International Mathematical Forum, Vol. 9, 2014, no. 20, 953 - 965
HIKARI Ltd, www.m-hikari.com
http://dx.doi.org/10.12988/imf.2014.4491

Frobenius Series Solution of Fuchs Second-Order Ordinary Differential Equations via Complex Integration

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Abstract

A method is presented (with standard examples) based on an elementary complex integral expression, for developing Frobenius series solutions for second-order linear homogeneous ordinary Fuchs differential equations. The method reduces the task of finding a series solution to the solution, instead, of a system of simple equations in a single variable. The method is straightforward to apply as an algorithm, and eliminates the manipulation of power series, so characteristic of the usual approach [14]. The method is a generalization of a procedure developed by Herrera [4] for finding Maclaurin series solutions for nonlinear differential equations.

Mathematics Subject Classification: 30B10, 30E20 34A25, 34A30

Keywords: Frobenius, Series solution, Fuchs differential equations, complex integrals

1. Introduction

We consider the Fuchs second-order linear ordinary differential equation (ODE)

$$f^{(2)}(z) + P(z)f^{(1)}(z) + Q(z)f^{(0)}(z) = 0 \qquad (1.1)$$

with the superscript numbers in brackets denoting differentiation with respect to z, the zeroth derivative being the function $f(z)$ itself. As usual, we assume [14]

$$p(z) = (z - z_0)P(z) \quad \text{and} \quad q(z) = (z - z_0)^2 Q(z) \tag{1.2}$$

are *analytic* functions of the independent variable z, with z_0 being a regular singular point of (1.1). The class of linear ODE represented by equation (1.1) contains many of the important equations of classical Mathematical Physics and its solution is of sufficient practical importance to warrant further study.

The usual [6, 14] Frobenius power series solution for the linear ODE (1.1) is

$$f(z) = \sum_{m=0}^{\infty} a_m (z - z_0)^{m+r} \tag{1.3}$$

with the unknown coefficients $\{a_m\}_{m=0}^{\infty}$ and the index r to be determined by substituting (1.3) into (1.1), which gives rise to a recurrence relation for the $\{a_m\}_{m=0}^{\infty}$. In particular, the index r is obtained as the solution(s) of the indicial equation

$$r(r-1) + p(0)r + q(0) = 0 \tag{1.4}$$

which arises form the *leading term* ($a_0 \neq 0$) on substituting (1.3) into (1.1) [6, 14]. This, the standard approach, involves (in general) the manipulation of infinite series. However, by an elementary device, the necessity for manipulating infinite series directly, when solving linear ODE, can be side-stepped and, in the resulting format, manipulations are reduced to solving simple equations along with some basic algebra.

The basis of this alternative approach to the determination of Frobenius power series, is the well-known formula from complex variable theory [5] (all closed-contour integrals that occur below are assumed evaluated in the counter-clockwise or positive direction)

$$\oint_C (z - z_0)^n \, dz = \begin{cases} 2\pi i, & n = 1 \\ 0, & \text{otherwise} \end{cases} \tag{1.5}$$

where z is a complex variable and z_0 a fixed point within the closed-contour C. The relation (1.5) is used to 'knock-out' all but one term from (1.3) through a combined differentiation/integration process, described in detail in section 2, which leads to a complex integral representation, equation (2.2) below, for the coefficients $\{a_m\}_{m=0}^{\infty}$ of (1.3).

Once equation (2.2) is established, it becomes a routine matter to apply it to the solution of wide classes of ODE, especially those of the Fuchs' class that are the main topic of discussion here. Assuming a solution of the Frobenius form (1.3), one multiplies through the given differential equation by $1/(z-z_0)^{n+1}$, applies a contour integration (round a particular closed-path) throughout the resulting expression and applies equation (2.2), term by term. The result of this process, is that the original differential equation is *transformed* into the recurrence relation for the coefficients, $\{a_n\}_{n=0}^{\infty}$, of (1.3), with the substitution $n=0$ yielding the indicial equation *tout court*. (The process, or algorithm, transforms the dummy variable m into the dummy variable n.) From the operational point of view, the basic problem of solving the ODE is reduced, in practice, to the repeated solution of a *simple algebraic equation in one variable*! From the historical point of view, the method is a generalization of a method originated by Herrera [4] for dealing with Maclaurin series solutions to nonlinear ODE.

The path of the paper proceeds as follows. In section 2 we derive the basic formula, (2.2), as a generalization, to the case of a Frobenius power series, of Herrera's formula for the coefficients of a Maclaurin series [4]. Next, in section 3, we apply the algorithm to some standard ODE from mathematical physics: the Bessel equation of order ν [1, 14], the hypergeometric equation [5, 14] and Heun's equation [2]. In these examples, and in the others that follow in sections 4 and 5, we consider the problem solved as soon as we have developed the recurrence relation, the rest of the solution process being well-documented in the literature. In section 4 the Riemann-Papperitz equation is considered and we see, explicitly, the effects of the standard transformation approach [3] to its solution on the transformation of the recurrence relations for the coefficients. Finally, section 5 provides a brief discussion of the method (its generic problems and other applications) and 'round things off' with a another couple of examples.

Note that, while we have applied the method to a standard range of 'classical' equations; many other equations will, naturally, yield a solution to this approach. Further, we have restricted ourselves to the consideration of an individual singular point in each example below: the singular point at the origin. Other singular points of the ODE may be transformed to the origin, if necessary.

2. Derivation of the Basic Formula

Progressing formally, suppose we start with an assumed Frobenius power series expansion of a function, $f(z)$, a regular singular point z_0, that is [14]

$$f(z) = \sum_{m=0}^{\infty} a_m (z-z_0)^{m+r} \tag{2.1}$$

for constant index r. Then, with C a closed-contour around z_0 avoiding other singularities of $f(z)$, we wish to show that, for $m = k, k+1, k+2, k+3, \ldots$, $k = 0,1,2,3,\ldots$

$$a_m = \frac{1}{[m+r]_k 2\pi i} \oint_C \frac{f^{(k)}(z)}{(z-z_0)^{m+r-k+1}} dz \tag{2.2}$$

where, following the notation of Ince [6], for positive integers k

$$[m+r]_k = (m+r)(m+r-1)(m+r-2)\cdots(m+r-k+1) = \frac{\Gamma(m+r+1)}{\Gamma(m+r-k+1)} \tag{2.3}$$

while $[m+r]_0 = 1$. First, we differentiate (2.1) k times, to find that

$$f^{(k)}(z) = \sum_{m=k}^{\infty} [m+r]_k a_m (z-z_0)^{m+r-k} \tag{2.4}$$

or

$$f^{(k)}(z) = [n+r]_k a_n (z-z_0)^{n+r-k} + \sum_{m=k\neq n}^{\infty} [m+r]_k a_m (z-z_0)^{m+r-k} \tag{2.5}$$

Next, if we divide through (2.5) by $(z-z_0)^{n+r-k+1}$ and integrate round a closed-contour C, containing z_0 while avoiding singularities of $f(z)$, we get, as the index r *cancels-out*

$$\oint_C \frac{f^{(k)}(z)}{(z-z_0)^{n+r-k+1}} dz = [n+r]_k a_n \oint_C \frac{dz}{(z-z_0)} = [n+r]_k a_n 2\pi i \tag{2.6}$$

so that

$$a_n = \frac{1}{[n+r]_k 2\pi i} \oint_C \frac{f^{(k)}(z)}{(z-z_0)^{n+r-k+1}} dz \tag{2.7}$$

or, changing dummy variables, we have (2.2). Finally, looking back, we find $(n =)m = k, k+1, k+2, k+3, \ldots$, $k = 0,1,2,3,\ldots$. We note that if r is zero, then (2.2) becomes the original formula of Herrera [4], that is

$$a_m = \frac{(m-k)!}{m!\, 2\pi i} \oint_C \frac{f^{(k)}(z)}{(z-z_0)^{m-k+1}} dz \tag{2.8}$$

We move on, now, to some standard examples. We note that series convergence is discussed, e.g., in [6], [7] and [14] and is not considered here.

3. The Solution of Some Standard Second-Order Equations

In this section, we will restrict the discussion to equations with regular singular points at the origin. Notice that we will always write the general recurrence relation in terms of a_n, which is as it *must* be written [6, 14]. As a first example, we solve the Bessel equation of order ν, that is [1, 6, 14]

$$z^2 f^{(2)}(z) + zf^{(1)}(z) + (z^2 - \nu^2) f^{(0)}(z) = 0 \qquad (3.1)$$

Next, Assuming (2.1), we divide through equation (3.1) by z^{n+r+1} and integrate round the closed-contour C, to get

$$\oint_C \frac{f^{(2)}(z)}{z^{n+r-1}} dz + \oint_C \frac{f^{(1)}(z)}{z^{n+r}} dz + \oint_C \frac{f^{(0)}(z)}{z^{n+r-1}} dz - \nu^2 \oint_C \frac{f^{(0)}(z)}{z^{n+r+1}} dz = 0 \qquad (3.2)$$

and compare the powers of the denominators of the integrands of (3.2) with that of (2.2) to get four equations for the dummy-variable m, one for each value of k (two, one, zero and zero, respectively), that is

$$m + r - k + 1 = m + r - 2 + 1 = n + r - 1 \ \ or\ \ m = n \qquad (3.3a)$$

and

$$m + r - k + 1 = m + r - 1 + 1 = n + r \ \ or\ \ m = n \qquad (3.3b)$$

and

$$m + r - k + 1 = m + r - 0 + 1 = n + r - 1 \ \ or\ \ m = n - 2 \qquad (3.3c)$$

and

$$m + r - k + 1 = m + r - 0 + 1 = n + r + 1 \ \ or\ \ m = n \qquad (3.3d)$$

Having identified the appropriate values of k and m, we use (2.2), again, to rewrite equation (3.2), after cancelling and re-arranging, as

$$[(n + r)^2 - \nu^2] a_n + a_{n-2} = 0, \quad n = 0,1,2,3,,\ldots \qquad (3.4)$$

Now, when $n = 0$, as $a_0 \neq 0$, we must have $a_{-2} = 0$ and

$$r^2 - \nu^2 = 0 \qquad (3.5)$$

which is the indicial equation, with solutions $r = \pm\nu$. Also, when $n = 1$, we must have $a_{-1} = 0$ and so we must set $a_1 = 0$. With $a_0 \neq 0$ an arbitrary constant, we recognize (3.4) as the recurrence relation(s) for $J_\nu(z)$. The third and final step is to solve the recurrence relation (3.4) and obtain the Frobenius series solution explicitly. As this is well-known [1, 6, 14], we stop here Now, before we consider our next example, we pause to point-out that the index r has, once more, *cancelled-*

out of the dummy-variable (m) calculations. This appears to be a characteristic of the method as a whole (*pace* section 2).

For our second example, we solve the hypergeometric equation, that is [5, 14]

$$z(1-z)f^{(2)}(z)+[\gamma-(\alpha+\beta+1)z]f^{(1)}(z)-\alpha\beta f^{(0)}(z)=0 \qquad (3.6)$$

where α, β and γ are constants. As before, assuming (2.1), we divide through equation (3.6) by z^{n+r+1} and integrate round the closed-contour C, to get

$$\oint_C \frac{f^{(2)}(z)dz}{z^{n+r}} - \oint_C \frac{f^{(2)}(z)}{z^{n+r-1}}dz + \gamma\oint_C \frac{f^{(1)}(z)}{z^{n+r+1}}dz$$
$$-(\alpha+\beta+1)\oint_C \frac{f^{(1)}(z)}{z^{n+r}}dz - \alpha\beta\oint_C \frac{f^{(0)}(z)}{z^{n+r+1}}dz = 0 \qquad (3.7)$$

and compare the powers of the denominators of the integrands of (3.7) with that of (2.2) to get five equations for the dummy variable m, one for each value of k (two, two, one, one and zero, respectively), that is

$$m+r-k+1=m+r-2+1=n+r \ \ or \ \ m=n+1 \qquad (3.8a)$$
and
$$m+r-k+1=m+r-2+1=n+r-1 \ \ or \ \ m=n \qquad (3.8b)$$
and
$$m+r-k+1=m+r-1+1=n+r+1 \ \ or \ \ m=n+1 \qquad (3.8c)$$
and
$$m+r-k+1=m+r-1+1=n+r \ \ or \ \ m=n \qquad (3.8d)$$
and
$$m+r-k+1=m+r-0+1=n+r+1 \ \ or \ \ m=n \qquad (3.8e)$$

Having identified the appropriate values of k and m, we use (2.2), again, to rewrite equation (3.7), after cancelling and re-arranging, as

$$(n+r+1)(n+r+\gamma)a_{n+1}-(n+r+\alpha)(n+r+\beta)a_n=0 \qquad (3.9a)$$
or
$$(n+r)(n+r-1+\gamma)a_n-(n+r-1+\alpha)(n+r-1+\beta)a_{n-1}=0 \qquad (3.9b)$$

with $n=0,1,2,3,\ldots$. Now, when $n=0$, as $a_0\neq 0$, we must have $a_{-1}=0$ and, in this case, the indicial equation is

$$r(r+\gamma-1)=0 \qquad (3.10)$$

so that $r=0$, or $r=1-\gamma$. Apparently, the first root, $r=0$, will yield an ordinary power series. With $a_0\neq 0$ an arbitrary constant, we recognize, in (3.9), the recurrence relation(s) for the Frobenius series solution of the hypergeometric equation [5, 14] and, as before, we terminate the example here.

As our final example in this section, we consider the Frobenius series solution of Heun's equation, that is [2]

$$f^{(2)}(z)+[\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\varepsilon}{z-c}]f^{(1)}(z)+\frac{\alpha\beta z-q}{z(z-1)(z-c)}f^{(0)}(z)=0 \qquad (3.11)$$

with $c\neq 0,\alpha,\beta,\gamma,\delta,\varepsilon$ and q constants and where $\gamma+\delta+\varepsilon=\alpha+\beta+1$. Clearing fractions' in (3.11), we then multiply-out and collect like terms to get

$$(z^3-(1+c)z^2+cz)f^{(2)}(z)$$
$$+[(\gamma+\delta+\varepsilon)z^2-(\gamma(1+c)+c\delta+\varepsilon)z+c\gamma]f^{(1)}(z)+(\alpha\beta z-q)f^{(0)}(z)=0 \quad (3.12)$$

Next, assuming (2.1), we divide through equation (3.12) by z^{n+r+1} and integrate round the closed-contour C, assuming (2.1), to get

$$\oint_C \frac{f^{(2)}(z)dz}{z^{n+r-2}}-(1+c)\oint_C \frac{f^{(2)}(z)}{z^{n+r-1}}dz+c\oint_C \frac{f^{(2)}(z)}{z^{n+r}}dz$$

$$+(\gamma+\delta+\varepsilon)\oint_C \frac{f^{(1)}(z)}{z^{n+r-1}}dz-(\gamma(1+c)+c\delta+\varepsilon)\oint_C \frac{f^{(1)}(z)}{z^{n+r}}dz+c\gamma\oint_C \frac{f^{(1)}(z)}{z^{n+r+1}}dz$$

$$+\alpha\beta\oint_C \frac{f^{(0)}(z)}{z^{n+r}}dz-q\oint_C \frac{f^{(1)}(z)}{z^{n+r+1}}dz=0 \qquad (3.13)$$

and compare the powers of the denominators of the integrands of (3.13) with that of (2.2) to get *eight* equations for the dummy variable *m*, one for each value of *k* (two, two, two, one, one, one, zero and zero, respectively), that is

$$m+r-k+1=m+r-2+1=n+r-2 \ \ or \ \ m=n-1 \qquad (3.14a)$$
and
$$m+r-k+1=m+r-2+1=n+r-1 \ \ or \ \ m=n \qquad (3.14b)$$
and
$$m+r-k+1=m+r-2+1=n+r \ \ or \ \ m=n+1 \qquad (3.14c)$$
and
$$m+r-k+1=m+r-1+1=n+r-1 \ \ or \ \ m=n-1 \qquad (3.14d)$$
and
$$m+r-k+1=m+r-1+1=n+r \ \ or \ \ m=n \qquad (3.14e)$$
and
$$m+r-k+1=m+r-1+1=n+r+1 \ \ or \ \ m=n+1 \qquad (3.14f)$$

and
$m+r-k+1=m+r-0+1=n+r \;\; or \;\; m=n-1$ (3.14g)
and
$m+r-k+1=m+r-0+1=n+r+1 \;\; or \;\; m=n$ (3.14h)

Having identified the appropriate values of k and m, we use (2.2), again, to rewrite equation (3.12), after cancelling and re-arranging, as

$$c(n+r+1)(n+r+\gamma)a_{n+1}-[(n+r)[(n+r-1+\gamma)(1+c)+c\delta+\varepsilon]+q]a_n$$

$$+(n+r-1+\alpha)(n+r-1+\beta)a_{n-1}=0 \qquad (3.15a)$$

or

$$c(n+r)(n+r-1+\gamma)a_n-[(n+r-1)[(n+r-2+\gamma)(1+c)+c\delta+\varepsilon]+q]a_{n-1}$$

$$+(n+r-2+\alpha)(n+r-2+\beta)a_{n-2}=0 \qquad (3.15b)$$

with $n=0,1,2,3,\ldots,$ and where we have used the fact that $\gamma+\delta+\varepsilon=\alpha+\beta+1.$

In this case, from (3.15b), when $n=0$ we must have $a_{-2}=a_{-1}=0$; also, as $a_0\neq 0$ and $c\neq 0,$ the indicial equation is (as in the previous example)

$$r(r+\gamma-1)=0 \qquad (3.16)$$

so that $r=0,$ or $r=1-\gamma.$ Again, the first root, $r=0,$ will yield an ordinary power series. Equation (3.15) constitutes the recurrence relation(s) for the Frobenius series solution of the Heun equation (see [2] and references therein) and, as usual, we terminate the explicit calculations here. However, we see that we have a *three-term* recurrence relation, (3.16), for the Heun Frobenius series solution. For a brief discussion of this matter of three-term recurrence relations see Figueiredo [2]. Further, if we set $n=1$ in (3.15b), we find a_1 given in terms of $a_0\neq 0$ and we have, indeed, only one solution.

4. The Riemann-Papperitz Equation

In this section we consider the Riemann-Papperitz equation and its relation to the hypergeometric equation [3]. To facilitate the discussion, we consider, first, the following. The indicial equation has its own defining differential equation

$$(z-z_0)^2 f^{(2)}(z)+p_0(z-z_0)f^{(1)}(z)+q_0 f^{(0)}(z)=0 \qquad (4.1)$$

To see this, we apply the method, with $n=0,$ to (4.1), when we find that (4.1) transforms to

$$r(r-1)a_0 + p_0 r a_0 + q_0 a_0 = 0 \qquad (4.2)$$

or, as $a_0 \neq 0$,

$$r(r-1) + p_0 r + q_0 = 0 \qquad (4.3)$$

which is, indeed, the indicial equation for (1.1). The equation (4.1) is just Euler's second-order homogeneous equation and is obtained (asymptotically) from (1.1) when (1.2) is rewritten as

$$P(x) = \frac{p_0}{(z-z_0)} + \sum_{m=1}^{\infty} p_m (z-z_0)^m \qquad (4.4a)$$

and

$$Q(x) = \frac{q_0}{(z-z_0)^2} + \frac{q_1}{(z-z_0)} + \sum_{m=2}^{\infty} q_m (z-z_0)^m \qquad (4.4b)$$

and then the leading terms are 'extracted' as $z \to z_0$.

We are in a position, now, to examine the Riemann-Papperitz equation, that is

$$f^{(2)}(z) + \left[\frac{A_1}{z-a} + \frac{A_2}{z-b} + \frac{A_3}{z-c}\right] f^{(1)}(z)$$
$$+\left[\frac{B_1}{z-a} + \frac{B_2}{z-b} + \frac{B_3}{z-c}\right] \frac{f^{(0)}(z)}{(z-a)(z-b)(z-c)} = 0 \qquad (4.5)$$

where a, b, c and the $A's$ and $B's$ are constants and $A_1 + A_2 + A_3 = 2$. There are three regular singular points at a, b and c, but we will seek a Frobenius series solution to (4.5) by first moving the regular singular points to $0, 1$ and ∞. The argument is a standard one [3] and involves the determination of the $A's$ and $B's$ in terms of the roots of the indicial equations corresponding to the three regular singular points at a, b and c.

So, if the roots of (4.3) corresponding to $z = a$ are denoted by α_1, α_2, the roots corresponding to $z = b$ by β_1, β_2 and the roots corresponding to $z = c$ are by γ_1, γ_2, then, on shifting the singular points by taking

$$a = 0, b = \infty, c = 1 \qquad (4.6)$$

the Riemann-Papperitz equation may be rewritten, on clearing fractions, as [3]

$$z^2(1-z)^2 f^{(2)}(z) + z(1-z)(1-\alpha_1-\alpha_2-(1+\beta_1+\beta_2)z) f^{(1)}(z)$$
$$+(\alpha_1\alpha_2 - (\alpha_1\alpha_2 + \beta_1\beta_2 - \gamma_1\gamma_2)z + \beta_1\beta_2 z^2) f^{(0)}(z) = 0 \qquad (4.7)$$

If we apply the method to (4.7), for the regular singularity at the origin ($a = 0$), then we find, after cancelling and re-arranging, that (4.7) transforms to

$$(n+r-\alpha_1)(n+r-\alpha_2)a_n - [(n+r-1-\alpha_1)(n+r-1-\alpha_2)$$
$$+(n+r-1+\beta_1)(n+r-1+\beta_2)+\gamma_1\gamma_2]a_{n-1}$$
$$+(n+r-2+\beta_1)(n+r-2+\beta_2)a_{n-2} = 0 \qquad (4.8)$$

Setting $n = 0$ in (4.8), we get the indicial equation for (4.7) as

$$(r-\alpha_1)(r-\alpha_2)a_0 = 0 \qquad (4.9)$$

as $a_{-1} = 0$ and $a_{-2} = 0$ identically. The relation (4.9) provides a consistency check on the calculation, as, with $a_0 \neq 0$, the roots of (4.9) must be α_1 and α_2, as required.

As a further consistency check on (4.8), without actually writing out the series, we note, following [3] again, that (4.7) reduces to the hypergeometric equation, (3.6), 'after reduction', when we choose

$$\alpha_1 = 0, \alpha_2 = 1-\gamma \text{ with } \beta_1 = \alpha, \beta_2 = \beta \text{ and } \gamma_1 = 0, \gamma_2 = \gamma-\alpha-\beta \qquad (4.10)$$

In this reduction process, we expect (4.8) to 'reduce' to (3.9) and this is the case. Further, making the substitutions (4.10) in (4.8), we find that, after some algebra

$$(n+r)(n+r-1+\gamma)a_n - (n+r-1+\alpha)(n+r-1+\beta)a_{n-1}$$
$$-(n+r-1)(n+r-2+\gamma)a_{n-1} + (n+r-2+\alpha)(n+r-2+\beta)a_{n-2} = 0 \qquad (4.11)$$

which is satisfied if

$$-(n+r-1)(n+r-2+\gamma)a_{n-1} + (n+r-2+\alpha)(n+r-2+\beta)a_{n-2} = 0 \qquad (4.12a)$$

as then

$$(n+r)(n+r-1+\gamma)a_n - (n+r-1+\alpha)(n+r-1+\beta)a_{n-1} = 0 \qquad (4.12b)$$

also, and we have retrieved the recurrence relation (3.9b) for (3.6), as required.

5. Conclusions and Discussion

While the modern trend is for solving ODE using computer algebra systems [13] (which is perfectly natural and necessary in most cases) the facility with which the present method produces the coefficients for Frobenius series solutions to linear ODE is exceptionable. The technical problems involved in finding the indicial equation and the recurrence equation in the standard approach (see, e.g., [14]) are replaced, now, with the solution of simultaneous simple equations, along with the necessary algebra, a fact that holds true in general and not just for the examples presented above. Indeed, in the examples presented in the previous section we have considered one singular point only. But, as mentioned above, when other singular points exist, as with the hypergeometric and Heun equations, a change of independent variable ($w = z - z_0$) transforms these singularities to the origin and the method proceeds as before (this applies to 'the point at infinity' also [9]).

Further examples can be generated by considering the confluent hypergeometric and Heun equations [5, 9] with the same type of procedure dealing with these equations with equal facility as with the examples presented in section 3. Additionally, as pointed out by Hildebrand [5], many second-order equations are special cases of

$$(1 + R_M z^M) f^{(2)}(z) + \frac{1}{z}(P_0 + P_M z^M) f^{(1)}(z) + \frac{1}{z^2}(Q_0 + Q_M z^M) f^{(0)}(z) = 0 \quad (5.1)$$

for particular choices of the constants M, P_0, P_M, Q_0, Q_M and R_M. In fact, special cases of (5.1) include [5] the Bessel equation, Legendre's equation, Gauss's hypergeometric equation, the confluent hypergeometric equation, the Hermite equation, the Chebyshev equation and the Jacobi polynomial equation. Naturally, it is possible to 'attack' (5.1) using the present approach also.

At this point in the discussion we turn to some other, related, matters in the solution of (1.1). There are two immediate problems to discuss: the necessity to solve, or at least deal with, three-or more-term recurrence relations, and the determination of the 'second solution' to (1.1) – which will not, in general, be another Frobenius series solution, if the roots of the indicial equation differ by an integer [12, 14].

In the case of *three*-term recurrence relations, one of the standard means of avoiding three-term recurrence relations is to transform ODE to an alternative form, in the hope that the transformed ODE has a power series solution with a *two-term* recurrence relation. For examples of this, see reference [14]. Failing this, such three-or more-term recurrence relations must be tackled on a case-by-case basis and this is where computer algebra systems come back into consideration. (See, also, reference [2] for a brief overview of the ideas behind the solution of three-term recurrence relations.)

As for the determination of the 'second solution' to (1.1), we have, at least in principle, the general solutions to our equations through the 'reduction of order' method [14]. Alternatively, the special 'derivative method' for finding a second solution (given a Frobenius series solution), as explained, for example, in the standard textbook of Rainville [12]. can be attempted. Or, again, it is possible to derive an ODE for the 'series part' of the known form [14] of the second solution and solve this using the present method. The actual choice of which method to attempt will depend on the particular circumstances and the details of the 'first solution'.

Naturally, the method presented here can be applied to higher-order ODE. Suppose we consider the example of the ℓ^{th}-order linear homogeneous ODE with four singularities analysed, recently, by Kruglov [10]. that is

$$\sum_{i=0}^{\ell}(A_i z^2 + B_i z + C_i) z^i f^{(i)}(z) = 0 \qquad (5.2)$$

'Applying the method', we find that (5.2) transforms to

$$\sum_{i=0}^{\ell}\left[A_i \oint_C \frac{f^{(i)}(z)}{z^{n+r-i-1}}dz + B_i \oint_C \frac{f^{(i)}(z)}{z^{n+r-i}}dz + C_i \oint_C \frac{f^{(i)}(z)}{z^{n+r-i+1}}dz\right] = 0 \qquad (5.3)$$

so that, with $k = i$ and using (2.2), we get the general recurrence relation

$$\sum_{i=0}^{\ell}(A_i[n+r-2]_i a_{n-2} + B_i[n+r-1]_i a_{n-1} + C_i[n+r]_i a_n) = 0 \qquad (5.4)$$

and, when $n = 0,$ we get the indicial equation (as $a_{-2} = a_{-1} = 0$ while $a_0 \neq 0$)

$$\sum_{i=0}^{\ell} C_i[r]_i = 0 \qquad (5.5)$$

and we have Kruglov's scheme [10], so we finish here.

Finally, we wish to draw attention to a certain 'family relationship' between (2.2)/(2.7) with $r = 0,$ and the Caputo fractional derivative [8]

$$D^n f(x) = \frac{1}{\Gamma(k-n)}\int_0^{x} \frac{f^{(k)}(t)}{(x-t)^{n-k+1}}dt, \quad n > 0,\ x > 0;\ k-1 < n < k, \text{ integer } k \qquad (5.6)$$

though (5.6) is applied to different types of equations and in a different way [8].

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Received: April 30, 2014