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Similarity Solutions to Unsteady Pseudoplastic Flow Near a Moving Wall

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Abstract

It is shown that similarity solutions to a partial differential boundary value problem for power law fluids may be generated by the application of a multi-parameter stretching group. Furthermore, the resulting ordinary differential boundary value problem arising from the initial group analysis is itself transformed to an initial value problem by the application of a further stretching (one-parameter) group transformation. The resultant initial value problem is solved by a standard forward marching method.

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1. Introduction

In this paper, we present a similarity group analysis of the equation of motion of a semi-infinite body of pseudoplastic (power law) fluid occupying the half-space $y > 0$ [2, 8], that is

$$\frac{\rho}{m}\frac{\partial u}{\partial t}+\frac{\partial}{\partial y}\left(-\frac{\partial u}{\partial y}\right)^{n}=0 \tag{1.1}$$

where $u(y,t)$ is the x-component of velocity in a cartesian system. The fluid is set in motion at time $t=0$ by imparting a constant velocity U on the boundary $y=0$; since the effects of this velocity is expected to decay and eventually vanish as we move into the body of the fluid, equation (1.1) is to be solved with the boundary conditions

$$u(0,t)=U,\quad t>0;\qquad u(\infty,t)=0,\quad t>0 \tag{1.2}$$

In equation (1.1), ρ/m is the ratio of the fluid density ρ to the empirical constant m, characteristic of the fluid, defined via equation (2) of [2]; the parameter n also is characteristic of the fluid [2]. The solution process proposed here for the partial differential equation (1.1) with boundary conditions (1.2), and where the novelty lies, involves the *complete reduction* of the original problem (1.1)/(1.2) to an initial value problem *entirely* by stretching group transformations.

The solution process and paper are organized as follows. The first part of the process, presented in section 2, involves the identification of a 3-parameter stretching group transformation that the original problem (1.1)/(1.2) is invariant under [5, 6]. This is achieved by demanding [5, 6] that the original problem (1.1)/(1.2) be invariant under the more basic stretching [9] 5-parameter group

$$\bar{y}=a_1 y,\quad \bar{t}=a_2 t,\quad \bar{u}=a_3 u,\quad \overline{\left(\frac{\rho}{m}\right)}=a_4\left(\frac{\rho}{m}\right),\quad \bar{U}=a_5 U \tag{1.3}$$

$a_k>0, k=1,2,\ldots,5$. Next, once the 3-parameter stretching group is determined, it is then used, in section 3, to transform the *partial* differential boundary value problem (1.1)/(1.2) into an *ordinary* differential boundary value problem. To do this, we assume the existence of an implicit solution [4] to the original problem (1.1)/(1.2) of the form

$$g(u,y,t,(\rho/m),U)=0 \tag{1.4}$$

with g an arbitrary function, which is to be *invariant* under the 3-parameter group of section 2. This invariance requirement on (1.4) leads to two systems of partial differential equations having identifiable [5, 6] *similarity solutions*, which, in turn, allows a *similarity transformation* of the *partial* differential boundary value problem (1.1)/(1.2) into an *ordinary* differential boundary value problem. The final part of the reduction process, presented in section 4, uses another [1, 3, 7] stretching group transformation to transform the ordinary differential *boundary* value problem, obtained from the invariant solution analysis of section 3, into an ordinary differential *initial value* problem. Finally, in section 5, the resultant initial value problem is solved by a standard fourth-order Runge-Kutta forward marching method and the results of our analysis compared with previous work on the problem, that is, (1.1)/(1.2).

2. Determining the r-parameter group

Following Moran [5] and Moran and Marshek [6], we will use the 5-parameter stretch transformation (with the group parameters $a_1,\ldots,a_5$ positive real variables)

$$\bar{y} = a_1 y, \quad \bar{t} = a_2 t, \quad \bar{u} = a_3 u, \quad \overline{\left(\frac{\rho}{m}\right)} = a_4\left(\frac{\rho}{m}\right), \quad \bar{U} = a_5 U \tag{2.1}$$

to construct an r-parameter group transformation, which we write as a group, G_r, and a subgroup, S_r [5,6], that is

$$G_r : \begin{cases} S_r : \begin{cases} \bar{y} = a_1^{b_{11}} a_2^{b_{12}} \cdots a_r^{b_{1r}}\, y \\ \bar{t} = a_1^{b_{21}} a_2^{b_{22}} \cdots a_r^{b_{2r}}\, t \\ \overline{(\rho/m)} = a_1^{b_{31}} a_2^{b_{32}} \cdots a_r^{b_{3r}}\, (\rho/m) \\ \bar{U} = a_1^{b_{41}} a_2^{b_{42}} \cdots a_r^{b_{4r}}\, U \end{cases} \\ \bar{u} = a_1^{c_1} a_2^{c_2} \cdots a_r^{c_r}\, u \end{cases} \tag{2.2}$$

The demand that equation (1.1) and the boundary conditions (1.2) be invariant under the 5-parameter group transformation (2.1), leads to relations that determine both the order r of the transformation group (2.2) and the values of the group indices $b_{11},\ldots,c_r$. It will be apparent that the r-parameter group (2.2) is simply another way of writing the 5-parameter group (2.1), that is, the 5 parameters are *not* independent and so $r < 5$. This means that equation (1.1) and the boundary conditions (1.2) will be invariant under the r-parameter group transformation *a fortiori*.

Applying the transformations (2.1) and the chain rule to equation (1.1), we find (subscripts on u or $\bar{u}$ implying partial differentiation)

$$a_1^{n+1}a_3^{-n}[\overline{(\rho/m)}\,\overline{u_t} + (\overline{(-u_y)^n})_{\bar{y}}] = a_1^{n+1}a_2^{-1}a_3^{1-n}a_4(\rho/m)u_t + ((-u_y)^n)_y \quad (2.3)$$

If we require equation (1.1) to be invariant under the transformation (2.1), then from (2.3) we must have

$$a_1^{n+1}a_2^{-1}a_3^{1-n}a_4 = 1 \qquad (2.4)$$

Also, if we demand that the boundary conditions (1.2) be invariant under the transformations (2.1), then we require

$$a_5 = a_3 \qquad (2.5)$$

From (2.4) and (2.5), we see that we may take a_1, a_2 and a_3 as the basic *independent* group parameters, with the transformation (2.2) now taking the form of a 3-parameter group transformation, which we write as a group, G_3, and a subgroup, S_3 [5, 6], that is

$$G_3 : \begin{cases} S_3 : \begin{cases} \bar{y} = a_1^{+1}a_2^{0}a_3^{0}y \\ \bar{t} = a_1^{0}a_2^{+1}a_3^{0}t \\ \overline{(\rho/m)} = a_1^{-(n+1)}a_2^{+1}a_3^{n-1}(\rho/m) \\ \overline{U} = a_1^{0}a_2^{0}a_3^{+1}U \end{cases} \\ \bar{u} = a_1^{0}a_2^{0}a_3^{+1}u \end{cases} \qquad (2.6)$$

In the next section we show that by determining the *group invariants* associated with S_3 and G_3 [5, 6], that the partial differential boundary problem (1.1)/(1.2) can be transformed to a corresponding ordinary differential boundary problem, (3.3) below.

3. The 3-parameter group analysis

The next stage of the analysis, is the reduction of the *partial* differential boundary value problem (1.1)/(1.2) into an *ordinary* differential boundary value problem. Critical to this analysis is the determination of the *group invariants* of S_3 and G_3 [5, 6] and *invariant solutions* of (1.1)/(1.2) under G_3 [5, 6]. If we express the solution to our partial differential boundary value problem *implicitly*, then we may introduce the function g such that the implicit solution is written as [4, 8]

$$g(u, y, t, (\rho/m), U) = 0 \tag{3.1}$$

For (3.1) to be invariant under the group G_3, we seek invariants, g, of the form

$$g(\bar{u}, \bar{y}, \bar{t}, \overline{(\rho/m)}, \bar{U}) = g(u, y, t, (\rho/m), U) \tag{3.2}$$

So, by partial differentiation of (3.2) with respect to a_1, a_2 and a_3, any (*absolute* [5, 6]) group invariant for the group G_3 must satisfy the partial differential system (where we may drop the bars)

$$y\frac{\partial g}{\partial y} - (n+1)(\rho/m)\frac{\partial g}{\partial(\rho/m)} = 0 \tag{3.3a}$$

$$t\frac{\partial g}{\partial \bar{t}} + (\rho/m)\frac{\partial g}{\partial(\rho/m)} = 0 \tag{3.3b}$$

$$u\frac{\partial g}{\partial \bar{u}} + (n-1)(\rho/m)\frac{\partial g}{\partial(\rho/m)} + U\frac{\partial g}{\partial U} = 0 \tag{3.3c}$$

That is, we have three *independent* linear homogeneous partial differential equations, in five variables, which has *two* solutions [6].

However, if we consider the *subgroup* S_3, we seek an invariant, η, of the form

$$\eta(\bar{y}, \bar{t}, \overline{(\rho/m)}, \bar{U}) = \eta(y, t, (\rho/m), U) \tag{3.4}$$

leading to three *independent* linear homogeneous partial differential equations, in four variables, that is, by partial differentiation of (3.4) with respect to a_1, a_2 and a_3, η must satisfy

$$y\frac{\partial \eta}{\partial y} - (n+1)(\rho/m)\frac{\partial \eta}{\partial(\rho/m)} = 0 \tag{3.5a}$$

$$t\frac{\partial \eta}{\partial \bar{t}} + (\rho/m)\frac{\partial \eta}{\partial(\rho/m)} = 0 \tag{3.5b}$$

$$(n-1)(\rho/m)\frac{\partial \eta}{\partial(\rho/m)} + U\frac{\partial \eta}{\partial U} = 0 \tag{3.5c}$$

(where we drop the bars again) which has *one* solution [6]. Apparently the sole solution η to (3.5) is *also* one of the two solutions to the G_3 equations (3.3).

From the general theory of the solution of equations such as (3.3) and (3.5) [6], we know that we may look for *particular* group invariants of the form

$$\eta = y[t]^{b_1}][\rho/m]^{b_2}[U]^{b_3}, \qquad \hat{g} = u[t]^{c_1}][\rho/m]^{c_2}[U]^{c_3} \tag{3.6}$$

The parameters $b_1,\ldots,c_3$ are evaluated by substituting that the relations (3.6) into equations (3.3) and (3.5): this leads to two sets of simultaneous equations for (b_1,b_2,b_3) and (c_1,c_2,c_3), which are solved to give the *particular* invariants

$$\eta = y\left[\frac{\rho}{mtU^{n-1}}\right]^{\frac{1}{n+1}}, \qquad \hat{g} = \frac{u}{U} \tag{3.7}$$

As the invariants (3.7) are *functionally independent*, the *general* solution, g, of (3.3) is a (differentiable) function [6], *f* say, of η and $\hat{g}$, so that

$$g(u,y,t,(\rho/m),U) = f(\hat{g},\eta) = 0 \tag{3.8}$$

Solving the implicit relation (3.8) for $\hat{g}$ in terms of, η, we see that we may rewrite (3.7) in the form

$$\eta = y\left[\frac{\rho}{mtU^{n-1}}\right]^{\frac{1}{n+1}}, \qquad \frac{u}{U} = F(\eta) \tag{3.9}$$

with $F(\eta)$ (currently) an arbitrary function.

Finally, since $u = UF(\eta)$, we can now reduce the original partial differential equation (1.1), with boundary conditions (1.2), to a boundary value problem in one variable only. That is, an ordinary differential equation with appropriately transformed boundary conditions. Indeed, under the similarity transformation (3.9), equations (1.1) and (1.2) are transformed into the following ordinary differential boundary value problem

$$(-F)^{n-1}F'' + \frac{1}{n(n+1)}F' = 0; \qquad F(0) = 1, \quad F(\infty) = 0 \tag{3.10}$$

where the dash denotes differentiation with respect to η.

The relations (3.9) and (3.10) are essentially those obtained by Bird [2] from *ad hoc* dimensional considerations. In fact, $\eta = (n+1)r$, where r is defined by equation (5) of reference [2]. In what follows, it proves convenient to transform the system (3.10) into the form obtained by Bird [2] by making the transformation $\eta \to \eta/(n+1)$, to get (the dash denotes differentiation with respect to 'new' η)

$$(-F')^{n-1}F'' + \frac{(n+1)^n}{n}\eta F' = 0; \qquad F(0) = 1, \quad F(\infty) = 0 \tag{3.11}$$

4. Reduction to an initial value problem: further group analysis

We wish to solve the system (3.11) to find the dimensionless velocity, F, from which we can get the x-component of the velocity, u, from UF. That is, we wish to solve the ordinary differential boundary value problem

$$(-F')^{n-1}F'' + \frac{(n+1)^n}{n}\eta F' = 0; \qquad F(0) = 1, \quad F(\infty) = 0 \tag{4.1}$$

Bird solved (4.1) using a semi-analytic method.[2]. In what follows, we solve the system (4.1) for F, following Phan-Thien [8], by turning (4.1) into an initial value problem. To do this we introduce, first, the change of variable $G = 1 - F$, when the system (4.1) becomes (the *variable* G is not to be confused with the transformation group)

$$(G')^{n-1}G'' + \frac{(n+1)^n}{n}\eta G' = 0; \qquad G(0) = 0, \quad G(\infty) = 1 \tag{4.2}$$

The boundary value problem (4.2) may be transformed into an initial value problem now by a one-parameter stretching group transformation (see, for example, the books by Na [7] or Bluman and Coles [3] or Ames [1]). If we define

$$\overline{G} = A^{\alpha_1}G, \quad \overline{\eta} = A^{\alpha_2}\eta \tag{4.3}$$

then the equation for $G = 1 - F$ in (4.2) will transform as

$$A^{-\alpha_1}[(\overline{G'})^{n-1}\overline{G''} + \frac{(n+1)^n}{n}\overline{\eta}\,\overline{G'}]$$
$$= A^{(n-1)\alpha_1-(n-1)\alpha_2}(G')^{n-1}G''\frac{(n+1)^n}{n}\eta G' \tag{4.4}$$

and, for invariance, we require

$$(n-1)\alpha_1 - (n-1)\alpha_2 = 0 \tag{4.5}$$

To determine α_1 and α_2, we require another equation, and we obtain this by forcing a condition on the *unknown* initial condition $G'(0)$. Let

$$G'(0) = A \tag{4.6}$$

then, transforming (4.6), we will have

$$A^{\alpha_1 - \alpha_2}\,\overline{G'(0)} = A \tag{4.7}$$

Now, by setting

$$\alpha_1 - \alpha_2 = 1 \tag{4.8}$$

equation (4.7) becomes $\overline{G'(0)} = 1$ and the simultaneous equations (4.5) and (4.8) may be solved, for α_1 and α_2, to give

$$\alpha_1 - \frac{n+1}{2}, \qquad \alpha_2 - \frac{n-1}{2} \tag{4.9}$$

It remains only to find the group parameter A. To find A, we consider the boundary condition at infinity, $G(\infty) = 1$. Transforming the boundary condition at infinity, we find that

$$A = \left[\frac{1}{\overline{G}(\infty)}\right]^{\frac{2}{n+1}} \tag{4.10}$$

With everything now in place, the solution procedure is as follows. First, for any n, we solve the initial value problem

$$(\overline{G'})^{n-1}\overline{G''} + \frac{(n+1)^n}{n}\eta\overline{G'} = 0; \qquad \overline{G}(0) = 0, \quad \overline{G'}(0) = 1 \tag{4.11}$$

to find (an approximation) to $\overline{G'}(\infty)$. Next, we use the information from the solution to the initial value problem (4.11), that is, $\overline{G'}(\infty)$, to solve the required system for F, from (4.2) with $F = 1 - G$, that is

$$(-F')^{n-1}F'' + \frac{(n+1)^n}{n}\eta F' = 0; \qquad F(0) = 1, \quad F'(0) = -\left[\frac{1}{\overline{G}(\infty)}\right]^{\frac{2}{n+1}} \tag{4.12}$$

5. Results and conclusions

The initial value problems (4.11) and (4.12) were solved using a standard fourth-order Runge-Kutta routine and the results of this solution process are presented in Table 1 and Figure 1, for the range of values of n considered by Bird [2].

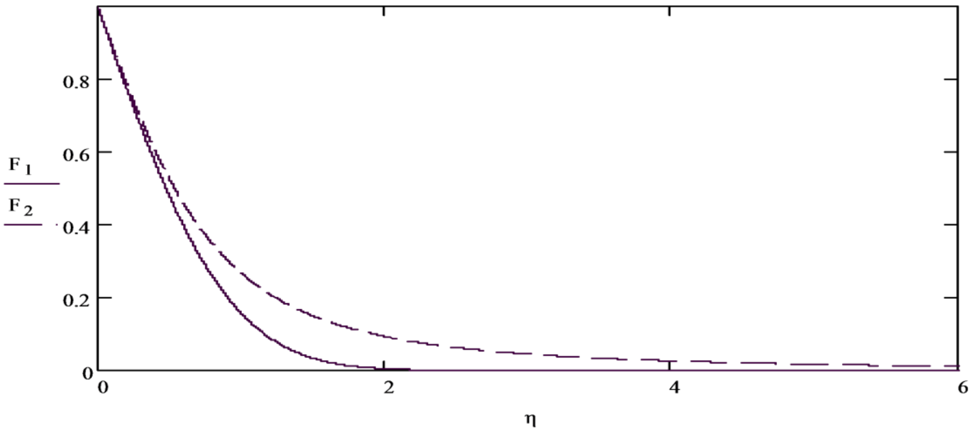
Table 1. Numerical values for calculating velocity profiles.

n	$\overline{G}(\infty)$	$F'(0) = -\left[\frac{1}{\overline{G}(\infty)}\right]^{\frac{2}{n+1}}$	$\eta_{0.01}$ (Bird [2])
$\frac{1}{3}$	0.95317347	– 1.07458823	6.53 (6.57)
$\frac{5}{12}$	0.98843042	– 1.01656445	5.28
$\frac{1}{2}$	1.00364937	– 0.99515481	4.31 (4.30)
$\frac{7}{12}$	1.00416224	– 0.99476708	3.59
$\frac{2}{3}$	0.99364570	– 1.00767882	3.04 (3.05)
$\frac{3}{4}$	0.97477920	– 1.02962378	2.62
$\frac{5}{6}$	0.94959087	– 1.05804857	2.29 (2.29)
$\frac{11}{12}$	0.91965714	– 1.09132859	2.03
1	0.88622693	– 1.12837916	1.82 (1.83)

The results of Table 1 provide a comparison with the work of Bird [2], the column headed $\eta_{0.01}$ being the values of the reduced variable η for which the fluid

velocity has fallen-off to 1% of the velocity of the moving wall (the xz-plane). The numbers in brackets in Table 1 refer directly to Bird's results. The comparison with Bird's results in Table 1 help to validate the solution method.
Figure 1 presents two representative solution curves, for the 'extreme values' $n = 1$ (F_1 - solid line) and $n = 1/3$ (F_2 - dashed line).

Figure 1. Typical reduced velocity curves for $n = 1$ (F_1 - solid line) and $n = 1/3$ (F_2 - dashed line).



The approach here is restricted to a basic similarity analysis through the medium of stretching groups. Phan-Thien [8] has developed a Lie-group solution process for problems slightly more general than (1.1)/(1.2), from which similarity solutions may be extracted. The method developed here is, hopefully, simpler.

In conclusion, an example of a similarity approach to the group analysis of partial differential equations has been presented, the partial differential equation, describing unsteady flow in a pseudo-plastic fluid, being reduced to an ordinary differential equation. The complete transformation, including boundary conditions, has led to an ordinary differential boundary value problem which in turn was transformed to an initial value problem and solved in a standard manner.

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