations following equation (37) are contained in Area *et al* [1], except for the determination of $b_n(z)$ and K_n^-, which is elementary and is omitted.

Moving on, we will now elucidate the relationship between equations (1), (19) and (24) and the *hypergeometric equation* (for constants α, β and γ) [2]

$$x(x-1)y''(x)+[(\alpha+\beta+1)x-\gamma)]y'(x)+\alpha\beta y(x)=0 \tag{38}$$

First, we consider equation (1), and write-out $p(z)$, $q(z)$ and λ_n explicitly as

$$p(z)=p_0z^2+p_1z+p_2,\quad q(z)=q_1z+q_2 \text{ and } \lambda_n=-n[(n-1)p_0+q_1] \tag{39}$$

with the p_k's and q_k's constants. Next, in (1), following [7], we make the linear substitution $z=ax+b$, with a and b constants, to transform (1) into

$$x(x-1)y''(x)+\left[\frac{q_1}{p_0}x+\frac{(2p_0q_2-p_1q_1)\pm q_1\sqrt{p_1^2-4p_0p_2}}{\mp 2p_0\sqrt{p_1^2-4p_0p_2}}\right]y'(x) \\ -\frac{n}{p_0}[(n-1)p_0+q_1]y(x)=0 \tag{40}$$

provided

$$a=\mp\frac{\sqrt{p_1^2-4p_0p_2}}{p_0} \quad \textit{and} \quad b=\frac{-p_1\pm\sqrt{p_1^2-4p_0p_2}}{2p_0} \tag{41}$$

Then, comparing (38) and (40) [7], we see that (40) is a special case of (38) with

$$\alpha=-n,\ \beta=n-1+\frac{q_1}{p_0} \text{ and } \gamma=\frac{2p_0q_2-p_1q_1\pm q_1\sqrt{p_1^2-4p_0p_2}}{\mp 2p_0\sqrt{p_1^2-4p_0p_2}} \tag{42}$$

It follows now [7], that the solution to the hypergeometric-type differential equation (1) can be written in terms of the hypergeometric equation (38).

By utilizing the above change of independent variable, we can *extend* the methodology of [7], and elucidate the relationship between equation (19) and the *hypergeometric equation*. First, we write-out the $q_m(z)$ and λ_n^m explicitly as

$$q_m(z)=(2mp_0+q_1)z+mp_1+q_2 \text{ and } \lambda_n^m=(m-n)[(m+n-1)p_0+q_1] \tag{43}$$

Next, in (19), we follow [7] again and make the linear substitution $z = ax + b$, to transform (19) into

$$x(x-1)[y_n^{(m)}(x)]'' + \left[\frac{2mp_0 + q_1}{p_0}x + \frac{(2p_0q_2 - p_1q_1) \pm (2mp_0 + q_1)\sqrt{p_1^2 - 4p_0p_2}}{\mp 2p_0\sqrt{p_1^2 - 4p_0p_2}}\right][y_n^{(m)}(x)]' + \frac{m-n}{p_0}[q_1 + (m+n-1)p_0]y_n^m(x) = 0 \tag{44}$$

provided a and b are determined by (41), as before. Then, comparing (38) and (44) we see that (44) is a special case of (38) with

$$\alpha = m - n,\ \beta = (m+n-1) + \frac{q_1}{p_0} \tag{45a}$$

and

$$\gamma = \frac{(2p_0q_2 - p_1q_1) \pm (2mp_0 + q_1)\sqrt{p_1^2 - 4p_0p_2}}{\mp 2p_0\sqrt{p_1^2 - 4p_0p_2}} \tag{45b}$$

It follows now, that the solution to the hypergeometric-type differential equation (19) can be written in terms of the hypergeometric equation (38).

Finally, we have the solution of the *associated equation* (24) in terms of that of the hypergeometric equation, as we know already that

$$y_{n,m}(z) = (-1)^m p^{\frac{m}{2}}(z) y_n^{(m)}(z) \tag{46}$$

Naturally, there are details in the *writing-out* of the relationships between the various equations we have considered and the hypergeometric equation [7], but we have finished the important part of the problem and the rest of the problem is a matter of notation. For, example, the development of the solution of equation (1) in terms of a solution of the equation (38) is presented in detail in [7].

In summary, a new systematic approach to the solution of the hypergeometric-like differential equation and its associated equation has been developed. Further, a novel analysis of a class of integrals determining the generalized Rodrigues formulae arising through the solution methodology has been given and the relationships between equations (1), (19) and (24) and equation (38) presented.

References

[1] Area, I., Godoy, E., Ronveaux, A. and Zarzo, A.: *Hypergeometric-type differential equations*: second *kind solutions and related integrals.* Journal of Computational and Applied Mathematics **157** (2003) 93–106.

[2] Beukers, F.: *Gauss' Hypergeometric Function*. Progress in Mathematics **260** (2007) 23–42.

[3] Chenaghlou, A. and Fakhri, H.: *Supersymmetry Approaches to the Radial Bound States of the Hydrogen-like Atoms.* International Journal of Quantum Chemistry **101** (2005) 291–304.

[4] H. Fakhri, H. and Chenaghlou, A.: *Ladder operators and recursion relations for the associated Bessel polynomials.* Physics Letters A **358** (2006) 345–353.

[5] Gonçalves, J. V.: *Sur La Formule de Rogrigues*. Portugaliae Mathematica **4** (1943) 52–64.

[6] Jafarizadeh M. A. and Fakhri H.: *Supersymmetry and shape invariance in differential equations of mathematical physics.* Physics Letters A **230** (1997) 164-170.

[7] Koepf, W and Masjed-Jamel, M.: *A generic polynomial solution for the differential equation of hypergeometric type and six sequences of orthogonal polynomials related to it.* Integral Transforms and Special Functions **17** (2006) 559–576.

[8] Nikiforov A. F. and Uvarov V. B.: *Special Functions of Mathematical Physics*. (Birkhauser) 1988.

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