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Series Solution of Second-Order Linear Homogeneous Ordinary Differential Equations via Complex Integration

W. Robin

Engineering Mathematics Group
Edinburgh Napier University
10 Colinton Road, EH10 5DT, UK

Abstract

A method is presented, with standard examples, based on an elementary complex integral expression, for developing, in particular, series solutions for second-order linear homogeneous ordinary differential equations. Straightforward to apply, the method reduces the task of finding a series solution to the solution, instead, of a system of simple equations in a single variable. The method eliminates the need to manipulate power series and balance powers, which is a characteristic of the usual approach. The method originated with Herrera [3], but was applied to the solution of certain classes of *nonlinear* ordinary differential equations by him.

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1. Introduction

One of the commonest means of seeking a solution of a linear homogeneous ordinary differential equation (ODE) is to attempt to find an infinite series solution.

This is a well understood process, but can still be a messy business when attempting to develop the recurrence relation after substituting the assumed form of the infinite series into the ODE [7, 8, 9]. In this paper we will introduce a method for finding power series solutions to ODE by direct integration in the complex plane. (All contour integrals that occur below are assumed evaluated in the counter-clockwise (positive) direction.) The basis of this integration method, due to Herrera [3], is the elementary result [4] that, if n is an integer

$$\oint_C (z-z_0)^n \, dz = \begin{cases} 2\pi i, & n=-1 \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

where z is a complex variable and z_0 a fixed point within the closed contour C. The relation (1.1) is used to derive an integral expression [3]

$$a_m = \frac{(m-k)!}{m!\, 2\pi i} \oint_C \frac{f^{(k)}(z)}{(z-z_0)^{m-k+1}} \, dz \tag{1.2}$$

for the coefficients of an assumed power series expansion for a solution, $f(z)$, to our ODE, that is (see, for example, [7, 8, 9])

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^m \tag{1.3}$$

The basic idea is best illustrated by a simple example, with further details presented as the paper is developed.

We will solve the defining equation of the (negative) exponential function, that is, we seek a series solution, about $z_0 = 0$, of the *first-order* equation

$$f^{(1)}(z) + f^{(0)}(z) = 0 \tag{1.4}$$

with the superscript numbers giving the *order* of the derivative. Assuming the series solution (1.3), we divide through equation (1.4) by z^{n+1} (with n becoming the new dummy variable in the series solution) and integrate round the closed contour C, avoiding any singularities of $f(z)$, to get

$$\oint_C \frac{f^{(1)}(z)}{z^{n+1}} \, dz + \oint_C \frac{f^{(0)}(z)}{z^{n+1}} \, dz = 0 \tag{1.5}$$

and compare the powers in the denominators of the integrands of (1.5) with that of (1.2) to get two equations for the dummy variable m, one for each value of k (one and zero, respectively), that is

$$m-k+1=m-1+1=n+1 \quad or \quad m=n+1 \tag{1.6a}$$

and

$$m-k+1=m-0+1=n+1 \quad or \quad m=n \tag{1.6b}$$

Having identified the appropriate values of k and m, we use (1.2) again to rewrite (1.5) as, after cancelling

$$(n+1)a_{n+1}+a_n=0, \quad n=0,1,2,3,\ldots \tag{1.7}$$

With $a_0 \neq 0$ an arbitrary constant, we recognize the recurrence relation for $a_0 e^{-z}$, but with *new dummy variable* n, so that after the third part of the solution process, solving (1.7), we would write

$$f(z)=a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n \tag{1.8}$$

(Note that varying the index in z^{n+1} 'shifts' the recurrence subscript accordingly.)

The above procedure can be followed in each situation where we are sure a power series solution exists for (first- and) second-order linear ODE (at least) and it becomes apparent, when a few examples have been worked through, that the solution of such linear ODE in series is reduced to the solution of one equation in one unknown *repeatedly*. In the above simple example no *explicit* use was made of (1.1), but the *derivation* of (1.2), presented in section 2 below, relies on (1.1).

The method itself was introduced by Herrera [3] to produce series solutions for *nonlinear* ODE, but its application to linear ODE, as developed below, proves remarkably efficient and the methodology should be more widely known, which is one of the main purposes of the current work. The paper is organized as follows. In section 2, for completeness, we provide a derivation of (1.2), as Herrera [3] simply stated formula (1.2) outright. Then, in section 3, we provide a selection of examples of the application of the Herrera method to the production of power series solutions to some of the basic equations of mathematical physics [1, 2, 4, 6, 7, 8, 9]. In the final section, section 4, we discuss the problem of higher-order recurrence relations in the series solution of second-order linear homogeneous ODE. In particular, the series solution of the second-order linear homogeneous ODE with constant coefficients requires some care, due to the possible occurrence of a three-term recurrence relation. Also, at the end of section 4, we consider (*very* briefly) an application of the Herrera method to a third-order ODE.

2. Derivation of the Basic Formula

Our approach is basically a formal one. Suppose we start with the power series expansion of $f(z)$ about the non-singular point z_0, that is [7, 8, 9]

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^m \tag{2.1}$$

valid within an assumed non-zero radius of convergence. If we differentiate (2.1) k times, we find that

$$f^{(k)}(z) = \sum_{m=k}^{\infty} \frac{m!}{(m-k)!} a_m (z - z_0)^{m-k} \tag{2.2}$$

or

$$f^{(k)}(z) = \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k} + \sum_{m=k\neq n}^{\infty} \frac{m!}{(m-k)!} a_m (z - z_0)^{m-k} \tag{2.3}$$

Dividing through (2.3) by $(z - z_0)^{n-k+1}$ and integrating round a closed contour C containing z_0 while avoiding any singularities of $f(z)$, we get, from (1.1)

$$\oint_C \frac{f^{(k)}(z)}{(z - z_0)^{n-k+1}} dz = \frac{n!}{(n-k)!} a_n \oint_C \frac{dz}{(z - z_0)} = \frac{n!}{(n-k)!} a_n 2\pi i \tag{2.4}$$

so that

$$a_n = \frac{(n-k)!}{n!\, 2\pi i} \oint_C \frac{f^{(k)}(z)}{(z - z_0)^{n-k+1}} dz \tag{2.5}$$

On changing dummy variables $(n \to m)$ in (2.5), we have equation (1.2). Finally, looking back, we find $(n =)m = k, k+1, k+2, k+3, \ldots,$ while $k = 0,1,2,3,\ldots.$ This completes the derivation of (1.2).

We now apply the Herrera method to derive power series solutions to some standard second-order linear homogeneous ODE. In the rest of the paper we will always take $z_0 = 0,$ as the equations we consider are known to have such power series [1, 2, 4, 6, 7, 8, 9] and we can *transform* back to the origin by a change of independent variable anyway. Further, we will consider the problem to be solved on obtaining the recurrence relation for the series solutions; the equations involved being well-known, the process of solving the recurrence relations is widely available in many textbooks and monographs (including those referenced below). This philosophy will be taken to apply to convergence issues also.

3. The Solution of Some Standard Second-Order Equations

The first of our examples, involves solving the Airy equation [1]

$$f^{(2)}(z) - z f^{(0)}(z) = 0 \tag{3.1}$$

Assuming (1.3), we divide through equation (3.1) by z^{n+1} and integrate round the closed contour C, to get

$$\oint_C \frac{f^{(2)}(z)}{z^{n+1}} dz - \oint_C \frac{f^{(0)}(z)}{z^n} dz = 0 \tag{3.2}$$

and compare the powers of the denominators of the integrands of (3.2) with that of (1.2) to get two equations for the dummy variable m, one for each value of k (two and zero, respectively), that is

$$m - k + 1 = m - 2 + 1 = n + 1 \ \ or \ \ m = n + 2 \tag{3.3a}$$

and

$$m - k + 1 = m - 0 + 1 = n \ \ or \ \ m = n - 1 \tag{3.3b}$$

Having identified the appropriate values of k and m, we use (1.2), again, to rewrite equation (3.2), after cancelling and re-arranging, as the recurrence relation

$$(n+1)(n+2)a_{n+2} - a_{n-1} = 0, \quad n = 0,1,2,3,\ldots \tag{3.4}$$

Setting $n = 0$ in (3.4), we must have $a_{-1} = 0$ (there are no negative indices) and therefore $a_2 = 0$ *also*. With $a_0 \neq 0$ and $a_1 \neq 0$ arbitrary constants, we have [1] the recurrence relation for the Airy function. The third and final step, to solve the recurrence relation (3.4) and obtain the power series solution explicitly, is well-known [1] and we move on to our next example.

As a second example, we consider the Bessel equation of order zero, that is

$$zf^{(2)}(z) + f^{(1)}(z) + zf^{(0)}(z) = 0 \tag{3.5}$$

Assuming (1.3), we divide through equation (3.5) by z^{n+1} and integrate round the closed contour C, to get

$$\oint_C \frac{f^{(2)}(z)}{z^n} dz + \oint_C \frac{f^{(1)}(z)}{z^{n+1}} dz + \oint_C \frac{f^{(0)}(z)}{z^n} dz \tag{3.6}$$

and compare the powers of the denominators of the integrands of (3.6) with that of (1.2) to get three equations for the dummy variable m, one for each value of k (two, one and zero, respectively), that is

$$m - k + 1 = m - 2 + 1 = n \ \ or \ \ m = n + 1 \tag{3.7a}$$

and

$$m-k+1=m-1+1=n+1 \ \ or \ \ m=n+1 \qquad (3.7b)$$

and

$$m-k+1=m-0+1=n \ \ or \ \ m=n-1 \qquad (3.7c)$$

Having identified the appropriate values of k and m, we use (1.2), again, to rewrite equation (3.6), after cancelling and re-arranging, as

$$(n+1)^2 a_{n+1}+a_{n-1}=0, \quad n=0,1,2,3,\ldots \qquad (3.8)$$

Now, setting $n=0$ in (3.8), we must have $a_{-1}=0$ (there are no negative indices) and therefore $a_1=0$ *also*. So there are no odd powers in the power series. With $a_0 \neq 0$ an arbitrary constant, we recognize in (3.8), for the recurrence relation for the Bessel function of order zero. The third and final step is to solve the recurrence relation (3.8) and obtain the power series solution explicitly. As this is well-known [8], we take this step as read and consider our next example.

So, for our third example, we solve the confluent hypergeometric equation (also called the Kummer equation) [7]

$$zf^{(2)}(z)+(\gamma-z)f^{(1)}(z)-\alpha f^{(0)}(z)=0 \qquad (3.9)$$

where α and γ are constants. As before, we divide through equation (3.9) by z^{n+1} and integrate round the closed contour $C,$ to get

$$\oint_C \frac{f^{(2)}(z)dz}{z^n}+\gamma\oint_C \frac{f^{(1)}(z)}{z^{n+1}}dz-\oint_C \frac{f^{(1)}(z)}{z^n}dz-\alpha\oint_C \frac{f^{(0)}(z)}{z^{n+1}}dz=0 \qquad (3.10)$$

and compare the powers of the denominators of the integrands of (3.10) with that of (1.2) to get four equations for the dummy variable m, one for each value of k (two, one, one and zero, respectively), that is

$$m-k+1=m-2+1=n \ \ or \ \ m=n+1 \qquad (3.11a)$$

and

$$m-k+1=m-1+1=n+1 \ \ or \ \ m=n+1 \qquad (3.11b)$$

and

$$m-k+1=m-1+1=n \ \ or \ \ m=n \qquad (3.11c)$$

and

$$m-k+1=m-0+1=n+1 \ \ or \ \ m=n \qquad (3.11d)$$

Having identified the appropriate values of k and m, we use (1.2), again, to rewrite equation (3.10), after cancelling and re-arranging, as

$$(n+1)(n+\gamma)a_{n+1}-(n+\alpha)a_n=0, \quad n=0,1,2,3,\ldots \qquad (3.12)$$

With $a_0 \neq 0$ an arbitrary constant, we recognize the recurrence relation for the power series solution of the confluent hypergeometric equation [7] and, as before, we terminate the example here.

Another important equation, which is our last example in this section, is the hypergeometric-type equation [6]

$$(az^2 + bz + c)f^{(2)}(z) + (dz + e)f^{(1)}(z) + \lambda f^{(0)}(z) = 0 \qquad (3.13)$$

with a, b, c, d, e and λ constants. Again, applying the method as before to (3.13), we get, *sans* details, a *three-term* recurrence scheme for the a_n, that is

$$c(n+1)(n+2)a_{n+2} + (bn+e)(n+1)a_{n+1} + (n[(n-1)a+d]+\lambda)a_n = 0 \qquad (3.14)$$

In this case, the three-term recurrence relations (3.14) can be reduced to two-term recurrence relations in two separate ways. First, the requirement that

$$c = 0 \qquad (3.15)$$

in (3.14), leaves us with

$$(bn+e)(n+1)a_{n+1} + (n[(n-1)a+d]+\lambda)a_n = 0 \qquad (3.16)$$

For example, (3.16) is satisfied by the Laguerre equation, the generalized Laguerre equation and the Bessel polynomial equation [6].

Secondly, if in (3.14)

$$b = e = 0 \qquad (3.17)$$

we are left with

$$c(n+1)(n+2)a_{n+2} + (n[(n-1)a+d]+\lambda)a_n = 0 \qquad (3.18)$$

For example, the Hermite, Legendre and Chebyshev equations [6] satisfy the requirement (3.18).

Finally, for both the Romanovsky and Jacobi polynomial equations [6] we are left, still, with the *full* three-term recurrence relation(s) (3.14), as this time either b or e (or both) is nonzero. However, the recurrence relations will, indeed, terminate provided, for some integer $n \geq 0$

$$\lambda = \lambda_n = -n[(n-1)a+d] \qquad (3.19)$$

and polynomial solutions will emerge. The relation (3.19) is the well- known [6] restriction on λ for the extraction of orthogonal polynomial solutions from (3.13).

Note that, all the above ODE may be solved *individually*. This concludes our brief survey of power series solutions to important ODE from mathematical physics.

4. Conclusions and Discussion

It must be said that the facility with which the Herrera method produces power series solutions to linear homogeneous ODE is nothing short of remarkable. All problems involving balancing powers in adjacent series in the standard approach (see, for example, [8]) simply ‘melt away’, to be replaced with the solution of simultaneous simple equations (along with the usual algebra, of course). In principle we have, in fact, the general solutions to our equations, as the second solution is always obtainable (if necessary) from the standard construction [8] of a second solution from a known solution. In the rest of this section, we discuss, through examples, the problem of dealing with the occurrence of three-or-more-term recurrence relations. The existence of such recurrence relations is well-known and generally unavoidable, as shown by the general theory [8]. (For a brief discussion of three-term recurrence relations see reference [2].)

As our first example, involving the occurrence of three-or-more-term recurrence relations, we consider the classic second-order linear homogeneous ODE with constant coefficients (κ and ω_0), that is [5, 8]

$$f^{(2)}(z) + 2\kappa f^{(1)}(z) + \omega_0^2 f^{(0)}(z) = 0 \tag{4.1}$$

Applying the method as before (again *sans* details) to (4.1), we do indeed get a *three-term* recurrence relation (and not an easier to solve two-term recurrence relation, which yields a closed-form solution). On making the substitution

$$f(z) = e^{-\kappa z} g(z) \tag{4.2}$$

however, we find that (4.1) is reduced to

$$g^{(2)}(z) + \omega^2 g^{(0)}(z) = 0 \tag{4.3}$$

where

$$\omega^2 = \omega_0^2 - \kappa^2 \tag{4.4}$$

Applying the method, equation (4.3) now yields the *two-term* recurrence scheme

$$(n+1)(n+2)a_{n+2} + \omega^2 a_n = 0 \qquad (4.5)$$

and, on inspection of (4.5), we may take *both* $a_0 \neq 0$ and $a_1 \neq 0$ as arbitrary constants. As usual [5, 8], given (4.5), there are three cases to consider.

(1) If $\omega^2 = \omega_0^2 - \kappa^2 > 0$, on replacing a_1 with ωa_1, we get the recurrence schemes for $a_0 \cos(\omega z)$ and $a_1 \sin(\omega z)$, so that $g(z) = a_0 \cos(\omega z) + a_1 \sin(\omega z)$.

(2) If $\omega^2 = \omega_0^2 - k^2 < 0$, on replacing ω^2 with $-\omega^2$ and a_1 with ωa_1, we get the recurrence schemes for $a_0 \cosh(\omega z)$ and $a_1 \sinh(\omega z)$, so that, in this case we have $g(z) = a_0 \cosh(\omega z) + a_1 \sinh(\omega z)$.

(3) If $\omega^2 = \omega_0^2 - \kappa^2 = 0$, we have, on integrating (4.3) directly, $g(z) = a_0 + a_1 z$.

For each of the above three cases, the solution, $f(z)$, of (4.1) is given, now, by equation (4.2).

In this last example, we see one of the standard means of avoiding three-term recurrence relations, that is, the ODE is transformed in the hope that the transformed ODE has a power series solution with a *two-term* recurrence relation. For other examples of this transformation technique, see, for example, reference [8]. Generally speaking, three-or-more-term recurrence relations are difficult to avoid otherwise. This fact is made even more clear in our next example. With $c_0, \ldots, d_2$ constants, the equation

$$(c_0 z + d_0) f^{(2)}(z) + (c_1 z + d_1) f^{(1)}(z) + (c_2 z + d_2) f^{(0)}(z) = 0 \qquad (4.6)$$

is a basic generalization of the Kummer equation, (3.9). Applying the method, to equation (4.6), we get the general *four*-term recurrence relation ($n = 0,1,2,3,\ldots$)

$$(n+1)(n+2)d_0 a_{n+2} + (n+1)(nc_0 + d_1)a_{n+1} + (nc_1 + d_2)a_n + c_2 a_{n-1} = 0 \qquad (4.7)$$

with, as $a_{-1} = 0$, an 'initializing' *three*-term recurrence relation

$$2d_0 a_2 + d_1 a_1 + d_2 a_0 = 0 \qquad (4.8)$$

The occurrence of three-or-more-term recurrence relations is simply a fact of life.

This fact brings us to our concluding remarks. It is apparent that the method can be applied to higher-order ODE than (first and) second-order ODE, the main problem being the development of the series coefficients from the higher-order recurrence relations that are bound to arise with higher-order ODE. Even for the second-order ODE, as we have seen, three-or-more-term recurrence relations occur and it is not clear that any closed-form solution for the series coefficients, even for three-term recurrence relations, is available. So, if we tackle third and higher- order linear homogeneous ODE using the present method, it will certainly prove necessary to enlist the aid of some form of 'computer algebra' package in the enumeration of the power series coefficients.

However, to finish on a positive note, we present an example of a higher-order ODE which does not have this problem, that is

$$f^{(3)}(z)+4zf^{(1)}(z)+2f^{(0)}(z)=0 \tag{4.9}$$

which has a *two-term* recurrence relation, as the reader may verify for himself.

References

[1] Boyce W. E. and DiPrima R.C.: *Elementary Differential Equations and Boundary Value Problems*, Fifth Edition. John Wiley and Sons, Singapore (1992).

[2] Figueiredo B. D. B.: *Generalized spheroidal wave equation and limiting cases*. J. Math. Phys. **48** (2007) 013503.

[3] Herrera J. C.: *Power series solutions in nonlinear mechanics*. Brookhaven National Laboratory Report 37494. Undated.

[4] Hildebrand F. B.: *Advanced Calculus for Applications*. Prentice-Hall, New Jersey (1962).

[5] Jeffrey A.: *Mathematics for Engineers and Scientists*, Fifth Edition. Chapman and Hall, London (1996).

[6] Nikiforov A. F. and Uvarov V. B.: *Special Functions of Mathematical Physics*. Birkhauser, Basel (1988).

[7] Rainville E.D. and Bedient P.E.: *Elementary Differential Equations*, Seventh Edition. Macmillan Publishing Company, New York (1989).

[8] Simmons G. F.: *Differential Equations with Applications and Historical Notes*, Second Edition. McGraw-Hill, London (1991).

[9] Zill D.G.: *A First Course in Differential Equations*, Fifth Edition PWS-KENT, Boston (1993).

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