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Ladder-Operator Factorization and the Bessel Differential Equations

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Abstract

We present an alternative approach to the discussion of Bessel equations and Bessel functions, through an elementary factorization method. The various Bessel equations are represented by a single parameterized form and, after a standard transformation of the dependent variable, a transformed parameterized (Bessel) equation is factorized in terms of raising and lowering ladder-operators. Once constructed, the ladder-operators for the transformed parameterized equation determine the ladder-operators that factorize the various Bessel equations and enable the determination of the various recurrence relations between the Bessel functions. In particular the construction of the Rayleigh formulae for the Bessel functions becomes particularly straightforward. However, 'starting' Bessel functions for the ladder operators and iterative and Rayleigh formulae must still be obtained as series solutions of particular Bessel equations.

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1. Introduction

Arising from many important problems in science and engineering, the Bessel equations form one of the linchpins of the theory of special functions [4]. It is usual for the various Bessel functions, and their properties, arising from the solution of the Bessel equations to be developed either from the well-known series solutions of the Bessel equations [6], or their complex integral representation [4]. Here, however, we offer an alternative approach to the solution of the various Bessel equations, and the properties of the resultant Bessel functions, through an application of a factorization technique.

To describe the basic idea, we represent the generic Bessel function by $Z_n(x)$ and consider the generic Bessel equation to be of the form

$$x^2 \frac{d^2 Z_n(x)}{dx^2} + kx \frac{dZ_n(x)}{dx} + (ax^2 + bn^2 + cn) Z_n(x) = 0 \qquad (1.1)$$

with n an integer and x a real variable. The *constants* a, b, c and k determine the particular type of Bessel equation (and Bessel function) and are defined when each Bessel equation is encountered below, in sections 3 and 4.

The general method is developed as follows. If $Z_n(x)$ is one of the Bessel functions, then we consider a *Lommel transformation* of the dependent variable in the corresponding Bessel equations of the form [2, 6]

$$Z_n(x) = x^n w_n(x) \qquad (1.2)$$

which transforms the generic Bessel equation (1.1) into another generic form

$$x \frac{d^2 w_n(x)(x)}{dx^2} + (k + 2n) \frac{dw_n(x)}{dx} + axw_n(x) = 0 \qquad (1.3)$$

provided

$$b = -1, \quad k = 1 - c \qquad (1.4)$$

Note that we could have taken $b = -1$ from the start. However, the fact that the transformation *requires* $b = -1$, makes our starting point *marginally* more general. Equation (2.3) is now rewritten as

$$-\frac{1}{ax}\left(x \frac{d^2 w_n(x)}{dx^2} + (k + 2n) \frac{dw_n(x)}{dx} \right) = w_n(x) \qquad (1.5)$$

which form suggest that we *assume* that (1.3) can be written in *factorized* forms, in terms of raising and lowering operators [3].. This programme is developed below, in subsequent sections.

Finally, before we consider the details of the method, we note that the Riccati-Bessel equation

$$x^2 \frac{d^2 Z_n(x)}{dx^2} + [x^2 - n(n+1)] Z_n(x) = 0 \qquad (1.6)$$

reduces to the spherical Bessel equation when $Z_n(x)$ is replaced with $\pm x Z_n(x)$ [1] and will not be considered further here.

2. The General Method

Our discussion above suggests looking for raising and lowering operators (with the φ's *independent* of x)

$$\left(p(x)\frac{d}{dx} + \varphi_n^+ \right) w_n(x) = w_{n+1}(x) \qquad (2.1a)$$

and

$$\left(q(x)\frac{d}{dx} - \varphi_n^- \right) w_n(x) = w_{n-1}(x) \qquad (2.1b)$$

when we *assume* that (1.3) can be written in the *factorized* forms [3]

$$\left(p(x)\frac{d}{dx} + \varphi_{n-1}^+ \right)\left(q(x)\frac{d}{dx} - \varphi_n^- \right) w_n(x) = w_n(x) \qquad (2.2a)$$

or

$$\left(q(x)\frac{d}{dx} - \varphi_n^- \right)\left(p(x)\frac{d}{dx} + \varphi_{n-1}^+ \right) w_{n-1}(x) = w_{n-1}(x) \qquad (2.2b)$$

By comparing (1.5) with (2.2), we *hope* to determine the unknown functions $p(x)$, $q(x)$, φ_n^+, and φ_n^-. We find that the following sets of consistency conditions must be satisfied:

$$p(x)q(x) = -\frac{1}{a} \qquad (2.3a)$$

and

$$\varphi_{n-1}^+ \varphi_n^- = 0 \qquad (2.3b)$$

and

$$p(x)[q'(x) - \varphi_n^-] + q(x)\varphi_{n-1}^+ + \frac{k+2n}{ax} = 0 \qquad (2.3c)$$

and

$$q(x)[p'(x) + \varphi_{n-1}^+] - p(x)\varphi_n^- + \frac{k+2(n-1)}{ax} = 0 \qquad (2.3d)$$

It is apparent that we may eliminate between (2.3a) and (2.3c) and between (2.3a) and (2.3d) to get two Riccati equations, one for $p(x)$ and one for $q(x)$, that is

$$p'(x) + ap^2(x)\varphi_n^- - \frac{k+2(n-1)}{x} p(x) + \varphi_{n-1}^+ = 0 \tag{2.4a}$$

and

$$q'(x) - aq^2(x)\varphi_{n-1}^+ - \frac{k+2n}{x} q(x) - \varphi_n^- = 0 \tag{2.4b}$$

It is a straightforward matter to pick-out particular solutions to equations (2.4), especially when we keep (1.5) and (2.3a) and (2.3b) in mind. The general solutions, should they be required, can then be written down straight away [6]. Indeed, by inspection, we find equations (2.4a,b) satisfied identically if

$$p(x) = -\frac{1}{ax}, \quad q(x) = x, \quad \varphi_{n-1}^+ = 0 \quad \text{and} \quad \varphi_n^- = -(2n+k-1) \tag{2.5}$$

We now have the transformed Bessel equation, (1.3), in *factorized* form(s), (2.2), and have developed a *ladder-operator formalism*, (2.1), for the corresponding transformed Bessel functions, $w_n(x)$. To complete the factorization or ladder-operator formalism for the original Bessel equation, (1.1), and its corresponding Bessel function, $Z_n(x)$, we

(a) substitute back using (1.2) and obtain the formalism in terms of the original set-up, and

(b) determine a *particular* solution $Z_n(x)$ for any integer n (usually $n = 0$.)

The ladder-operator formalism enables us to find all other $Z_n(x)$ from the given particular function (which particular function will be a *solution in series*). Further, the ladder-operator formalism enables the development of other relations between the Bessel functions, like the three-term recurrence relations and the Rayleigh formulae, and so on. Note that each individual Bessel equation has *two* fundamental particular solutions associated with it for each value of n, so each ladder-operator representation has two starting points: one starting point for each set of fundamental particular solutions. In addition, other functions defined in terms of the two fundamental particular solutions, the Hankel functions say [4],will also satisfy the same fundamental relations as the defining Bessel functions.

The series solution to (1.3) is easy to obtain. However, all that we require is

$$w_0(x) = Z_0(x) = \sum_{r=0}^{\infty} d_r x^r \tag{2.6a}$$

with the coefficients d_r given by

$$d_1 \equiv 0 \qquad (2.6b)$$

and

$$d_{r+2} = \frac{-ad_r}{(r+2)(r+k+1)}, \quad \mathrm{r} = 1,2,3,\ldots \qquad (2.6c)$$

From (2.6b) and (2.6c), we see that all *odd-numbered* coefficients *vanish* and we are left with even-numbered coefficients only. If we shift the dummy index r down two units and replace r with $2m$, then the non-zero coefficients satisfy the adjusted recurrence relation

$$d_{2m} = \frac{-ad_{2m-2}}{2m(2m+k-1)}, \quad \mathrm{m} = 1,2,3,\ldots \qquad (2.7)$$

with the initial coefficient, d_0, remaining as an *arbitrary* coefficient.

Interestingly, k takes on the two values 1 and 2 only (see sections 3 and 4 below). When $k=1$, (2.7) becomes

$$d_{2m} = \frac{-ad_{2m-2}}{2^2 m^2}, \quad \mathrm{m} = 1,2,3,\ldots \qquad (2.8a)$$

while for $k = 2$, (2.7) becomes

$$d_{2m} = \frac{-ad_{2m-2}}{2m(2m+1)}, \quad \mathrm{m} = 1,2,3,\ldots \qquad (2.8b)$$

Now, when $k=1$ we make the *conventional* choice of $d_0 = 1$ to find, from (2.8a)

$$d_{2m} = \frac{(-a)^m}{2^{2m}(m!)^2}, \quad \mathrm{m} = 1,2,3,\ldots \qquad (2.9a)$$

and then

$$w_0(x) = Z_0(x) = \sum_{m=0}^{\infty} \frac{(-a)^m x^{2m}}{2^{2m}(m!)^2} \qquad (2.9b)$$

On the other hand, when $k = 2$ the *conventional* choice of $d_0 = 1$ in (2.8b) gives

$$d_{2m} = \frac{(-a)^m}{(m+1)!}, \quad \mathrm{m} = 1,2,3,\ldots \qquad (2.10a)$$

and then

$$w_0(x) = Z_0(x) = \sum_{m=0}^{\infty} \frac{(-a)^m x^{2m}}{(m+1)!} \qquad (2.10b)$$

The second solutions corresponding to (2.9b) and (2.10b), along with other relevant functions, are obtained in a standard manner as discussed, briefly in section 5 below.

3. The Basic Bessel Equations

Suppose the Bessel function $Z_n(z)$ is a particular solution of a Bessel equation of order n, or

$$x^2 \frac{d^2 Z_n(x)}{dx^2} + x\frac{dZ_n(x)}{dx} + (ax^2 - n^2)Z_n(x) = 0 \qquad (3.1)$$

with $a = 1$ giving the standard Bessel equation and $a = -1$ giving the modified Bessel equation. Following our discussion in sections 1 and 2, we compare (3.1) with (1.1) and note that for (3.1) $c = 0$ and $k = 1$, so the functions

$$Z_n(x) = x^n w_n(x) \qquad (3.2)$$

satisfy the differential equation (see, also, Bernardini and Natalini [2])

$$x\frac{d^2 w_n(x)}{dx^2} + (1+2n)\frac{dw_n(x)}{dx} + axw_n(x) = 0 \qquad (3.3)$$

To find the ladder-operators for (3.3), we substitute $k = 1$ into equations (2.4) and solve the resulting two Riccati equations to get, in particular, from (2.5)

$$\left(-\frac{1}{ax}\frac{d}{dx}\right)w_n(x) = w_{n+1}(x) \qquad (3.4a)$$

and

$$\left(x\frac{d}{dx} + 2n\right)w_n(x) = w_{n-1}(x) \qquad (3.4b)$$

in agreement with the results of Bernardini and Natalini [2].

The differential recurrence relations (3.4), on substituting $w_n(x) = x^{-n}Z_n(x)$ from (3.2), reduce to their ‘usual’ forms [6]

$$-\frac{dZ_n(x)}{dx}+\frac{n}{x}Z_n(x)=aZ_{n+1}(x) \qquad (3.5a)$$

and

$$\frac{dZ_n(x)}{dx}+\frac{n}{x}Z_n(x)=Z_{n-1}(x) \qquad (3.5b)$$

where we have replaced n with $n-1$ in the final form of equation (3.5b). We may now factorize the Bessel's equation of (integral) order n, since, from equations (3.5), it follows that equation (3.1) is identical to (for example)

$$\left[\frac{d}{dx}+\frac{n+1}{x}\right]\left[-\frac{d}{dx}+\frac{n}{x}\right]Z_n(x)=aZ_n(x) \qquad (3.6)$$

We may infer further relations from equations (3.5). For example, on eliminating the derivative terms from equations (3.5), we find that

$$2nZ_n(x)=zZ_{n+1}(x)+azZ_{n-1}(x) \qquad (3.7)$$

which is the 'usual' [6] three term recurrence relation for Bessel functions. Or, again, if we examine (3.5a) we have, by mathematical induction

$$\left(-\frac{1}{ax}\frac{d}{dx}\right)^n[w_0(x)]=w_n(x) \qquad (3.8)$$

which, by (3.2), becomes

$$Z_n(x)=(-1)^n x^n\left(\frac{1}{ax}\frac{d}{dx}\right)^n[Z_0(x)] \qquad (3.9)$$

and (3.8) is the *Rayleigh formula* for the Bessel function(s).

The question remains of finding the starting function(s), $Z_0(x)$, from which we may generate the $Z_n(x)$, using either of equations (3.5 a) or (3.8). In this, with (3.9) in mind, we set $n=0$ in equation (3.3) and look for a *solution in series*. Of course, the well-known series solution(s) for $Z_0(z)$ is obtained from (2.9b) as

$$Z_0(x)\equiv\sum_{m=0}^{\infty}\frac{(-a)^m x^{2m}}{2^{2m}(m!)^2} \qquad (3.10)$$

and we are finished, or, at least, this is as far as we are going here.

4. The Spherical Bessel Equations

Suppose, $Z_n(x)$ is a spherical Bessel function, that is, $Z_n(x)$ is a particular solution of the spherical Bessel equation of order n, or

$$x^2 \frac{d^2 Z_n(x)}{dx^2} + 2x \frac{dZ_n(x)}{dx} + [ax^2 - n(n+1)] Z_n(x) = 0 \qquad (4.1)$$

with $a = 1$ giving the standard spherical Bessel equation and $a = -1$ giving the modified spherical Bessel equation. Following our discussion in sections 1 and 2, we compare (3.1) with (1.1) and note that for (3.1) $c = -1$ and $k = 2$, so the functions

$$Z_n(x) = x^n w_n(x) \qquad (4.2)$$

satisfy the differential equation

$$x \frac{d^2 w_n(x)}{dx^2} + 2(n+1) \frac{dw_n(x)}{dx} + axw_n(x) = 0 \qquad (4.3)$$

To find the ladder-operators for (4.3), we substitute $k = 2$ into equations (2.4) and solve the resulting two Riccati equations to get, in particular, from (2.5)

$$\left(-\frac{1}{ax} \frac{d}{dx} \right) w_n = w_{n+1} \qquad (4.4a)$$

and

$$\left(x \frac{d}{dx} + (2n+1) \right) w_{n+1} = w_n \qquad (4.4b)$$

The differential recurrence relations (4.4), on substituting from (4.2), reduce to their ‘usual’ forms [1] for $Z_n(x)$

$$-\frac{dZ_n(x)}{dx} + \frac{n}{x} Z_n(x) = aZ_{n+1}(x) \qquad (4.5a)$$

and

$$\frac{dZ_n(x)}{dx} + \frac{n+1}{x} Z_n(x) = Z_{n-1}(x) \qquad (4.5b)$$

where we have replaced n with $n-1$ in the final form of equation (4.5b).

We may now factorize the spherical Bessel’s equation of (integral) order n, since, from equations (4.5), it follows that equation (4.1) is identical to (for example)

$$\left[\frac{d}{dx}+\frac{n+2}{x}\right]\left[-\frac{d}{dx}+\frac{n}{x}\right]Z_n(x)=aZ_n(x) \qquad (4.6)$$

Further, as before, eliminating the derivative terms from the equations (4.5), we find that

$$(2n+1)Z_n(x)=xZ_{n-1}(x)+axZ_{n+1}(x) \qquad (4.7)$$

which is the 'usual' [1] three term recurrence relation for spherical Bessel functions. And again, from (4.4a) and mathematical induction, we have

$$\left(-\frac{1}{ax}\frac{d}{dx}\right)^n[w_0(x)]=w_n(x) \qquad (4.8)$$

which, by (3.2), becomes the Rayleigh formula

$$Z_n(x)=(-1)^n x^n\left(\frac{1}{ax}\frac{d}{dx}\right)^n[Z_0(x)] \qquad (4.9)$$

The question remains, again, of finding the starting function, $Z_0(x)$, from which we may generate the $Z_n(x)$, $n\geq 1$, using equation (4.8a). In this case, it is necessary to set $n=0$ in equation (4.3) and look for a *solution in series*. Of course, the well-known series solution for $Z_0(z)$ is obtained from (2.10b) as

$$Z_0(x)=\sum_{m=0}^{\infty}\frac{(-a)^m x^{2m}}{(m+1)!} \qquad (4.10)$$

and we are finished.

5. Conclusions and Brief Discussion

We have presented an alternative factorization method for the various Bessel functions, denoted *generically* here by $Z_n(x)$, and provided we start with the most basic infinite series representation – that of $Z_0(x)$ – the rest of the series representations and relationships between the $Z_n(x)$ follow directly. Now, while we have developed 'starting' functions for the usual 'first' solutions to the Bessel equations, the construction of the 'second' solutions from the first solutions is well-known [4, 6] and, given *both* particular solutions to any Bessel equation, the construction of the various 'higher-order' Bessel functions follows immediately also [4, 6].

The method presented here has been developed as an alternative to two other ladder-operator approaches in the literature: that of Bernardini and Natalini [2] and the direct operator factorization of the Mexican school, represented, for example in Reyes et al [5]. Bernardini and Natalini [2] start from a general theorem 'in the abstract' and apply it to the Bessel equation and the modified Bessel equation. However, Bernardini and Natalini [2] are forced to take one of their ladder-operators from the manipulation of the series solution for, in the standard notation [4], $J_n(x)$ ($I_n(x)$), but our sole use of series solutions is to start the ladder-operations off, the various recurrence relations following from the general method, independent of the series solutions to the Bessel equations.

Reyes et al [5] factorize the Bessel equation *directly*, when it is written expressly as

$$\frac{d^2 J_n(x)}{dx^2} + \frac{1}{x^2}\frac{dJ_n(x)}{dx} + \left(1 - \frac{n^2}{x^2}\right) J_n(x) = 0 \qquad (5.1)$$

This direct factorization of (5.1), which results in the same ladder-operators as in section 3 (with $a = 1$), is a special case of the general method [5] of the Mexican school for factorizing the differential equations specifying the special functions and, of course, can be applied to the other types of Bessel equation also. However, even the approach of Reyes et al [5] requires a series representation of a particular Bessel function to start it off. There is no escape from a series solution somewhere 'in the mix'.

Overall, by way of comparison, the main advantage of the process presented here, for factorizing the Bessel equation and developing the standard relations between Bessel equations, is probably the basic simplicity of its approach.

References

[1] Abramowitz, M. and Stegun, I. A.: *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. New York: Dover, 1972.

[2] Bernardini, A and Natalini, P.: *Factorization of differential operators for some special functions* Soochow Journal of Mathematics, Volume 29, No. 2, pp. 147-156, April 2003.

[3] Jafarizadeh M. A. and Fakhri H., Physics Letters A **230** (1997) 164

[4] Nikiforov A. F. and Uvarov V. B., *Special Functions of Mathematical Physics* (Birkhauser) 1988.

[5] Reyes M.A., Jimenez D. and Rosu H.C., *Rev. Mex. Fis*. **49** (2003) 358-363.

[6] Watson, G.N.: *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, 1966.

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