Some New Uniform Approximate Analytical Representations of the Blasius Function

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Abstract

New uniform approximations of the Blasius velocity profile and Blasius function are developed from existing rational approximations of the Blasius velocity profile and of the Blasius function itself. Essentially, the existing rational approximations are rescaled (when necessary) and one of two methodologies [1, 11] is applied to the rescaled rational approximations to determine corresponding uniform approximations to the Blasius velocity profile and Blasius function. A brief discussion is presented on the merits of the results.

Indexing terms/Keywords

Boundary layer, Nonlinear ODE, Blasius equation, Blasius profile, Collocation method

SUBJECT CLASSIFICATION

Mathematics Subject Classification; 34A34, 34B16, 34B40, 65L60

INTRODUCTION

In the light of a recent interest in approximate analytical representations of the Blasius function and its derivative or velocity profile (see, for example, [1, 3, 5–14]), we present in this paper new uniform (valid for $x \in [0, \infty)$) approximations of the Blasius velocity profile and the Blasius function itself. Basically, these new uniform approximations of the Blasius velocity profile and the Blasius function originate from *given* rational approximations to the Blasius velocity profile or the Blasius function which are *not* themselves *uniform* approximations, but may be developed into such as described below [1, 11]. While all of the rational approximations that we start with have been taken from the general literature [1, 8], it is to be noted that, in sections 3 and 4 *before* we apply one of the methods of extending the rational approximations into uniform approximations, the *given* rational approximations [8] are first *scaled* to provide *new* rational approximations to either the Blasius velocity profile or the Blasius function, which improves the final 'fit' of the uniform approximation also. Before describing the methodology (in general terms) we remind ourselves of the relevant background of the Blasius problem, which arises from boundary layer theory [4].

As is well known [2], the Blasius function $F(x)$, with x a dimensionless distance, is the solution of the nonlinear ordinary differential equation

$$F'''(x) + \frac{1}{2}F(x)F''(x) = 0 \tag{1.1}$$

with the boundary/initial conditions [2]

$$F(0) = F'(0) = 0;\ F'(\infty) = 1 \tag{1.2}$$

The initial/boundary value problem (1.1) and (1.2) leads, immediately, to a boundary value problem for the *dimensionless velocity* $f(x) = F'(x)$, that is, the nonlinear ordinary differential equation

$$f'' + \frac{1}{2}Ff' = 0 \tag{1.3}$$

along with the boundary/initial conditions

$$f(0) = 0,\ f(\infty) = 1 \tag{1.4}$$

As well as the profile of $f(x) = F'(x)$ being well known [2], we see that we may also write

$$F(x) = \int_0^x f(t)dt \tag{1.5}$$

which is important to the overall method describe below.

Further aspects of the Blasius problem are summarized in Table 1, which contains information, taken from reference [6], on the initial and asymptotic properties of the Blasius function critical to the production of new rational approximations from given rational approximations, as well as providing functional material for the production of uniform approximations in conjunction with the original rational approximations.

Table 1. Basic Properties of the Blasius Function [6]

Symbol	Definition	Numerical Value
κ	$F''(0)$	0.33205733621519630
B	$\lim_{x\to\infty}(F(x)-x)$	1.7207876575205038
Q	$\lim_{x\to\infty}\exp\left(\frac{x}{4}(x+2B)\right)F''(x)$	0.111483754997889

Before launching into the 'meat' of the paper, we now describe, in general terms of course, the basic methodologies presented in detail in sections 2 to 4.

Two methods of obtaining uniform approximations from rational approximations are presented below. First, we describe a 'combination' method. Suppose we start from a, if necessary scaled (see above), rational approximation to the Blasius velocity profile. Then, we obtain a *uniform* approximate velocity profile by combining the rational approximation with the derivative of the asymptotic expression for $F(x)$ obtained from row two of Table 1; the combination method is that developed by Yun [11]. Alternatively, if we start from an approximation of the Blasius function itself, either one obtained from our rational approximation to the Blasius velocity profile and (1.5) or a given (scaled again) rational approximation of the Blasius function, then we obtain a *uniform* approximate Blasius function by combining the approximate Blasius function with the asymptotic expression for $F(x)$, obtained from row 2 of Table 1 again; the method of combination is, again, that developed by Yun [11].

The second method of producing a uniform approximation from a given approximation is obtained by starting, as before, from a, if necessary scaled, rational approximation to the Blasius velocity profile. The rational approximation to the Blasius velocity profile is then extended to a uniform rational approximation, following Ahmad and Al-Barakati [1], by the inclusion of an asymptotic factor – similar to that of row three of Table 1 – to both the numerator and denominator of our (scaled) rational approximation to the Blasius velocity profile. This new rational approximation to the velocity profile is then substituted into (1.5) to get a uniform analytical approximation of the Blasius function.

The body of the paper is arranged as follows. In section 2, we develop the first of our new approximate velocity function/Blasius function pairs by starting with the rational approximation to the Blasius velocity profile, $f(x)=F'(x)$, of Ahmad and Al-Barakati [1] and applying the methodology of Yun [11], mentioned above, to obtain new uniform approximations of the velocity function/Blasius function pair. Note, again, that the starting functions are not uniform approximations, but have a finite radius of application. We emphasise that this is true of all the starting approximate function pairs that we use in the present analysis. Next, in section 3, we adapt the rational approximate solutions of the Blasius problem derived in Noghrehabadi et al [8], to obtain uniform approximations of the velocity function/Blasius function pair. In order to do this, it is necessary to first rescale the Noghrehabadi et al [8] velocity function/Blasius function pair to accommodate the 'missing' initial condition $F''(0)$, presented in Table 1. Once this is done, new uniform approximations of the velocity function/Blasius function pair follow on using Yun's method [11] again. In section 4, we extract a result from section 3 and produce new uniform rational approximations of the velocity function/Blasius function pair by adapting an idea of Ahmad and Al-Barakati [1] to the *scaled* Blasius velocity profile approximation of section 3 to obtain further new uniform approximations of the velocity function/ Blasius function pair.

2. THE AHMAD AND AL-BARAKATI VELOCITY PROFILE

In this section we consider the uniform approximations arising from the Ahmad and Al-Barakati velocity profile, that is [1] (with κ given in Table 1)

$$f_A(x)=F_A'(x)=\frac{\kappa x+\frac{3}{560}\kappa^2x^4}{1+\frac{11}{420}\kappa x^3} \tag{2.1}$$

As explained in the introduction, we will use relation (2.1) in conjunction with the asymptotic form (from Table 1) $F(x) \to x + B$ and its derivative form $f(x) \to B$ to produce uniform approximations of the velocity function/Blasius function pair. A critical point in this construction is the existence of the 'weight function' [11]

$$\theta(x) = \frac{\left(\frac{x}{b}\right)^k}{1+\left(\frac{x}{b}\right)^k}, \quad 0 \le x < \infty \tag{2.2}$$

which, for 'sufficiently large' k, approaches the Heaviside step function [11]

$$H(x-b) = \begin{cases} 0, & x < b, \\ 1, & x > b. \end{cases} \tag{2.3}$$

so that we may conjoin two functions $\varphi_\alpha(x)$ and $\varphi_\beta(x)$, smoothly, by taking a weighted average of the form [11]

$$(1-\theta(x))\varphi_\alpha(x) + \theta(x)\varphi_\beta(x), \quad 0 \le x < \infty \tag{2.4}$$

The point is that $\varphi_\alpha(x)$ predominates for $0 \le x < b$, while $\varphi_\beta(x)$, predominates for $x > b$, thus ensuring the conjunction (2.4) gives a 'smooth' coverage of the whole half-line $0 \le x < \infty$. Naturally the choice of the parameters b and k determine just how good the fit of the conjunction (2.4) is to the problem in question.

In this instance, we wish to 'mix' (2.1) with $f'(x) \to B$ and also 'mix' the Blasius function corresponding to (2.1) with $F(x) \to x + B$. Following some 'numerical experimentation', we find that $b = 5$ and $k = 20$ lead to the best overall coverage for the velocity function/Blasius function pair. This leads to the two uniform approximations, one for the velocity profile

$$f_1(x) = (1-\theta(x))\, f_A(x) + \theta(x)B, \quad 0 \le x < \infty \tag{2.5}$$

and one (implicitly) for the Blasius function

$$F_1(x) = (1-\theta(x))F_A(x) + \theta(x)(x+B), \quad 0 \le x < \infty \tag{2.6}$$

where, from (1.5)

$$F_A(x) = \int_0^x f_A(t)dt \tag{2.7}$$

In Table 2 we present the results of the evaluation of (2.6) for selected values over the range $0 \le x < 100$. Also shown in Table 2, for comparison, are the results of the numerical solution of the Blasius problem (1.1) and (1.2) [4]. The results presented show that all values calculated from (2.6) lie within 0.29% of the numerical solution of the Blasius problem [4]. Finally, note that Ahmad and Al-Barakati [1] develop uniform approximations of the velocity function/Blasius function pair from (2.1) also, but they use a different method of approach (see the discussion of section 4 below).

3. THE NOGHREHABADI ET AL VELOCITY PROFILE I

In this section we consider the uniform approximation arising from the velocity profile of Noghrehabadi et al [8], which we write as

$$f_N(x) = F_N'(x) \frac{\frac{5622567066000.00}{16834174156800.00}x + \frac{32563000180.00}{16834174156800.00}x^4 + \frac{114789608.70}{16834174156800.00}x^7}{1+\frac{214631643070.73}{16834174156800.00}x^3 + \frac{812490706.83}{16834174156800.00}x^6 + \frac{961489.31}{16834174156800.00}x^9} \tag{3.1}$$

The approximate velocity profile (3.1) arose as an approximate solution to the Blasius problem and, because of this, unlike (2.1), (3.1) does *not* satisfy the condition $f_N'(0) = F_N''(0) = \kappa$. Instead, we find that

$$f_N'(0) = F_N''(0) = \frac{5622567066000.00}{16834174156800.00} \approx 0.3339972019 \tag{3.2}$$

which is the leading term in the denominator of (3.1) (c.f. (2.1)).

Table 2. Approximate Representations of the Blasius Function

x	$F_1(x)$	$F_2(x)$	$F_3(x)$	$F_4(x)$	'Exact' [4]
0	0	0	0	0	0
0.4	0.0266	0.0266	0.0266	0.0266	0.0266
0.8	0.1061	0.1061	0.1061	0.1061	0.1061
1.2	0.2379	0.2379	0.2379	0.2379	0.2379
1.6	0.4203	0.4203	0.4203	0.4203	0.4203
2.0	0.6500	0.6500	0.6499	0.6500	0.6500
2.4	0.9223	0.9223	0.9221	0.9223	0.9223
2.8	1.2309	1.2309	1.2306	1.2310	1.2310
3.2	1.5688	1.5691	1.5684	1.5691	1.5691
3.6	1.9285	1.9295	1.9284	1.9296	1.9295
4.0	2.3031	2.3056	2.3039	2.3059	2.3057
4.4	2.6866	2.6917	2.6894	2.6927	2.6924
4.6	2.8806	2.8872	2.8847	2.8887	2.8882
4.8	3.0764	3.0839	3.0815	3.0859	3.0853
5.0	3.2744	3.2816	3.2796	3.2841	3.2833
5.2	3.4745	3.4803	3.4789	3.4828	3.4819
5.4	3.6755	3.6797	3.6787	3.6820	3.6809
5.6	3.8766	3.8794	3.8788	3.8815	3.8803
5.8	4.0775	4.0793	4.0790	4.0812	4.0799
6.0	4.2781	4.2793	4.2790	4.2809	4.2796
6.4	4.6788	4.6792	4.6791	4.6802	4.6794
6.8	5.0790	5.0792	5.0792	5.0796	5.0793
7.0	5.2791	5.2792	5.2792	5.2794	5.2792
7.4	5.6792	5.6792	5.6792	5.6793	5.6792
8.0	6.2792	6.2792	6.2792	6.2792	6.2792
10	8.2792	8.2792	8.2792	8.2792	8.2792
20	18.2792	18.2792	18.2792	18.2792	18.2792
100	98.2792	98.2792	98.2792	18.2792	98.2792

To overcome this deficiency, we 'rescale' (3.1) by replacing the coefficient of x in (3.1) by κ from Table 1 (set $a = 0$ in (4.1) below). So, after replacing the coefficient of x in (3.1) by κ, we again look for two uniform approximations (now labelled by κ)

$$f_2(x) = (1-\theta(x))\, f_{N\kappa}(x) + \theta(x)B, \quad 0 \le x < \infty \tag{3.3}$$

for the velocity profile and (implicitly)

$$F_2(x) = (1-\theta(x))F_{N\kappa}(x) + \theta(x)(x+B), \quad 0 \le x < \infty \tag{3.4}$$

for the Blasius function where, from (1.5)

$$F_{N\kappa}(x) = \int_0^x f_{N\kappa}(t)dt \tag{3.5}$$

As in section 2, after some 'numerical experimentation', we find that $b = 5$ and $k = 20$ lead, again, to the best overall coverage for the velocity function/ Blasius function pair. In Table 2 we present the results of the evaluation of (the appropriate form of) (3.4) for selected values over the range $0 \le x \le 100$. The results presented show that all values calculated from (2.6) lie within about 0.046% of the numerical solution [4] of the Blasius problem.

In fact, Noghrehabadi et al [8] also develop a rational approximation for $F(x)$. If we apply the same scaling to the Noghrehabadi et al $F(x)$ (equation (36-b) of [8]) as we did to (3.1), and form the same weighted average as in (3.4), we get another uniform approximation to the Blasius function which we label $F_3(x)$. The results of the usual evaluation of $F_3(x)$ are presented in Table 2 and all lie within 0.13% of Howarth's [4] numerical solution. (Of course, a similar analysis can be developed with $F'(x)$ to produce an approximate velocity profile $f_3'(x)$.)

4. THE NOGHREHABADI ET AL VELOCITY PROFILE II

Actually, there is another means of extending the effective range of a given rational approximation to encompass the real half-line $0 \le x < \infty$. This process, devised by Ahmad and Al-Barakati [1], bears a resemblance to the second limit in row three of Table 1. Using a limiting argument, Ahmad and Al-Barakati [1] attach an exponential factor to their rational approximation (2.1) of $f(x)$ in an attempt to overcome the limited range of validity of (2.1). In this section, we will attempt to do the same for the corrected velocity profile of Noghrehabadi et al [8].

Specifically, we propose the form (note that we are basing (4.1) on the scaled form of (3.1), with the coefficient of x in (3.1) replaced by the value of κ from Table 1)

$$f_{N2}(x) = F'_{N2}(x) = \frac{0.3320573362x + \frac{32563000180.00}{16834174156800.00}x^4 + \frac{114789608.70}{16834174156800.00}x^7 + ax^{10}e^{\frac{x^2}{4}-1}}{1 + \frac{214631643070.73}{16834174156800.00}x^3 + \frac{812490706.83}{16834174156800.00}x^6 + \frac{961489.31}{16834174156800.00}x^9 + ax^{10}e^{\frac{x^2}{4}-1}} \quad (4.1)$$

where the parameter a is to be determined. In practice the simplest way to determine a is by *collocation*. In fact, similar to [1], we determine a by fixing the value of

$$F_{N2}(x) = \int_0^x f_{N2}(t)dt \quad (4.2)$$

for $x = 8$ at 6.2792, which is taken the last column of Table 2 and *assumed* exact.

The result of the collocation is that we find that

$$a \approx 7.255588531 \times 10^{-12} \quad (4.3)$$

which we then substitute into (4.1) and use in (4.2) to produce another (implicit) uniform approximation for the Blasius function which we label $F_4(x)$, that is

$$F_4(x) = (1-\theta(x))F_{N2}(x) + \theta(x)(x+B), \quad 0 \le x < \infty \quad (4.4)$$

with $\theta(x)$ as before. We present the results of the evaluation of $F_4(x)$ in Table 2 again. The results presented in Table 2 show that all values calculated from (2.6) lie within about 0.032% of the numerical solution [4] of the Blasius problem.

5. DISCUSSION AND CONCLUSIONS

We have presented three new *uniform* rational algebraic approximations to the Blasius velocity profile, along with corresponding *uniform* approximations to the Blasius function. In addition, a new uniform *rational* approximation to the Blasius function itself has been developed directly. All of these new functions were developed from existing approximations in the literature [1, 8] which were rescaled (when necessary) and combined according to the ideas of Yun [11] or Ahmad and Al-Barakati [1]. The numerical results for the four new (implicit) Blasius function approximations are presented in Table 2 and compared with the 'standard' numerical results of Howarth [4], also presented in Table 2.

A detailed examination of the results presented in Table 2 shows that the best uniform fit to the Blasius function is (4.4). The improvement of (4.4) over, say, (3.4) is probably [1] due to the fact that we have collocated using the 'exact' value for $x = 8$ from Howarth's results [4]. Also, (4.4) is a different type of improvement over the original (rescaled) rational *algebraic* approximation than was used in the other three cases; although both types of improvement rely on knowledge of the asymptotic form [1, 11] of the Blasius function and/or its first derivative.

As is usual in these discussions, we have concentrated on quoting the results of the numerical computation of the approximate Blasius functions only. This is probably due to the (apparent) convention of comparing the results of such analyses with the original numerical results of Howarth [4]. Naturally, there is no great difficulty in evaluating the velocity functions and, as there is sufficient information provided above, the interested readers can construct the full functional form of any of the approximate velocity profiles themselves. The velocity profiles are, one and all, sigmoid-shaped (as is only to be expected).

As a last word, we note that while the approximations presented here clearly approximate the solution of the Blasius problem, they are not solutions of the Blasius equation or of the equation for the velocity function. In every case, on substituting one of the approximations into the relevant equation, there will be a *residual* to account for. So, for example, if we substitute $F_4(x)$ into (1.1) we find that the residual is nonzero, that is

$$R_4(x) = F_4'''(x) + \frac{1}{2} F_4(x) F_4''(x) \neq 0 \tag{5.1}$$

However, this is normal in such work.

REFERENCES

[1] Ahmad, F. and Al-Barakati, W.H.: An approximate analytic solution of the Blasius problem. Communications in Nonlinear Science and Numerical Simulation **14** (2009) 1021–1024.

[2] Boyd, J.P.: The Blasius Function: Computations Before Computers, the Value of Tricks, Undergraduate Projects, and Open Research Problems. SIAM Review **50** (2008) 791–804.

[3] He, J.: Approximate Analytical Solution of Blasius' Equation. Communications in Nonlinear Science & Numerical Simulation **3** (1998) 260–263.

[4] Howarth L. On the solution of the laminar boundary layer equations. Proceedings of the London Mathematical Society A **164 (**1938) 547–79.

[5] Kinaci, O.K. and Usta, O.: A Method of Moments Approach for Laminar Boundary Layer Flows. International Journal of Ocean System Engineering **3** (2013) 111–115.

[6] Lal S.A. and Neeraj P.M.: An Accurate Taylors Series Solution with High Radius of Convergence for the Blasius Function and Parameters of Asymptotic Variation. Journal of Applied Fluid Mechanics, **7** (2014) 557–564.

[7] Lin, J.: New Approximate Iteration Solution of Blasius Equation. Communications in Nonlinear Science and Numerical Simulation **4** (1999) 91–94.

[8] Noghrehabadi, A., Ghalambaz, M. and Samimi, A.: Approximate solution of laminar thermal boundary layer over a thin plate heated from below by convection. Journal of Computation and Applied Research in Mechanical Engineering **2** (2013) 4 –57.

[9] Parand, K. and Taghavi, A.: Rational scaled generalized Laguerre function collocation method for solving the Blasius equation. Journal of Computational and Applied Mathematics **233** (2009) 980–989.

[10] Savaş, Ö.: An approximate compact analytical expression for the Blasius velocity profile. Communications in Nonlinear Science and Numerical Simulation **17** (2012) 3772–3775.

[11] Yun, B.I.: Algebraic Type Approximation to the Blasius Velocity Profile. Journal of Applied Mathematics Volume 2014, Article ID 754910, 5 pages.

[12] Yun, B.I.: Approximate Analytical Solutions Using Hyperbolic Functions for the Generalized Blasius Problem. Abstract and Applied Analysis Volume 2012, Article ID 581453, 10 pages.

[13] Yun, B.I.: Constructing Uniform Approximate Analytical Solutions for the Blasius Problem. Abstract and Applied Analysis Volume 2014, Article ID 495734, 6 pages.

[14] Yun, B.I.: Intuitive approach to the approximate analytical solution for the Blasius problem. Applied Mathematics and Computation **215** (2010) 3489–3494.