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On the Procedure for the Series Solution of Certain General-Order Homogeneous Linear Differential Equations via the Complex Integration Method

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Abstract

The theory of series solutions for two important classes of the general higher-order linear homogeneous ordinary differential equation is developed *ab initio*, using an elementary complex integral expression derived and applied in previous papers [10, 11], based on the original work of Herrera [5]. As well as producing general expressions for the recurrence relations for higher-order equations with analytic coefficients or the general-order Fuchs' equation, the complex integral method is straight-forward to apply as an algorithm on its own. 'Benchmark' examples from the general mathematic literature, are presented and a brief discussion of 'logarithmic' solutions is included.

Mathematics Subject Classification: 30B10, 30E20, 33C10, 34A25, 34A30

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1. Introduction

In this paper, we continue the project [10, 11] of developing power series and Frobenius series solutions of ordinary differential equations (ODE) using a particular complex integration procedure. As before we find that the technique reduces the solution of the original ODE, through the complex integral transformation [5, 10, 11], to a system of simple equations for the indices of the series coefficients that define the series recurrence relation. In fact, as well as presenting further examples of the technique, we apply the complex integral methodology to series solutions of both the general or N$^{\text{th}}$-order linear homogeneous ODE ([6], chapter XV)

$$f^{(N)}(z)+\sum_{j=0}^{N-1}p_j(z)f^{(j)}(z)=0 \tag{1.1}$$

and the general or N$^{\text{th}}$-order Fuchs' equation ([6], chapter XVI)

$$(z-z_0)^N f^{(N)}(z)+\sum_{j=1}^{N}(z-z_0)^{N-j}\,p_j(z)f^{(N-j)}(z)=0 \tag{1.2}$$

(notice the different summation order). As for the notation, in equations (1.1) and (1.2) the superscript numbers in brackets denote differentiation with respect to z, the zeroth derivative being the function $f(z)$ itself, and the $p_j(z), j=0,1,2,3,\dots N$, are *analytic* functions (coefficients) of the independent variable z, that is [6]

$$p_j(z)=\sum_{i=0}^{\infty}p_{j,i}(z-z_0)^i, \quad j=0,1,2,3,\dots N \tag{1.3}$$

for given constants $\{p_{j,i}\}_{i=0}^{\infty}$, with z_0 being an ordinary point of (1.1) or a regular singular point of (1.2).

In the solution process, there are, of course, two cases. First, for ODE (1.1), we seek a power series solution of (1.1) about the ordinary point z_0, that is [12]

$$f(z)=\sum_{m=0}^{\infty}a_m(z-z_0)^m \tag{1.4}$$

and, in this case, the series coefficients, $\{a_m\}_{m=0}^{\infty}$, in (1.4) are determined via [5, 10]

$$a_m = \frac{(m-k)!}{m!\,2\pi\hat{i}} \oint_C \frac{f^{(k)}(z)}{(z-z_0)^{m-k+1}}\,dz \tag{1.5}$$

with $\hat{i}^2 = -1$ and k a non-negative integer. In the other case, when z_0 is a regular singular point of (1.2), we seek a Frobenius series solution of (1.2) about z_0, that is [6, 9, 12], with r the usual Frobenius index

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^{m+r} \tag{1.6}$$

when the series coefficients, $\{a_m\}_{m=0}^{\infty}$, are determined via the contour integral [11]

$$a_m = \frac{1}{[m+r]_k\,2\pi\hat{i}} \oint_C \frac{f^{(k)}(z)}{(z-z_0)^{m+r-k+1}}\,dz \tag{1.7}$$

where, following the notation of Ince [6]

$$[m+r]_k = (m+r)(m+r-1)(m+r-2)\cdots(m+r-k+1) = \frac{\Gamma(m+r+1)}{\Gamma(m+r-k+1)} \tag{1.8}$$

for non-negative integers k. Note that $[m+r]_0 = 1$.

It is to be noted that in both (1.5) and (1.7) the contour C is a closed contour, taken in the positive or anti-clockwise direction, encircling the point z_0, but avoiding any (other) singularities. This is a formal approach, as the Frobenius procedure usually is, and we do not consider technical problems involving multi-valued functions, Riemann sheets and cuts and so forth. Further, regardless of the type of ODE considered, (1.1) or (1.2), we will consider the problem (ODE) solved once the recurrence relation for the series coefficients, $\{a_m\}_{m=0}^{\infty}$, has been obtained, though, in some cases, we may write-out the first few terms of the series. Finally, it is to be emphasised that the general methodology presented below, in sections 2, 3 and 4, whereby (1.1) and (1.2) are solved, yields not only the general series solutions to (1.1) or (1.2), but also 'expresses' itself as an algorithm for the solution of arbitrary ODE of the form of (1.1) or (1.2).

The paper is organized as follows. In section 2 we solve (1.1) for the case of z_0 being an ordinary point and present the general series solution; the algorithm is then exemplified through two third-order homogeneous ODE [3, 8]. In section 3 we solve (1.2) with z_0 being, now, a regular singular point and present the Frobenius series solution; the solution is then exemplified via a problem requiring the solution of the fourth-order Bessel-type ODE [2] and a sixth-order ODE from fluid mechanics [7]. Next, in section 4, we examine the case when certain solutions of (1.2), with z_0 a regular singular point, form a combination of both a Frobenius series and a 'logarithmic series', instead of just a Frobenius series. Although not related to the complex integral method we consider the standard procedure [6, 9] for procuring further solutions and apply it, in part, to the full solution of the fourth-order Bessel-type equation [2] again; the problem [7] from fluid mechanics is mentioned in passing. The motivation for considering the full solution of the Bessel-type equation becomes clearer when the mechanics of the full solution problem is examined in detail below, in sections 3 and 4. The paper finishes-off, in section 5, with some general remarks and conclusions.

2. Ordinary Points

Consider again the N$^{\text{th}}$-order linear homogeneous ODE [6]

$$f^{(N)}(z)+\sum_{j=0}^{N-1}p_j(z)f^{(j)}(z)=0 \tag{2.1}$$

with the superscript numbers in brackets denoting differentiation with respect to z, the zeroth derivative being the function $f(z)$ itself, and where the coefficients $p_j(z),\ j=0,2,3,\ldots,N-1$, are analytic functions for all positive integers N, that is

$$p_j(z)=\sum_{i=0}^{\infty}p_{j,i}(z-z_0)^i, \qquad j=0,2,3,\ldots,N-1 \tag{2.2}$$

for given constants $\{p_{j,i}\}_{i=0}^{\infty}$ and with the series expansion taken about some fixed point z_0. Substituting (2.2) into (2.1), we get the more explicit form of (2.1) as

$$f^{(N)}(z)+\sum_{j=0}^{N-1}\sum_{\mathrm{i}=0}^{\infty}p_{j,i}(z-z_0)^i f^{(j)}(z)=0 \tag{2.3}$$

The problem is to find a series solution, of the form (1.4), of (2.3). Following the method of [10], we first divide through (2.3) by $(z-z_0)^{n+1}$ and integrate through the resulting expression, the integration being round a closed contour C taken in the anti-clockwise direction while avoiding any singularities of $f(z)$, to get

$$\oint_C\frac{f^{(N)}(z)}{(z-z_0)^{n+1}}+\sum_{j=0}^{N-1}\sum_{\mathrm{i}=0}^{\infty}p_{j,i}\oint_C\frac{f^{(j)}(z)}{(z-z_0)^{n-i+1}}=0 \tag{2.4}$$

Next, with (1.4) in mind, we compare the denominators in (2.4) with that of (1.5), term by term, to get two equations for the dummy index m in terms of the dummy index n, one for each of the values of k (the order of the derivative of $f(z)$) in each integral (N and j, respectively). We find that (2.4) yields the two equations

$$m-k+1=m-N+1=n+1 \Leftrightarrow m=n+N \tag{2.5a}$$
$$m-k+1=m-j+1=n-i+1 \Leftrightarrow m=n+j-i \tag{2.5b}$$

Utilizing (1.5) again, with (1.4) and the results of (2.5) in hand, we find that our equation (2.4) transforms into

$$\frac{(n+N)!}{n!}a_{n+N}+\sum_{j=0}^{N-1}\sum_{i=0}^{n}\frac{(n+j-i)!}{(n-i)!}p_{j,i}\,a_{n+j-i}=0, \quad n=0,1,2,3,\ldots \tag{2.6}$$

after cancellation and recalling that $a_m=0, m<0.$ Further, on changing the dummy variable in (2.6), $i \to n-k,$ we get the simpler format of

$$\frac{(n+N)!}{n!}a_{n+N}+\sum_{j=0}^{N-1}\sum_{k=0}^{n}\frac{(k+j)!}{k!}p_{j,n-k}\,a_{k+j}=0, \quad n=0,1,2,3,\ldots \tag{2.7}$$

and our problem is solved 'in principle' (recall the remarks in the introduction).

For our first example of solving general-order ODE with this approach, we consider the series solution about the origin of (Dawkins [3])

$$f^{(3)}(z)+z^2f^{(1)}(z)+z\,f^{(0)}(z)=0 \tag{2.8}$$

which is a third-order linear homogeneous ODE. Substituting in (2.7) the fact that $N=3,$ and the only nonzero coefficients, $p_{j,i},$ are $p_{1,2}=1$ and $p_{0,1}=1,$ we find that the recurrence relation for (2.8) is, after cancellation and some algebra

$$(n+1)(n+2)(n+3)a_{n+3}+na_{n-1}=0, \quad \mathrm{n}=0,1,2,3,\ldots \tag{2.9}$$

with a_0, a_1 and a_2 arbitrary. The recurrence relation (2.9) agrees with the results of Dawkins' traditional series analysis [3], to which the reader is referred for the rest of the calculation yielding the three independent solutions of (2.8).

As a second example, we consider an equation from Nachbagauer [8]

$$f^{(3)}(z)+z^2f^{(2)}(z)+5z\,f^{(1)}(z)+3f^{(0)}(z)=0 \tag{2.10}$$

where $z_0=0$ and where the only nonzero coefficients, $p_{j,i},$ are $p_{2,2}=1,$ $p_{1,1}=5$ and $p_{01}=3,$ which facts, along with $N=3,$ we substitute in (2.7) to find that the recurrence relation for (2.10) is, after cancellation and some algebra

$$(n+2)a_{n+3} + a_n = 0, \quad n = 0, 1, 2, 3, \dots \tag{2.11}$$

with a_0, a_1 and a_2 arbitrary. The recurrence relation (2.11) agrees with the results the traditional series analysis of Nachbagauer [8], to which, again, the reader is referred for the rest of the calculation yielding the three independent solutions of (2.10).

3. Regular Singular Points

Consider again the N$^{\text{th}}$-order linear homogeneous Fuchs' ODE [6]

$$(z-z_0)^N f^{(N)}(z) + \sum_{j=1}^{N} (z-z_0)^{N-j} p_j(z) f^{(N-j)}(z) = 0 \tag{3.1}$$

with the superscript numbers in brackets denoting differentiation with respect to z, the zeroth derivative being the function $f(z)$ itself, and where the coefficients $p_j(z)$, $j = 1, 2, 3, \dots, N$, are analytic functions for all positive integers N. In other words, we assume that

$$p_j(z) = \sum_{i=0}^{\infty} p_{j,i}(z-z_0)^i, \quad j = 1, 2, 3, \dots, N \tag{3.2}$$

for given constants $\{p_{j,i}\}_{i=0}^{\infty}$ and with the series expansion taken about some fixed (regular singular) point z_0. Substituting (3.2) into (3.1), we get the more explicit form of (3.1) as

$$(z-z_0)^N f^{(N)}(z) + \sum_{j=1}^{N} \sum_{i=0}^{\infty} p_{j,i}(z-z_0)^{i+N-j} f^{(N-j)}(z) = 0 \tag{3.3}$$

In this instance, instead of a series solution of (3.1), we search for a Frobenius series solution of (3.1). In other words, we assume [6, 9, 12] that for z_0 a regular singular point of $f(z)$, we have

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^{m+r} \tag{3.4}$$

where the index r and the coefficients $\{a_m\}_{m=0}^{\infty}$ are to be determined.

Following reference [11], to find a Frobenius series solution of (3.3) we divided through (3.3) by $(z-z_0)^{n+r+1}$ and integrate through the resulting expression, the integration being round a closed contour C taken in the anti-clockwise direction and containing z_0 while avoiding any other singularities of $f(z)$, to get

$$\oint_C \frac{f^{(N)}(z)}{(z-z_0)^{n-N+r+1}} + \sum_{j=1}^{N}\sum_{i=0}^{\infty} p_{j,i} \oint_C \frac{f^{(N-j)}(z)}{(z-z_0)^{n-N+r+j-i+1}} = 0 \qquad (3.5)$$

Next, with (3.4) in mind, we compare the denominators in (3.5) with that of (1.7), term by term, to get two equations for the dummy index m in terms of the dummy index n, one for each of the values of k (the order of the derivative of $f(z)$) in each integral, (N and $N-j$, respectively). So, we find that (3.5) and (1.7) yield the two equations

$$m+r-k+1 = m+r-2+1 = n+r-1 \Leftrightarrow m=n \qquad (3.6a)$$
$$m+r-k+1 = m+r-1+1 = n+r-i \Leftrightarrow m=n-i \qquad (3.6b)$$

Utilizing (1.7) again, with (3.4) and the results of (3.6) in hand, we see that equation (3.5) transforms, term by term, into

$$[n+r]_N a_n + \sum_{j=1}^{N}\sum_{i=0}^{n}[n+r-i]_{N-j}\, p_{j,i}\, a_{n-i} = 0, \quad n = 0,1,2,3,\ldots \qquad (3.7)$$

after cancellation and recalling that $a_m = 0, m<0$. Introducing the new dummy variable $k = n-i$ into (3.7) and collecting like terms in a_n, we can rewrite (3.7) as

$$([n+r]_N + \sum_{j=1}^{N}[n+r]_{N-j}\, p_{j,0})a_n + \sum_{j=1}^{N}\sum_{k=0}^{n-1}[k+r]_{N-j}\, p_{j,n-k}\, a_k = 0, \quad n = 0,1,2,3,\ldots$$

---------- (3.8)

Now, setting $n=0$ in (3.8) we get the *indicial equation* for (3.1), that is, if $a_n \neq 0$

$$[r]_N + \sum_{j=1}^{N}[r]_{N-j}\, p_{j0} = 0 \qquad (3.9)$$

With (3.8) and (3.9) our Frobenius problem is solved 'in principle'.

As a first example of this solution process, we consider the fourth-order Bessel-type equation (Das et al [2]), which is an example with $z_0 = 0$, that is

$$z^4 f^{(4)}(z) + 2z^3 f^{(3)}(z) - z^2(9+8M^{-1}z^2) f^{(2)}(z)$$
$$+ z(9-8M^{-1}z^2) f^{(1)}(z) - \Lambda z^4 f^{(0)}(z) = 0 \qquad (3.10)$$

with M and Λ constants. From (3.8), we get the recurrence relation, which, with $p_{1,0} = 2$, $p_{2,0} = -9$, $p_{2,2} = -8M^{-1}$, $p_{3,0} = 9$, $p_{3,2} = -8M^{-1}$, $p_{4,4} = -\Lambda$ and all the other $p_{j,i}$'s in (3.10) vanishing, is

$$([n+r]_4+2[n+r]_3-9[n+r]_2+9[n+r]_1)a_n$$
$$-8M^{-1}[n+r-2]_2 a_{n-2}-8M^{-1}[n+r-2]_1 a_{n-2}-\Lambda[n+r-4]_0 a_{n-4} \quad (3.11a)$$

or, with some simplification and with the dependence on r shown explicitly

$$(n+r+2)(n+r)(n+r-2)(n+r-4)a_n(r)$$
$$=8M^{-1}(n+r-2)^2 a_{n-2}(r)+\Lambda a_{n-4}(r) \quad (3.11b)$$

From (3.11b), we see that, (3.10) has indicial equation $(n=0, a_0 \neq 0)$

$$r(r+2)(r-2)(r-4)=0 \quad (3.12)$$

with solutions $r=0, r=-2, r=2$ and $r=4.$ Following Das et al [2], we find a *strictly* Frobenius series, associated with $a_0 \neq 0,$ is determined by the root $r=4.$ Correspondingly, from (3.11b), the recurrence relation for $r=4,$ is

$$(n+6)(n+4)(n+2)na_n=8M^{-1}(n+2)^2 a_{n-2}+\Lambda a_{n-4} \quad (3.13)$$

from which we find that all the odd-suffixed coefficients are identically zero while the first few terms of this Frobenius series solution to (3.10), with $a_0=1,$ are

$$f_4(z)=z^4+\frac{M^{-1}}{3}z^6+\frac{96M^{-2}+\Lambda}{1920}z^8+\cdots \quad (3.14)$$

in agreement with Das et al [2].

In fact, there is a second *strictly* Frobenius series, associated with $a_0 \neq 0,$ and determined by the root $r=0,$ with, from (3.11b), recurrence relation

$$(n+2)n(n-2)(n-4)a_n=8M^{-1}(n-2)^2 a_{n-2}+\Lambda a_{n-4} \quad (3.15)$$

from which the odd-suffixed coefficients are *zero* as before, while the first few terms of this second *strictly* Frobenius series solution to (3.10), with $a_0=1,$ are

$$f_0(z)=1-\frac{\Lambda M}{32}z^2-\frac{\Lambda^2 M}{12288}z^6+\cdots \quad (3.16)$$

When solving (3.15) to get (3.16), we find that a_0 and a_2 are arbitrary while

$$0\times a_4=8M^{-1}(2)^2 a_2+\Lambda a_0 \quad (3.17)$$

On setting $a_4 = 0$ in (3.17), we get a_2 in terms of a_0 and the rest of the series (3.16) follows. Series (3.16) appears to form the basis of the *logarithmic* solution quoted for $f_{-2}(z)$ by Das et al [2]. In fact both series (3.14) and (3.16) can be obtained from at least one of the other two roots, but only if the *strict* Frobenius restriction $a_0 \neq 0$ is lifted. This situation is quite common when indicial roots differ by an integer [9]. Indeed, the solution (3.14) can be obtained from root $r = 0$, if we take $a_2 = a_0 = 0$ and leave a_4 arbitrary, as is easily checked; this proves important when we construct further solutions to (3.10) in section 4 below.

As a second example with $z_0 = 0$, we consider the problem set and solved by Littlefield and Desai [7], that is, with m and Ra constants, the sixth-order ODE

$$\begin{aligned} &z^6 f^{(6)}(z) - 3z^5 f^{(5)}(z) + z^4(9 - 3m^2z^2) f^{(4)}(z) + z^3(-24 + 6m^2z^2) f^{(3)}(z) \\ &\quad + z^2(45 - 9m^2z^2 + (3m^4 - Ra)z^4) f^{(2)}(z) \\ &\quad\quad + z(-45 + 9m^2z^2 - (3m^4 - Ra)z^4) f^{(1)}(z) - m^6z^6 f^{(0)}(z) = 0 \end{aligned} \tag{3.18}$$

which, from (3.8) and inspection of (3.18), gives rise to the recurrence relation

$$\begin{aligned} &([n+r]_6 - 3[n+r]_5 + 9[n+r]_4 - 24[n+r]_3 + 45[n+r]_2 - 45[n+r]_1)a_n \\ &\quad + (-3m^2[n+r-2]_4 + 6m^2[n+r-2]_3 - 9m^2[n+r-2]_2 - 9m^2[n+r-2]_1)a_{n-2} \\ &\quad\quad + ((3m^4 - Ra)[n+r-4]_2 - (3m^4 - Ra)[n+r-4]_1)a_{n-4} \\ &\quad\quad\quad - m^6[n+r-6]_0 a_{n-6} = 0 \end{aligned} \tag{3.19a}$$

or, with the dependence on r shown explicitly

$$\begin{aligned} &(n+r)(n+r-2)^2(n+r-4)^2(n+r-6)a_n(r) \\ &\quad - 3m^2(n+r-2)(n+r-4)^2(n+r-6)a_{n-2}(r) \\ &\quad\quad + (3m^4 - Ra)(n+r-4)(n+r-6)a_{n-4}(r) - m^6 a_{n-6}(r) = 0 \end{aligned} \tag{3.19b}$$

From (3.19b) we see that the indicial equation for (3.19) is $(n = 0, a_0 \neq 0)$

$$r(r-2)^2(r-4)^2(r-6) = 0 \tag{3.20}$$

The recurrence relation (3.19b) corresponds to equation(s) (5.1) of Littlefield and Desai [7], while the indicial equation (3.20) is Littlefield and Desai's (2.22) [7].

As in the previous example, taken from Das et al [2], we find that all four roots of the indicial equation (3.20) are able, with judicious choice of values of arbitrary constants, to 'share' the same solution to (3.18). However, only three of the roots can accommodate the restriction $a_0 \neq 0$, the *double* roots $r = 2$ and $r = 4$ and the single root $r = 6$. We find, from (3.19b), that for all three roots, $r = 2$ and

$r = 4$ and $r = 6$, all coefficients with odd-numbered indices are identically zero. Further, if we set the arbitrary coefficient $a_0 = 1$, then we get, from (3.19b), the first few terms of the three *strictly* Frobenius series as

$$f_2(z) = z^2 + \frac{m^6}{9216}z^8 + \frac{3m^8}{737280}z^{10} + \cdots \tag{3.21}$$

for $r = 2$

$$f_4(z) = z^4 + \frac{Ra - 3m^4}{1152}z^8 + \frac{3m^2(Ra - 3m^4)}{92160}z^{10} + \cdots \tag{3.22}$$

for $r = 4$ and

$$f_6(z) = z^6 + \frac{3m^2}{48}z^8 + \frac{6m^4 + Ra}{1920}z^{10} + \cdots \tag{3.23}$$

for $r = 6$.

We move-on, now, to consider the rest of the solutions to our two problems.

4. Frobenius Series: Other Independent Particular Solutions

Given the basic Frobenius series solution(s), all (!) that is necessary to find further independent particular solutions to an ODE, when required, is for us to re-express the formalism of the previous section in such a manner that we bring it into line with the standard formalism presented in textbooks, in particular references [6] and [9]; but see also reference [12]. In principle, then, should be able to *quote* the required results or techniques and apply them to our examples. First, we define, as usual [6, 9], the linear operator L via

$$L[f(x)] = (z - z_0)^N f^{(N)}(z) + \sum_{j=1}^{N}(z - z_0)^{N-j} p_j(z) f^{(N-j)}(z) \tag{4.1}$$

However, the left-hand-side of the recurrence relation (3.8) is just the coefficient of $(z - z_0)^{n+r}$ in the infinite Frobenius series, so that we may write (4.1) as

$$L[f(z,r)] = ([r]_N + \sum_{j=1}^{N}[r]_{N-j}\, p_{j,0})a_0(z - z_0)^r +$$

$$\sum_{n=1}^{\infty}[([n+r]_N + \sum_{j=1}^{N}[n+r]_{N-j}\, p_{j,0})a_n + \sum_{j=1}^{N}\sum_{k=0}^{n-1}[k+r]_{N-j}\, p_{j,n-k}a_k](z-z)^{n+r} \tag{4.2}$$

on extracting the $n = 0$ term and with the dependence of $f(z,r)$ on r made explicit, that is

$$f(z,r) = \sum_{m=0}^{\infty} a_m(r)(z - z_0)^{m+r} \tag{4.3}$$

for arbitrary r. Whenever r is taken as a solution of the indicial equation (3.9), then $f(z,r)$ is a solution of the basic equation, that is $L[f(z,r)] = 0.$

Now, in the standard discussion of the required other solutions [6, 9], we *choose* in (4.2), as before, for *any* r and n ≥ 1

$$([n+r]_N + \sum_{j=1}^{N} [n+r]_{N-j}\, p_{j,0})a_n(r) + \sum_{j=1}^{N} \sum_{k=0}^{n-1} [k+r]_{N-j}\, p_{j,n-k} a_k(r) = 0 \tag{4.4}$$

leaving (4.2) as

$$L[f(z,r)] = ([r]_N + \sum_{j=1}^{N} [r]_{N-j}\, p_{j,0})a_0(z - z_0)^r \tag{4.5}$$

If we assume that the *indicial polynomial*

$$[r]_N + \sum_{j=1}^{N} [r]_{N-j}\, p_{j,0} \tag{4.6}$$

can be factorized (which, of course, it always can be) then, after we have found all basic Frobenius series solutions to (3.1), we can find any other required solutions to (3.1) by standard constructions; see, for example, the textbooks [6] and [9] for further details (reference [6] produces a general formula for the general solution of (3.1) involving powers of $\ln z$).

As our first example of this general process, we consider again the fourth-order Bessel equation (3.10) with recurrence relation (3.11b) and indicial equation (3.12), with roots $r = 0, r = -2, r = 2$ and $r = 4,$ so that in this case (4.5) becomes

$$L[f(z,r)] = r(r+2)(r-2)(r-4)a_0(z - z_0)^r \tag{4.7}$$

In section 3, we developed two independent series solutions to (3.10); series (3.14), corresponding to the root $r = 4,$ and series (3.16), corresponding to the root $r = 0.$ The other roots, $r = -2$ and $r = 2,$ did not give rise to any new independent series. So, we require two further independent particular solutions of (3.10) from which we may form its general solution. To obtain two further independent solutions to (3.10), we apply the following standard argument [6, 9].

First, we note that there are no repeated roots in (4.7), this makes a difference to the procedure [9]; see the next example below for a treatment of a repeated roots case. Next, we take the $a_n(r)$ as functions of an assumed *variable* r, with $a_0(r)$ a *given* function of r (see below) and the rest of the $a_n(r)$ determined by the recurrence relation (4.4). Finally, we differentiate (4.7) with respect to the now *variable* r, when we expect [6, 9] the limits

$$\left.\frac{\partial f(z,r)}{\partial r}\right|_{r=0,4} = \left(\ln z \sum_{n=0}^{\infty} a_n(r) z^{n+r} + \sum_{n=0}^{\infty} a_n'(r) z^{n+r} \right)_{r=0,4} \tag{4.8}$$

will provide us with two further independent solutions of (3.10). With (3.14) and (3.16) providing us with two independent solutions of (3.10) already, and (4.8) providing, in principle, a further two solutions of (3.10), we can construct the general solution of (3.10) via a linear superposition of these four independent particular solutions.

To pursue this course of action, it is apparent, from (4.8), that to construct further independent solutions of (3.10), we require a recurrence relation for the derivatives $a_n'(r)$. Differentiating the recurrence relation (3.11b)

$$\begin{aligned}(n+r+2)(n+r)(n+r-2)(n+r-4)a_n(r)& \\ = 8M^{-1}(n+r-2)^2 a_{n-2}(r) + \Lambda a_{n-4}(r)&\end{aligned} \tag{4.9}$$

with respect to r, gives, after a bit of algebra

$$\begin{aligned}&(n+r+2)(n+r)(n+r-2)(n+r-4)a_n'(r) \\ &\quad + 2[(n+r)^2(n+r-4) + (n+r-2)^2(n+r+2)]a_n(r) \\ &\quad = 8M^{-1}(n+r-2)^2 a_{n-2}'(r) + 16M^{-1}(n+r-2)a_{n-2}(r) + \Lambda a_{n-4}'(r)\end{aligned} \tag{4.10}$$

for $n \geq 1$, with (4.9) and (4.10) starting, respectively, from the *given* $a_0(r)$ and so $a_0'(r)$. With (4.8), (4.9) and (4.10) in place, we can now continue with our current analysis. We will restrict ourselves to a consideration of the root $r = 0$ only; the other solution, corresponding to the root $r = 4$, follows in a similar manner.

Suppose we consider the 'logarithmic series' part of (4.8) first. We solve (4.9) with $a_0(r) = a_0 r$ and taking the limit as $r \to 0$, when we get a 'logarithmic series' with the first few terms

$$\frac{a_0 \Lambda}{48}\left(z^4+\frac{M^{-1}}{3}z^6+\frac{96M^{-2}+\Lambda}{1920}z^8+\cdots\right)\ln z \tag{4.11}$$

Note that the series multiplying $\ln z$, and associated with $r=0$, is series (3.14) instead of (3.16), and with $a_4=\Lambda/48$; as before, there are no odd powers in this series.

This leaves the 'pure series' part of (4.8) to be determined, from (4.9) and (4.10). Once more, with $a_0(r)=a_0 r$, no odd powers appear in the series and if we set $a_4'(r)=0$, the leading terms of the 'pure series' part of (4.8), when $r=2$, are

$$\frac{a_0 \Lambda}{48}\left(1-\frac{13M^{-1}}{72}z^6+\cdots\right) \tag{4.12}$$

With a fourth particular solution for (3.10) following for the root $r=4$ in a similar manner to that of the root $r=0$, we now have a choice of another two independent solutions to (3.10) and we may form the general solution to (3.10).

Our second example of this general process, we consider in outline only. We return to the solution of equation (3.18) with recurrence relation (3.19b) and indicial equation (3.20), with roots $r=0$, $r=2$ (twice), $r=4$ (twice) and $r=6$, so that in this case (4.5) becomes

$$L[f(z,r)]=r(r-2)^2(r-4)^2(r-6)a_0(z-z_0)^r \tag{4.13}$$

and we can obtain further independent solutions, again by standard arguments [6, 9]. In section 3, we developed three independent solutions, out of six, for equation (3.18), that is, (3.21), (3.22) and (3.23), for $r=2,4$ and 6, respectively. Now, with $f(z,r)$ given by (4.3) and the $a_n(r)$ obtained this time from (3.19b), for *arbitrary* r and a_0 constant, then [6, 9] via (4.13)

$$\left.\frac{\partial f(z,r)}{\partial r}\right|_{r=2} \quad \text{and} \quad \left.\frac{\partial f(z,r)}{\partial r}\right|_{r=4} \tag{4.14}$$

give two further independent solutions to (3.18). To obtain the sixth, and final, independent solution to (3.18), we take $f(z,r)$ as given by (4.3), with the $a_n(r)$ obtained (still) from (3.19b) with *arbitrary* r, replace a_0 with $a_0(r)=a_0(r-6)$, in (4.3) and (4.13), when the limit [6, 9]

$$\left.\frac{\partial f(z,r)}{\partial r}\right|_{r=6} \tag{4.15}$$

provides us with the final independent solution of (3.18), along the lines of the previous example. With the six independent solutions, (3.21), (3.22), (3.23) and (implicitly), (4.14) and (4.15), we can construct the general solution of (3.18).

5. General Remarks and Conclusions

We begin this section with a few general remarks on the current method and follow this up with a brief discussion of the results of sections 2,3 and 4 in the light of the wok of previous authors on these problems. The section is rounded-off with a summary of the overall approach or general methodology and specific results developed throughout the paper.

Our first remark concerns the particular importance, in applications, of the case of second-order homogeneous ODE. The recurrence relations for these cases follow directly from the general formulae (2.7) for series about an ordinary point and (3.8) for series about a regular singular point, as we now show. First, setting $N=2$ in (2.7), we find that

$$(n+2)(n+1)a_{n+2}+\sum_{k=0}^{n}[(k+1)p_{1,n-k}a_{k+1}+p_{o,n-k}a_k]=0 \tag{5.1}$$

for $n=0,1,2,3,\ldots$, which is the usual formula for the recurrence relation for a series about an ordinary point (see Simmons [12]). Next, setting $N=2$ in (3.8), we find that

$$[(n+r+2)(n+r+1)+(n+r)p_{1,0}+p_{2,0}]a_n+\sum_{k=0}^{n-1}[(k+r)p_{1,n-k}+p_{2,n-k}]a_k=0 \tag{5.2}$$

for $n=0,1,2,3,\ldots$, which is the usual formula for the recurrence relation for a series about a regular singular point (again, see Simmons [12]).

Next, we note that the complex integral methodology can be extended to systems of equations, for example, by applying the complex integral method directly to the system itself, as Herrera did in his original paper [5]. For the application of the *Frobenius* method to general linear systems of ODE, the interested reader is referred to Barkatou et al [1].

A third general point, we recall from the introduction, is that the complex integral method, based, of course, on Herrera's original conception [5], can be used as an algorithm without the general formulae of sections 2 and 3. We take the equation in its original (standard [6]) form, (3.1) say, and apply the complex integral methodology, with (3.2) in mind, to (3.1) *directly*; the rest is just basic arithmetic and elementary algebra. The series manipulations are encapsulated in the complex integral formula (1.7), or its special case (when dealing with ordinary points) (1.5). In comparison, the usual Frobenius methodology, which is based on the use of the 'Euler operator' [6]

$$\delta = x\frac{d}{dz} \tag{5.3}$$

involves rewriting the basic equation (3.1), with (3.2) in mind, in terms of δ and then substituting the assumed Frobenius series solution, (3.4), into the resulting format; this is followed by the usual series manipulations until a recurrence equation (and so the indicial equation) is determined.

We now move-on to discuss, briefly, the results of sections 2, 3 and 4 in the light of the wok of previous authors on these problems. With Dawkins' [3] and Nachbagauer's [8] equations, dealt with in section 2, there is not much more to be said other than that the complex integral method simplifies the solution process in the usual manner [5, 10, 11]. The mechanics of producing the actual series solutions from the recurrence relations for Dawkins' [3] and Nachbagauer's [8] equations we have left to a consultation, by the reader, of the original calculations [3, 8].

As to the analysis of sections 3 and 4 on the series solution of (3.10), the equation analysed by Das et al [2], there are two main points of interest. First, we have presented, in section 3, two non-logarithmic *strict* ($a_0 \neq 0$) Frobenius series, while Das et al quote, explicitly, one series only (our equation (3.14)).
Two such series are *known* to exist, in the form of (almost) Bessel functions (see equation (1.4) of reference [2].) Secondly, from the analysis of the $r=0$ case at the end of section 4, it appears that the method for procuring a further solution from a given root of the indicial equation may pick-up an alternative solution (if one exists) in the 'logarithm term', other than the *strict* ($a_0 \neq 0$) Frobenius series. In our case, on applying the standard argument [9] to obtain a further solution with the root $r=0$, in section 4, we discovered (3.14) as the Frobenius series to go along with the 'logarithm term' in (4.8) instead of (3.16), which was the *strict* Frobenius series determined for $r=0$ in section 3. (Possibly this occurred when the algorithm Das et al applied [2] worked-out that, in their equation (3.4), the $r=-2$ logarithmic solution *automatically* goes along with what is probably our (3.16).) However, as the series in the 'logarithmic term' is only required to be a

(Frobenius) series associated with $r = 0,$ we could *choose* the series (3.16) to partner the logarithm in (4.8) and evaluate the terms in the other series, if this were required, accordingly.

In fact, an elementary ODE for the non-logarithmic 'other series' could be developed and solved by substituting (effectively) (4.8) into (3.18) since, as both (3.14) and (3.16) are particular solutions of (3.18), the logarithmic terms vanish identically [11]. This fact holds true for logarithmic 'other solutions' in general, as is easy to verify through the substitution of (4.8) into the generic equation (3.1).

Similar remarks, as those we have made about the work of Das et al [2], could be made about the work of Littlefield and Desai [7] and the solution, in sections 3 and 4, of equation (3.18). Littlefield and Desai [7] derive the recurrence relation, (3.19b) as their (5.1); naturally, this leads to the same indicial equations, our (3.20) and their (2.22) [7]. However, Littlefield and Desai do not write-out explicit solutions for their (5.1) (with their (2.22) in mind) but develop, instead their *own method* [7] for getting further solutions from a given Frobenius solution, when solving, for example, equation (3.18); however, these solutions are left implicit as *their* equations (5.1) to (5.6). In fact, Littlefield and Desai also solve (3.18) by factorizing the left hand side of (3.18) into a product of three Bessel operators, which yield six particular solutions to (3.18) as required for the general solution of (3.18) (see equation (6.9) of reference [7]). In our analysis, we have, corresponding to these Bessel function solutions, equations (3.21), (3.22) and (3.23), along with the potential logarithmic solutions from section 4, from which a general solution to (3.18) may be constructed; although, to match-up with relation (6.9) of reference [7] a division by z is required at some point in the calculation). And, on this point, we finish our general discussion.

In conclusion, we have developed, based, on Herrera's work [5], a complex integral technique [10, 11] for integrating general-order linear homogeneous ODE 'in series' for higher-order equations with analytic coefficients or the general-order Fuchs' equation. The method reduces the solution of general-order linear homogeneous ODE to the solution, instead, of a set of *uncoupled* simple linear equations (which have here mostly been *implied* rather than given explicitly) and whose solution determines the (subscripts of) the usual recurrence relations. General formulae for the recurrence relations for series solutions about both ordinary points and regular singular points were presented and applied to four standard examples. (The solution of recurrence relations is a major topic in itself and we have avoided it here; for a brief discussion of this matter, see reference [9].) A short discussion of the problems inherent in finding solutions other than the 'strict' Frobenius solution ($a_0 \neq 0$) has been given.

As our procedure has been essentially formal, we have not discussed the convergence of the solutions. This is covered, for example in reference [6]. Also we have not touched, here, on such concepts as 'the point at infinity'; again, the reader is referred, for example, to reference [9] for a discussion of this topic.
As a post script, we note that further examples of higher-order ODE, to which the present method may be applied, are given, for example, in the review article of Everitt [4].

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